On generalized weakly conharmonically symmetric manifolds

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Abstract. The object of the present paper is to study generalized weakly conharmonically symmetric manifold and obtained some of its geometric properties. Existence of such manifold is ensured by a non-trivial example.

Keywords: weakly symmetric manifold, generalized weakly symmetric manifold, conharmonic curvature tensor.

1. Introduction

Let M be a $n(\geq 3)$-dimensional manifold equipped with a Riemannian metric g, Levi-Civita connection ∇, Riemnnan curvature R, Ricci tensor S and scalar curvature r. Generalizing Chaki pseudo symmetry, Támassy and Binh [16] introduced the notion of weakly symmetric manifold (briefly, $(w.s)_n$). The above M is called $(w.s)_n$ if

$$\nabla R = \alpha \otimes R + \beta \otimes R + \beta \otimes R + \gamma \otimes R + \gamma \otimes R.$$

For details we refer to see the paper of A. A. Shaikh and his co-authors ([3], [6], [7], [8], [9], [10], [11], [14], etc) and also the references therein.

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Generalizing $(w.s)_n$, recently, Baishya [1] introduced the notion of generalized weakly symmetric space (briefly g.w.s) defined by the following equations

$$\begin{aligned} \nabla R &= \alpha_1 \otimes R + \beta_1 \otimes R + \beta_1 \otimes R + \gamma_1 \otimes R + \gamma_1 \otimes R \\ &+ \alpha_2 \otimes G + \beta_2 \otimes G + \beta_2 \otimes G + \gamma_2 \otimes G + \gamma_2 \otimes G, \end{aligned} \tag{1}$$

where G is Guassian curvature tensor and α_1, β_1, γ_1, α_2, β_2 and γ_2 are non-zero 1-forms defined by $\alpha(X) = g(X, \theta_1)$, $\beta(X) = g(X, \phi_1)$ and $\gamma(X) = g(X, \pi_1)$, $\alpha_1(X) = g(X, \theta_2)$, $\beta_1(X) = g(X, \phi_2)$ and $\gamma_1(X) = g(X, \pi_2)$ for all X. As a special subgroup of the conformal transformation group,Y. Ishii [2] introduced the notion of the conharmonic transformation and defined a curvature tensor, called, conharmonic curvature tensor $\overline{C}$.

The object of the present paper is to study the notion of g.w.s with $\overline{C}$. The manifold M is called generalized weakly conharmonically manifolds if the conharmonic curvature tensor $\overline{C}$ satisfies the following relation:

$$\begin{aligned} &(\nabla_X \overline{C})(Y, Z, U, V) = \alpha_3(X)\overline{C}(Y, Z, U, V) + \beta_3(Y)\overline{C}(X, Z, U, V) \\ &+ \beta_3(Z)\overline{C}(Y, X, U, V) + \gamma_3(U)\overline{C}(Y, Z, X, V) + \gamma_3(V)\overline{C}(Y, Z, U, X) \\ &+ \alpha_4(X)G(Y, Z, U, V) + \beta_4(Y)G(X, Z, U, V) + \beta_4(Z)G(Y, X, U, V) \\ &+ \gamma_4(U)G(Y, Z, X, V) + \gamma_4(V)G(Y, Z, U, X), \end{aligned} \tag{2}$$

where α_3, β_3, γ_3, α_4, β_4 and γ_4 are non-zero 1-forms defined as $\alpha_3(X) = g(X, \overline{\theta_1})$, $\beta_3(X) = g(X, \overline{\phi_1})$ and $\gamma_3(X) = g(X, \overline{\pi_1})$, $\alpha_4(X) = g(X, \overline{\theta_2})$, $\beta_4(X) = g(X, \overline{\phi_2})$ and $\gamma_4(X) = g(X, \overline{\pi_2})$. Such an n-dimensional manifold is denoted by $(GW\overline{C}S)_n$. The nature of the scalar curvature is determined and shown that the 1-forms are co-directional. It is observed that a weakly conformally symmetric space is a $(GW\overline{C}S)_n$ and vice-versa. A non-trivial example is found out to ensure the existence of such space.

The structure of the paper is as follows. Section 2 deals with preliminaries. Section 3 is concerned with some basic properties of $(GW\overline{C}S)_n$. In section 4 we investigate some spaces which are $(GW\overline{C}S)_n$. Section 5 deals with Einstein $(GW\overline{C}S)_n$. The last section is concerned with a non-trivial example of $(GW\overline{C}S)_n$.

2. Preliminaries

The conharmonic curvature tensor $(\overline{C})$ defined by Ishii [2] is given as follows

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{3}$$

where $\wedge$ is the wedge product ([12], [13]) defined by

$$\begin{aligned} (A \wedge B)(X_1, X_2, Y_1, Y_2) &= A(X_1, Y_2)B(X_2, Y_1) + A(X_2, Y_1)B(X_1, Y_2) \\ &- A(X_1, Y_1)B(X_2, Y_2) - A(X_2, Y_2)B(X_1, Y_1), \end{aligned}$$

where A and B are two (0,2) tensors and X_1, X_2, Y_1 and $Y_2 \in \chi(M)$, $\chi(M)$ being the Lie algebra of all smooth vector fields on M.

The local expression of (2) is

$$\begin{aligned}\overline{C}_{mnpq,k} &= \alpha_{3k}\overline{C}_{mnpq} + \beta_{3m}\overline{C}_{knpq} + \beta_{3n}\overline{C}_{mkpq} + \gamma_{3p}\overline{C}_{mnkq} \\ &+ \gamma_{3q}\overline{C}_{mnpk} + \alpha_{4k}G_{mnpq} + \beta_{4m}G_{knpq} + \beta_{4n}G_{mkpq} \\ &+ \gamma_{4p}G_{mnkq} + \gamma_{4q}G_{mnpk}.\end{aligned} \tag{4}$$

If the 1-forms $\alpha_3 = \beta_3 = \gamma_3 = 0$, then the manifold reduces to a weakly conharmonically symmetric manifold [4], [5].

3. Some basic geometric properties of $(GW\overline{C}S)_n$

Now, contracting (2) we have

$$\begin{aligned}&-\frac{1}{n-2}g(Y,V)dr(X) \\ &= -\frac{1}{n-2}\alpha_3(X)rg(Y,V) - \frac{1}{n-2}rg(X,V)\beta_3(Y) \\ &- \beta_3(R(Y,X)V) - \frac{1}{n-2}[\beta_3(X)S(Y,V) + \beta_3(QX)g(Y,V) \\ &- \beta_3(Y)S(X,V) - g(X,V)\beta_3(QY)] \\ &+ \gamma_3(R(X,V)Y) - \frac{1}{n-2}[\gamma_3(X)S(Y,V) + \gamma_3(QX)g(Y,V) \\ &- g(Y,X)\gamma_3(QV) - \gamma_3(V)S(Y,X)] - \frac{1}{n-2}\gamma_3(V)rg(Y,X) \\ &+ (n-1)[\alpha_4(X)g(Y,V) + \beta_4(Y)g(X,V) + \gamma_4(V)g(Y,X)] \\ &+ [\beta_4(X)g(Y,V) - \beta_4(Y)g(X,V)] + [\gamma_4(X)g(Y,V) - \gamma_4(V)g(Y,X)].\end{aligned} \tag{5}$$

Contraction (5) over Y and V, we get

$$\begin{aligned}\frac{1}{n-2}ndr(X) &= \frac{1}{n-2}nr\alpha_3(X) + \frac{2}{n-2}r\beta_3(X) \\ &+ 2n - 2r\gamma_3(X) - (n-1)[n\alpha_4(X) + 2\beta_4(X) + 2\gamma_4(X)].\end{aligned} \tag{6}$$

Again taking contraction of (5) over Y and X, we get

$$\begin{aligned}-\frac{1}{n-2}dr(V) &= -\frac{1}{n-2}r\alpha_3(V) \\ &- \frac{1}{n-2}r\beta_3(V) - \frac{1}{n-2}r(n-1)\gamma_3(V) \\ &+ (n-1)[\alpha_4(V) + \beta_4(V) + (n-1)\gamma_4(V)].\end{aligned} \tag{7}$$

Contracting (5) over X and V gives us

$$
\begin{aligned}
(8)\qquad -\frac{1}{n-2}dr(Y) &= -\frac{1}{n-2}r\alpha_3(Y) \\
&- r\frac{(n-1)}{n-2}\beta_3(Y) - \frac{1}{n-2}r\gamma_3(Y) \\
&+ (n-1)[\alpha_4(Y) + (n-1)\beta_4(Y) + \gamma_4(Y)].
\end{aligned}
$$

From (6) and (7), we have

$$
(9)\qquad r = (n-1)(n-2)\frac{[\beta_4(X) + (n+1)\gamma_4(X)]}{[\beta_3(X) + (n+1)\gamma_3(X)]}.
$$

In view of (6) and (8) we get

$$
(10)\qquad r = (n-1)(n-2)\frac{[(n+1)\beta_4(X) + \gamma_4(X)]}{[(n+1)\beta_3(X) + \gamma_3(X)]}.
$$

From (7) and (8) we obtain

$$
(11)\qquad r = (n-1)(n-2)\frac{[\beta_4(X) - \gamma_4(X)]}{[\beta_3(X) - \gamma_3(X)]}.
$$

Also (9) and (11) yields

$$
(12)\qquad \gamma_4(X)\beta_3(X) = \beta_4(X)\gamma_3(X).
$$

This leads the following:

Theorem 3.1. *In a* $(GW\overline{C}S)_n$, *the* 1*-forms are related by the expression* (12).

Using (12) in (11) we have

$$
(13)\qquad r = (n-1)(n-2)\frac{\gamma_4(X)}{\gamma_3(X)}.
$$

Again by virtue of (12), (10) takes the form

$$
(14)\qquad r = (n-1)(n-2)\frac{\beta_4(X)}{\beta_3(X)}.
$$

Thus, we can state the following:

Theorem 3.2. *The scalar curvature of a* $(GW\overline{C}S)_n$ *is given by* (13) *and* (14).

If the scalar curvature of the manifold is constant, with the help (13) and (14) we obtain from (6) that

$$
(15)\qquad r\alpha_3(X) = (n-1)(n-2)\alpha_4(X).
$$

This gives the following:

Theorem 3.3. *If the scalar curvature of a* $(GW\overline{C}S)_n$ *is constant, then its associated 1-forms are related by* (15).

If the scalar curvature of $(GW\overline{C}S)_n$ is a non-zero constant, then we have $dr(X) = 0$ and hence from (6), (7) and (8) we obtain

$$\begin{aligned} &nr\alpha_3(X) + 2r\beta_3(X) + 2r\gamma_3(X) \\ -\;&(n-1)(n-2)\big[n\alpha_4(X) + 2\beta_4(X) + 2\gamma_4(X)\big] = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &r\alpha_3(X) + r\beta_3(X) + (n-1)r\gamma_3(X) \\ -\;&(n-1)(n-2)\big[\alpha_4(X) + \beta_4(X) + (n-1)\gamma_4(X)\big] = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &r\alpha_3(X) + r\beta_3(X) + (n-1)r\gamma_3(X) \\ -\;&(n-1)(n-2)\big[\alpha_4(X) + (n-1)\beta_4(X) + \gamma_4(X)\big] = 0. \end{aligned} \tag{18}$$

From the above relations we can state the following:

Theorem 3.4. *If the scalar curvature of a* $(GW\overline{C}S)_n$ *is constant, then its associated 1-forms are related by the expression (16), (17) and (18).*

Theorem 3.5. *If the scalar curvature of a* $(GW\overline{C}S)_n$ *is constant, then 1-forms* α_3*,* β_3 *and* γ_3 *are respectively co-directional with the 1-forms* α_4*,* β_4 *and* γ_4*.*

4. Weakly conformally symmetric space and $(GW\overline{C}S)_n$

We consider a Riemannian manifold which is weakly conformally symmetric [15], then we have

$$\begin{aligned} (\nabla_X C)(Y,Z,U,V) = {}& \alpha_5(X)C(Y,Z,U,V) + \beta_5(Y)C(X,Z,U,V) \\ &+ \beta_5(Z)C(Y,X,U,V) + \gamma_5(U)C(Y,Z,X,V) \\ &+ \gamma_5(V)C(Y,Z,U,X), \end{aligned} \tag{19}$$

where α_5, β_5, γ_5 are non-zero 1-forms and C is the conformal curvature tensor given by

$$C = \overline{C} + \frac{r}{(n-1)(n-2)}G.$$

Using above equation in (19) we get

$$\begin{aligned} (\nabla_X \overline{C})(Y,Z,U,V) = {}& \alpha_6(X)\overline{C}(Y,Z,U,V) + \beta_6(Y)\overline{C}(X,Z,U,V) \\ &+ \beta_6(Z)\overline{C}(Y,X,U,V) + \gamma_6(U)\overline{C}(Y,Z,X,V) + \gamma_6(V)\overline{C}(Y,Z,U,X) \\ &+ \alpha_7(X)G(Y,Z,U,V) + \beta_7(Y)G(X,Z,U,V) + \beta_7(Z)G(Y,X,U,V) \\ &+ \gamma_7(U)G(Y,Z,X,V) + \gamma_7(V)G(Y,Z,U,X), \end{aligned} \tag{20}$$

where $\alpha_6 = \alpha_5$, $\beta_6 = \beta_5$, $\gamma_6 = \gamma_5$, $\alpha_7 = \frac{1}{(n-1)(n-2)}[r\alpha_5 - dr]$, $\beta_7 = \frac{r\beta_5}{(n-1)(n-2)}$ and $\gamma_7 = \frac{r\gamma_5}{(n-1)(n-2)}$ are non-zero 1-forms. Conversely, if we consider the space as a $(GW\overline{C}S)_n$, then we can easily shown that the space is weakly conformally symmetric. This leads to the following:

Theorem 4.1. *A weakly conformally symmetric space is a $(GW\overline{C}S)_n$ and vice-versa.*

5. Einstein $(GW\overline{C}S)_n$

Let us consider a $(GW\overline{C}S)_n$ which is Einstein. Then we have

$$S(X,Y) = \frac{r}{n} g(X,Y), \tag{21}$$

which yields $(\nabla_Z S)(X,Y) = 0$ and $dr(X) = 0$. By virtue of (21), (3) yields

$$\overline{C} = R - \frac{2r}{n(n-2)} G. \tag{22}$$

Using (22) in (2) we have

$$\begin{aligned}
&(\nabla_X R)(Y,U,V,W) = \alpha_8(X)R(Y,U,V,W) + \beta_8(Y)R(X,U,V,W)\\
&+ \beta_8(U)R(Y,X,V,W) + \gamma_8(V)R(Y,U,X,W) + \gamma_8(W)R(Y,U,V,X)\\
&+ \alpha_9(X)G(Y,U,V,W) + \beta_9(Y)G(X,U,V,W) + \beta_9(U)G(Y,X,V,W)\\
&+ \gamma(V)_9 G(Y,U,X,W) + \gamma_9(W)G(Y,U,V,X),
\end{aligned} \tag{23}$$

where $\alpha_8 = \alpha_3$, $\beta_8 = \beta_3$, $\gamma_8 = \gamma_3$, $\alpha_9 = \alpha_4 - \frac{2r}{n(n-2)}\alpha_3$, $\beta_9 = \beta_4 - \frac{2r}{n(n-2)}\beta_3$ and $\gamma_9 = \gamma_4 - \frac{2r}{n(n-2)}\gamma_3$ are non-zero 1-forms and hence the manifold under consideration is a *g.w.s* [1].

This leads to the following:

Theorem 5.1. *An Einstein $(GW\overline{C}S)_n$ is a g.w.s.*

6. Existence of a $(GW\overline{C}S)_4$ space

Let $(\mathbb{R}^4, g)$ be a connected smooth 4-dimensional Riemannian space endowed with the Riemannian metric g given by

$$\begin{aligned}
&ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^{\frac{4}{3}}(dx^2)^2 + (x^1)^{\frac{4}{3}}(dx^3)^2 + (x^1)^{\frac{4}{3}}(dx^4)^2,\\
&\text{with} \quad x^1 > 0, \quad (i,j = 1,2,3,4).
\end{aligned} \tag{24}$$

$$\begin{cases}
\overline{C}_{1212} = \frac{2}{9(x^1)^{\frac{2}{3}}} = \overline{C}_{1313} = \overline{C}_{1414},\\
\overline{C}_{2323} = \frac{2(x^1)^{\frac{2}{3}}}{9} = \overline{C}_{2424} = \overline{C}_{3434},\\
r = \frac{4}{3(x^1)^2}.
\end{cases} \tag{25}$$

With the help of (25), we can find out

$$\begin{cases}
G_{1212} = -(x^1)^{\frac{4}{3}} = G_{1313} = G_{1414},\\
G_{2323} = -(x^1)^{\frac{8}{3}} = G_{2424} = G_{3434}.
\end{cases} \tag{26}$$

The non-vanishing components of covariant derivative of conharmonic curvature tensors are

$$\begin{cases} \overline{C}_{1212,1} = -\frac{4}{9(x^1)^{\frac{5}{3}}} = \overline{C}_{1313,1} = \overline{C}_{1414,1}, \\ \overline{C}_{2323,1} = -\frac{4}{9(x^1)^{\frac{1}{3}}} = \overline{C}_{2424,1} = \overline{C}_{3434,1}. \end{cases} \tag{27}$$

It is easy to check that the metric (24) is conformally flat.
We consider the 1-forms as follows:

$$\begin{cases} \alpha_3(\partial_i) = \alpha_{3i} = \begin{cases} \frac{7}{x^1}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \beta_3(\partial_i) = \beta_{3i} = \begin{cases} \frac{9}{2}(x^1)^2, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \gamma_3(\partial_i) = \gamma_{3i} = \begin{cases} 9x^1, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \end{cases} \tag{28}$$

$$\begin{cases} \alpha_4(\partial_i) = \alpha_{4i} = \begin{cases} \frac{2}{(x^1)^3}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \beta_4(\partial_i) = \beta_{4i} = \begin{cases} 1, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \\ \gamma_4(\partial_i) = \gamma_{4i} = \begin{cases} \frac{2}{x^1}, & \text{for} \quad i = 1, \\ 0, & \text{otherwise}, \end{cases} \end{cases} \tag{29}$$

where $\partial_i = \frac{\partial}{\partial u^i}$, u^i being the local coordinates of $\mathbb{R}^4$. By virtue of (25)-(29) it can be easily shown that the manifold considering above is a $(GW\overline{C}S)_4$ and hence we can state the following:

Theorem 6.1. *Let $M = (\mathbb{R}^4, g)$ be a Riemannian manifold equipped with the metric given by* (24)*. Then M is a $(GW\overline{C}S)_4$ with non-vanishing and non-constant scalar curvature.*

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