# Study on value distribution of L-functions sharing a small function

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**Abstract.** In this paper, we mainly concentrate on Nevanlinna theory and the properties of $L$-functions in the extended selberg class, we mainly study the uniqueness problems on $L$-functions related to Brück conjecture. This extend the result due to Q.C. Zhang.

**Keywords:** Nevanlinna theory, $L$-functions, integral function, meromorphic function, weighted sharing, differential polynomial.

## 1. Introduction

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We denote by $N_{k)}(r, \frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity at least $k$ and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Moreover, $f$ and $g$ are said to share a value $c$ CM (counting multiplicities) if they share the value $c$ and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM.

The Riemann hypothesis as one of the millenium problems has been given a lot of attention by mathematical workers for a long time. Selberg guessed that

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the Riemann hypothesis is also true for L-functions in the selberg class. Such an $L$-function based on Riemann zeta function as the prototype is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable $s = \sigma + it$ satisfying the following axioms:

(i) Ramanujan hypothesis. $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) Analytic continuation. there is a non-negative integer $m$ such that $(s-1)^m L(s)$ is an entire function of finite order.

(iii) Functional equation. $L$ satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers $Q, \lambda_j$ and complex numbers $\nu_j, \omega$ with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product. $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless $n$ is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Let $z_0$ be a zero of $f-a$ of multiplicity $p$ and a zero of $g-a$ of multiplicity $q$. We denote by $\overline{N}_L(r, a; f)$ the counting function of those $a$-points of $f$ and $g$ where $p > q \geq 1$, by $N_E^{1)}(r, a; f)$ the counting function of those $a$-points of $f$ and $g$ where $p = q = 1$ and by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those $a$-points of $f$ and $g$ where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g), \overline{N}_E^{(2}(r, a; g)$.

Let $k$ be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with weight $k$.

In connection to find the relation between an entire function with its derivative when they share one value CM, in 1996, in this direction the following famous conjecture was proposed by Brück [5]:

**Conjecture.** Let $f$ be a non-constant entire function such that the hyper order $\rho_2(f)$ is not a positive integer or infinite. If $f$ and $f'$ share a finite value $a$ CM, then $\frac{f'-a}{f-a} = c$, where $c$ is a nonzero constant.

Brück himself proved the conjecture for $a = 0$ and for $a = 1$, he showed that under the assumption $N(r, 0; f') = S(r, f)$ the conjecture was true.

**Theorem A** ([5]). *Let $f$ be a non-constant entire function. If $f$ and $f'$ share the value 1 CM and if $N(r, 0; f') = S(r, f)$, then $\frac{f'-1}{f-1}$ is a nonzero constant.*

However, for entire function of finite order, Yang [30] removed the supposition $N(r, 0; f') = 0$ and obtained the following result.

**Theorem B** ([30]). *Let $f$ be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If $f, f^{(k)}$ share the value $a$ CM, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

In 2005, Zhang [33] further extended the results of Lahiri-sarkar to a small function and proved the following result for IM sharing.

**Theorem C** ([33]). *Let $f$ be a non-constant meromorphic function and $k(\geq 1)$, and $l \geq 0$ be integer. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $(f)^{(k)} - a$ share $(0, l)$.*

If $l \geq 2$ and

$$2\overline{N}(r, f) + N_2\left(r, 0; (f)^{(k)}\right) + N_2\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, if $l = 1$

$$2\overline{N}(r, f) + N_2\left(r, 0; (f)^{(k)}\right) + 2\overline{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, $l = 0$ and

$$4\overline{N}(r, f) + 3N_2\left(r, 0; (f)^{(k)}\right) + 2\overline{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

for $r \in I$, where $0 < \lambda < 1$, and $I$ is a set of infinite linear measure, then $\frac{f^{(k)}-a}{(f)-a} = c$, for some constant $c \in C - \{0\}$.

Now, it is natural to ask the following question which is the motivation of the paper.

**Question.** Can Brück type conclusion be obtained when $f$ and $f^{(k)}$ is replaced by $[f^n]^{(k)}$ and $[L^n]^{(k)}$ in Theorem C.

Now, we have the following theorems

**Theorem 1.** *Let $f$ be a nonconstant entire function in $\mathbb{C}$, let $L$ be an non-constant $L$ - function, and let $n$ and $k$ be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(a(z), l)$, where $a(z)(\not\equiv 0, \infty)$ be a meromorphic function. Suppose one of the following conditions hold:*

*I.* $l \geq 2$ *and* $n > 2k + 4$ *;*

*II.* $l = 1$ *and* $n > \frac{5k+9}{2}$ *;*

*III.* $l = 0$ *and* $n > 5k + 7$.

*Then,* $\frac{(L^n)^{(k)}-1}{(f^n)^{(k)}-1} = C$.

## 2. Lemmas

Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by $H$ the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

**Lemma 1** ([25]). *Let $f$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), ..., a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

**Lemma 2** ([27]). *Suppose that $f$ is a nonconstant meromorphic function in the complex plane and $k$ is a positive integer. Then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty, f) + O(log(T(r, f)) + \log r),$$

*as $r \to \infty$, outside of a possible exceptional set of finite linear measure.*

**Lemma 3** ([33]) *Let $f$ be a nonconstant meromorphic function, and $p, k$ be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f). \tag{3}$$

**Lemma 4** ([4]). *Let $F$ and $G$ share $(1, l)$ and $\overline{N}(r, \infty; F) = \overline{N}(r, \infty; G)$ and $H \not\equiv 0$, where $F, G$ and $H$ are defined as earlier. Then*

$$\begin{aligned} N(r, \infty, H) \leq\ & \overline{N}(r, \infty; F) + \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + \overline{N}_L(r, 1; F) \\ & + \overline{N}_L(r, 1; G) + S(r, F). \end{aligned}$$

**Lemma 5** ([4]). *If $F$ and $G$ share $(1, l)$, then*

$$\begin{aligned} \overline{N}_L(r, 1; F) &\leq \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F) \text{ when } l \geq 1, \\ \overline{N}_L(r, 1; F) &\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ when } l = 0. \end{aligned}$$

**Lemma 6** ([4]). *Let $F$ and $G$ share $(1, l)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} \overline{N}(r, 1, F) + \overline{N}(r, 1, G) \leq\ & N(r, \infty; H) + \overline{N}_E^{(2}(r, 1; F) \\ & + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, F). \end{aligned}$$

## 3. Proof of the Theorem 1

Suppose that $d$ is the degree of $L$. Then $d = 2\sum_{i=1}^{k}\lambda_j$, where $k$ and $\lambda_j$ are respectively the positive integer and the positive real number in the axiom (iii) of the definition of $L$- function. Then, we have that

$$T(r,L) = \frac{d}{\Pi} r\log r + O(r)$$

(cf. [8] , P.150). Clearly, $f$ and $L$ are transcendental meromorphic functions (cf. [22] , P.43). Note that an $L$- function at most has one pole $z = 1$ in the complex plane.

Let $F = \frac{(f^n)^{(k)}}{a(z)}$ and $G = \frac{(L^n)^{(k)}}{a(z)}$. Then $F-1 = \frac{(f^n)^{(k)}-a}{a}$, $G-1 = \frac{(L^n)^{(k)}-a}{a}$. Since $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(a,l)$ it follows that $F$ and $G$ share $(1,l)$ except the zeros and poles of $a(z)$.

Now, we consider the following cases.

**Case 1.** Let $H \not\equiv 0$.

**Subcase 1.1.** Assume $l \geq 1$. Using the second fundamental Theorem and Lemmas 4, 6 we get

$$\begin{aligned}
(4)\quad T(r,F)+T(r,G) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)\\
&+\overline{N}(r,0;G)+N(r,H)+\overline{N}_E^{(2}(r,1;F)\\
(5)\qquad &+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\
&-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)\\
&\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N2(r,0;F)+N2(r,0;G)\\
&+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)\\
&+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,F).
\end{aligned}$$

**Subcase-1.1.1.** Next assume $l \geq 2$. Now, by using the inequality (4) and Lemma 3, we get

$$\begin{aligned}
T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)\\
&+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)\\
&+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,F)\\
&\leq 3\overline{N}(r,\infty;f)+N_2(r,0;F)\\
&+N_2(r,0;G)+N(r,1;F)+S(r,f)\\
T(r,G) &\leq 3\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^n)+k\overline{N}(r,\infty;f)\\
&+N_2(r,0;G)+S(r,f)\\
&\leq 3\overline{N}(r,\infty;f^n)+N_{k+2}(r,0;f^n)+k\overline{N}(r,\infty;f^n)\\
&+N_2(r,0;(L^n)^{(k)})+S(r,f)\\
T(r,L^n) &\leq 3\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^n)+k\overline{N}(r,\infty;f)
\end{aligned}$$

$$\begin{aligned}
&+ N_{k+2}(r,0;L^n) + k\overline{N}(r,L) + S(r,f)\\
&\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n)\\
&+ k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n) + S(r,f)\\
&\leq (k+2)T(r,f) + (k+2)T(r,L) + O(\log r)\\
nT(r,L) &\leq (2k+4)T(r) + O(\log r),
\end{aligned}$$

where $T(r) = \max\{T(r,f), T(r,L)\}$. In a similar way we can obtain

$$nT(r,f) \leq (2k+4)T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (2k+4)T(r) + O(\log r).$$

Since $n > 2k+4$, above equation leads to a contradiction.

**Subcase-1.1.2.** Next, we assume $l = 1$. Now inequality (4) and in view of Lemmas 3, 5 we get

$$\begin{aligned}
T(r,F) + T(r,G) &\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;F)\\
&+ 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F)\\
&\leq 2\overline{N}(r,\infty;F) + \frac{3}{2}\overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G)\\
&+ N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F)\\
&+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,F)\\
&\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_1(r,0;G) + N_2(r,0;F)\\
&+ N_2(r,0;G) + N(r,1;F) + S(r,f),\\
T(r,L^n) &\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_1(r,0;(L^n)^{(k)})\\
&+ N_2(r,0;(f^n)^{(k)}) + N_2(r,0;(L^n)^{(k)}) + S(r,f)\\
&\leq 2\overline{N}(r,\infty;f) + \frac{1}{2}N_{k+1}(r,0;L^n)\\
&+ N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n)\\
&+ \frac{3}{2}k\overline{N}(r,L) + S(r,f)\\
&\leq (k+2)T(r,f) + (\frac{3k}{2} + \frac{5}{2})T(r,L) + O(\log r),\\
nT(r,L) &\leq (\frac{5k}{2} + \frac{9}{2})T(r) + O(\log r)
\end{aligned}$$

where $T(r) = \max\{T(r,f), T(r,L)\}$ . In a similar way we can obtain

$$nT(r,f) \leq (\frac{5k}{2} + \frac{9}{2})T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (\frac{5k+9}{2})T(r) + O(\log r).$$

Since $n > \frac{5k+9}{2}$, above equation leads to a contradiction.

**Subcase-1.2.** Next we assume $l = 0$. Then by using the second fundamental Theorem and Lemmas 3, 4, 5 and 6 we get

$$\begin{aligned}
T(r,F) + T(r,G) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) \\
&+ \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) \\
&- N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G) \\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) \\
&+ N(r,\infty;H) + \overline{N}_E^{(2}(r,1;F) \\
&+ \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) \\
&- N_0(r,0;F') - N_0(r,0;G') + S(r,f) \\
&\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G) \\
&+ N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F) \\
&+ 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\
&\leq 3\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;G) \\
&+ \overline{N}(r,0;F) + N(r,1;F) + S(r,f), \\
T(r,L^n) &\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_2(r,0;(L^n)^{(k)}) \\
&+ \overline{N}(r,0;(f^n)^{(k)}) + 2\overline{N}(r,0;G) + S(r,f) \\
&\leq 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^n) + k\overline{N}(r,\infty;f) + N_{k+2}(r,0;L^n) \\
&+ N_{k+1}(r,0;f^n) \\
&+ k\overline{N}(r,f) + 2N_{k+1}(r,0;L^n) + 3k\overline{N}(r,L) + S(r,f) \\
&\leq (2k+3)T(r,f) + (3k+4)T(r,L) + O(\log r), \\
nT(r,L) &\leq (5k+7)T(r) + O(\log r),
\end{aligned}$$

where $T(r) = \max\{T(r,f), T(r,L)\}$.

In a similar way we can obtain

$$nT(r,f) \leq (5k+7)T(r) + O(\log r).$$

Combining the above two equations we see that

$$nT(r) \leq (5k+7)T(r) + O(\log r).$$

Since $n > 5k+7$, above equation leads to a contradiction

**Case 2.** If $H \equiv 0$, then on integration, we get

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{6}$$

where $C, D$ are constants and $C \neq 0$. From (3.2) it is clear that $F$ and $G$ share 1 CM. We first, assume that $D \neq 0$. Then, by (6) we get

$$\overline{N}(r, \infty; f) = S(r, f). \tag{7}$$

Now, we can write (6) as

$$\frac{1}{F-1} = \frac{D\left(G - 1 + \frac{C}{D}\right)}{G-1}. \tag{8}$$

Consequently,

$$\overline{N}\left(r, 1 - \frac{C}{D}; G\right) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r, F). \tag{9}$$

**Subcase-2.1.** If $\frac{C}{D} \neq 1$, by the second fundamental theorem, Lemma 3, we have

$$\begin{aligned}
T\left(r, G\right) &\leq \overline{N}(r, \infty; G) + N_1\left(r, 0; G\right) + \overline{N}\left(r, 1 - \frac{C}{D}; G\right) + S(r, G) \\
&\leq \overline{N}\left(r, 0; G\right) + S(r, f) \leq N_2\left(r, 0; G\right) + S(r, f), \\
T(r, G) &\leq N_{k+2}(r, 0; L) + S(r, f), \\
T(r, L^n) &\leq (k+2)\overline{N}(r, 0; L) + S(r, f),
\end{aligned}$$

i.e., $n \leq k+2$, which is a contradiction since $n > k+2$.

**Subcase-2.2.** If $\frac{C}{D} = 1$, we get from (6)

$$\left(F - 1 - \frac{1}{C}\right) G \equiv -\frac{1}{C} \tag{10}$$

i.e.

$$\frac{1}{L^n[(f^n)^{(k)} - (1 + \frac{1}{C})a]} \equiv -\frac{C}{a^2} \frac{[L^n]^{(k)}}{L^n}. \tag{11}$$

From (10) it follows that

$$N\left(r,0;[f^n]^{(k)}|\geq k+1\right)\leq N\left(r,0;[L^n]^{(k)}\right)$$
$$\leq N(r,0;G)\leq N(r,0;a)=S(r,f). \tag{12}$$

Applying the first fundamental theorem, (7), (11) and (12)

$$(n-k)T(r,f)+nT(r,L)=T\left(r,\frac{1}{L^n[(f^n)^{(k)}-(1+\frac{1}{C})a]}\right)$$
$$=m\left(r,\frac{(L^n)^{(k)}}{L^n}\right)+N\left(r,\frac{(L^n)^{(k)}}{L^n}\right)+S(r,f). \tag{13}$$

From (13) it follows that $(n-k)T(r,f)\leq S(r,f)$, which is impossible.

Hence, $D=0$ and $\frac{G-1}{F-1}=C$ or $\frac{(L^n)^{(k)}-1}{(f^n)^{(k)}-1}=C$.

## References


[1] A. Banerjee, *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci., 22 (2005), 3587-3598.

[2] A. Banerjee, *On a question of Gross*, J. Math. Anal. Appl, 327 (2007), 1273-1283.

[3] A. Banerjee, Goutam Haldar, *Certain non-linear differential polynomials having common poles sharing a non zero polynomial with finite weight*, Journal of Classical Analysis, 6 (2015), 167-190.

[4] Abhijit Banerjee, Bikash Chakraborty, *Further investigations on a question of Zhang and Lü*, Ann. Univ. Paedagog. Crac. Stud. Math., 14 (2015), 105-119.

[5] Brück, Rainer, *On entire functions which share one value CM with their first derivative,* Results Math., 30 (1996), 21-24.

[6] C. Meng, *On unicity of meromorphic function and its kth order derivative,* Journal of Mathematical Inequalities, 4 (2010), 151-159

[7] M.-L. Fang, *Uniqueness and value-sharing of entire functions,* Comput. Math. Appl, 44 (2002), 823-831.

[8] Fang LIU, Xiao-Min LI, Hong-Xun YI, *Value distribution of L-functions concerning shared values,* Proc. Japan Acad., 93 (2017).

[9] W.K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[10] W. K. Hayman, J. Miles, *On the growth of a meromorphic function and its derivatives,* Complex Variables Theory Appl., 12 (1989), 245-260.

[11] Jun-Feng Xu, Feng Lu, Hong-Xun Yi, *Fixed-points and uniqueness of meromorphic functions,* Computers and Mathematics with Applications, 59 (2010), 9-17.

[12] I. Lahiri, *Uniqueness of meromorphic functions as governed by their differential polynomials,* Yokohama Math. J., 44 (1997), 147-156.

[13] I. Lahiri, *Weighted sharing of three values and uniqueness of meromorphic functions,* Kodai Math. J., 24 (2001), 421-435.

[14] I. Lahiri, *On a question of Hong Xun Yi,* Arch. Math. (Brno), 38 (2002), 119-128.

[15] I. Laine, *Nevanlinna theory and complex differential equations,* de Gruyter Studies in Mathematics, 15, de Gruyter, Berlin, 1993.

[16] P. Li, C. C. Yang, *On the characteristics of meromorphic functions that share three values CM,* J. Math. Anal. Appl., 220 (1998), 132-145.

[17] B. Q. Li, *A result on value distribution of L-functions,* Proc. Amer. Math. Soc., 138 (2010), 2071-2077.

[18] X.-M. Li, H.-X. Yi, *Uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a polynomial,* Comput. Math. Appl., 62 (2011), 539–550.

[19] A. Z. Mokhonko, *On the Nevanlinna characteristics of some meromorphic functions,* in Theory of Functions, Functional Analysis and their Applications, Izd-vo Khar'kovsk. Un-ta, 14 (1971), 83-87.

[20] R. Nevanlinna, *Einige eindeutigkeitssatze in der theorie der meromorphen funktionen,* Acta Math., 48 (1926), 367-391.

[21] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series,* in Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367-385, Univ. Salerno, Salerno, 1992.

[22] J. Steuding, *Value-distribution of L-functions,* Lecture Notes in Mathematics, 1877, Springer, Berlin, 2007.

[23] J. M. Whittaker, *The order of the derivative of a meromorphic function,* J. London Math. Soc., S1-11 (1936), 82-87.

[24] W. Lv, T. Zhang, *On uniqueness of a meromorphic function with its derivative,* J. Anal. Appl., 6 (2008), 41-53.

[25] L.Q.Wang, X.D.Luo, *Uniqueness of meromorphic functions concerning fixed points,* Math. Slovaca, 62 (2012), 29-38.

[26] C.-C. Yang, X. Hua, *Uniqueness and value sharing of meromorphic functions,* Ann. Acad. Sci. Fenn. Math., 22 (1997), 395–406.

[27] C.-C. Yang, H.-X. Yi, *Uniqueness theory of meromorphic functions mathematics and its applications*, 557, Kluwer Acad. Publ., Dordrecht, 2003.

[28] L. Yang, *Normality for families of meromorphic functions,* Sci. Sinica Ser. A, 29 (1986), 1263–1274.

[29] L. Yang, *Value distribution theory,* translated and revised from the 1982 Chinese original, Springer, Berlin, 1993.

[30] Yang, Lian-Zhong, *Solution of a differential equation and its applications,* Kodai Math. J., 22 (1999), 458-464.

[31] J. Zhang, L. Yang, *A power of a meromorphic function sharing a small function with its derivative,* Ann. Acad. Sci. Fenn. Math., 34 (2009), 249-260.

[32] Q.C. Zhang, *Meromorphic functions sharing three values,* Indian J. Pure Appl. Math., 30 (1999), 667-682.

[33] Q.C. Zhang, *Meromorphic function that shares one small function with its derivative,* J.Inequal. Pure Appl. Math., 6 (2005), Art.116.

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