Generalized-pre-irresolute and λ-irresolute functions in closure spaces

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Abstract. In this paper, we introduce and study two new types of maps called generalized-pre-irresolute functions and λ-irresolute functions. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces,we find some relation between them.
Keywords: closure operator, closure space, generalized-pre-irresolute functions, λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions.

1. Introduction

In 1922, C. Kuratowski defined closure operator [18]. Let X be a set and $P(X)$ its power set. A kuratowski closure operator is a function $Cl : P(X) \rightarrow P(X)$ with the following properties:

1. $Cl(\phi) = \phi$;
2. $A \subseteq Cl(A)$, for every subset $A \subseteq X$;
3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$, for any subsets $A, B \subseteq X$;
4. $Cl(Cl(A)) = Cl(A)$, for every subset $A \subseteq X$.

Then Cl, together with the universal set X, is called closure space and is denoted by (X, Cl). His closure operator makes closed sets of topological space.

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After him mathematician called this closure operator the kuratowski closure operator. In 1937, Eduard Cech [1] defined closure operator as a Cech closure operator. After that, in 1964, P.C. Hammer [14] defined closure operator as followed:

Let X be a set and $P(X)$ its power set. A function $c : P(X) \to P(X)$ defined on $P(X)$ is called a closure operator on X, if it is satisfied the following conditions:

1. $c(\phi) = \phi$;
2. $A \subseteq c(A)$, for every subset $A \subseteq X$;
3. If $A \subseteq B$, then $c(A) \subseteq c(B)$, for any subsets $A, B \subseteq X$.

He said the pair (X, c) is called a neighborhood space [14]. But, J. Khampakdee [16] named this space as closure space. A closure operator is said to be idempotent [17] if $cc(A) = c(A)$, for every subset $A \subseteq X$. It is called additive if $c(A \cup B) = c(A) \cup c(B)$ for every subsets $A, B \subseteq X$. The additive closure operator c is Cech closure operator, and the idempotent Cech closure operator is kuratowski closure operator and (X, c) is a topological space, in this case, c is denoted by Cl, where Cl is closure operator with respect kuratowski closure. In several branches of Science, the concept of closure operator and closure spaces are very advantageous material, such as Topology [3], [4], [20], [22], Computer Science Theory [21] and Biochemistry [5].

Many mathematicians studied closure operator and closure spaces (see [2], [3],[4], [6], [7], [8], [9] and [11]). Furthermore, the concept of preopen sets and precontinuous functions are introduced by Mashhure et al [19]. The concepts of precontinuity and almost - continuity (in the sense Hussain [15] are same in topological spaces, while Halgwrd M. Darwesh showed the both concepts are independent in closure spaces*:* It is well known that there are many conditions and properties that are equivalent to the concept of precontinuity and preopenness topological spaces.

In Section 3, we introduce and study new type of functions called generalized-pre-irresolute function. We discuss the relation between generalized-pre-irresolute functions and generalized-pre-continuous functions.

In Section 4, we introduce and study new type of function called λ-irresolute function. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_p-spaces, T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces, we find some relation between them.

2. Preliminaries

Let (X, c) be a closure space, a subset $A \subseteq X$ is closed [16] if $c(A) = A$. It is called open [16], if its complement in X is closed. The empty set and the *whole space* are both *open* and *closed.* Let (X, τ) be a *topological space*, a subset $A \subseteq X$ is preopen [19] if $A \subseteq IntCl(A)$ where the interior operator $Int : P(X) \to P(X)$ corresponding to the *kuratowski closure operator* Cl on X with respect to the topology, is given by; $Int(A) = X \backslash Cl(X \backslash A)$.

Definition 2.1 ([10])**.** The interior operator $i : P(X) \to P(X)$ corresponding to the closure operator c on X is given by; $i(A) = X \backslash c(X \backslash A)$.

Definition 2.2 ([10])**.** A closure operator c on a set X is called idempotent if $cc(A) = c(A))$, for all $A \subseteq X$.

Definition 2.3 ([10])**.** A subset A of a *space* (X, c) is said to be a *preopen set*, if there exists an *open set* G such that $A \subseteq G \subseteq c(A)$. The family of all *preopen sets* is denoted by $PO(X, c)$. The complement of a *preopen set* is called *preclosed*. The family of all *preclosed sets* is denoted by $PC(X, c)$. A closure space (X, c) is said to be a T_p-space, if every preopen subset of (X, c) is open.

Theorem 2.4 ([10])**.** *Let c be a closure operator on X and $A \subseteq X$. If A is preopen, then $A \subseteq X \backslash c(X \backslash cA) = ic(A)$.*

Theorem 2.5 ([10])**.** *Let c be an idempotent closure operator on X. A subset A of X is preopen if and only if $A \subseteq X \backslash c(X \backslash cA) = ic(A)$.*

Definition 2.6 ([10])**.** Let (X, c_1) and (Y, c_2) be *closure spaces*. A *function* $f : (X, c_1) \to (Y, c_2)$ is said to be *precontinuous*, if the inverse image of every *open* set in (Y, c_2) under f is *preopen* in (X, c_1).

Theorem 2.7 ([10])**.** *Let (X, c_1) and (Y, c_2) be closure spaces. A function $f : (X, c_1) \to (Y, c_2)$ is precontinuous if and only if the inverse image of every closed set in (Y, c_2) under f is preclosed in (X, c_1).*

Definition 2.8 ([12])**.** Let (X, c_1) and (Y, c_2) be two *closure spaces*. A *function* $f : (X, c_1) \to (Y, c_2)$ is *preopen* if $f(G)$ is *preopen* in Y, for every *open subset* G of X.

Definition 2.9 ([13])**.** A subset B of a closure space (X, c) is called generalized-preopen, (briefly g-preopen), if there exists a preopen subset A of (X, c) such that $B \subseteq A \subseteq c(B)$ .A subset B of X is called generalized-preclosed, briefly g-preclosed, if its complement is g-preopen.

Remark 2.10.
1. The family of all $g-$preopen (resp. $g-$preclosed) subsets of (X, c) will be denoted by $g - PO(X, c)$ (resp. $g - PC(X, c)$).
2. If A is preopen (resp. preclosed) in a closure space (X, c), then A is g-preopen (resp. g-preclosed).

Proposition 2.11 ([13])**.** *Let B be a subset of T_p- closure space (X, c). If B is g-preopen, then B is open.*

Definition 2.12 ([13])**.** A subset B of a closure space (X, c) is called λ-open, if there exists an open subset G of X such that $B \subseteq G$ and $c(B) = c(G)$. A subset B of X is called λ-closed, if its complement is λ-open. The family of all λ-open (λ-closed) subsets of X denoted by $\lambda O(X, c)$ ($\lambda C(X, c)$).

Remark 2.13 ([13])**.** If G is open (respectively, closed) in (X, c), then G is λ-open (resp. λ-closed) in (X, c).

Remark 2.14 ([13])**.** If B is λ-open (resp. λ-closed) in a closure space (X, c), then B is preopen (resp. preclosed) in (X, c).

Remark 2.15 ([13])**.** It follows from Remark 2.10, Remark 2.13 and Remark 2.14, that for a subset G of a closure space (X, c), we have the following implications:

G is open $\rightarrow$ G is λ-open $\rightarrow$ G is preopen $\rightarrow$ G is g-preopen.

Definition 2.16 ([13])**.** A closure space (X, c) is said to be a T_p-space, if every preopen subset of (X, c) is open.

Definition 2.17 ([13])**.** A closure space (X, c) is said to be:

1. a T_λ-space, if every λ-open subset of (X, c) is open**.**
2. a $T_{p\lambda}$-space, if every preopen subset of $(X, c)$is λ-open.
3. a T_{gp}-space, if every g-preopen subset of (X, c) is preopen.

Proposition 2.18 ([13])**.** *Let (X, c) be an idempotent closure space and B be a subset of X. If (X, c) is a $T_{p\lambda}$-space and B is g-preopen, then B is λ-open.*

Remark 2.19 ([13])**.** For a function $f : (X, c_1) \rightarrow (Y, c_2)$, where (X, c_1) and (Y, c_2) are closure spaces, the following implications hold:

f is continuous $\rightarrow$ f is λ-continuous $\rightarrow$ f is pre-continuous $\rightarrow$ f is g-pre-continuous.

3. Generalized-pre-irresolute functions in closure spaces

In this section, we introduce and study new type of functions called generalized-pre-irresolute function. We discuss the relation between generalized-pre-irresolute functions and generalized-pre-continuous functions.

Definition 3.1. A function $f : (X, c_1) \rightarrow (Y, c_2)$ is said to be generalized-pre-irresolute (briefly, g-pre-irresolute), $f^{-1}(B)$ is a g-preopen subset of (X, c_1) for every g-preopen subset B of (Y, c_2).

The proof of the following two results are easy, so they are omitted.

Proposition 3.2. *A function $f : (X, c_1) \rightarrow (Y, c_2)$ is g-pre-irresolute if and only if the inverse image of every g-preclosed subset of (Y, c_2) under f is g-preclosed in (X, c_1).*

Proposition 3.3. *Let* $f : (X, c_1) \to (Y, c_2)$ *and* $g : (Y, c_2) \to (Z, c_3)$ *be functions. If* f *and* g *are g-pre-irresolute, then* gof *is also g-pre-irresolute.*

Remark 3.4. The concepts of pre-irresolute functions and g-pre-irresolute functions are independent as shown by the following examples:

Example 3.5. Consider the closure operator $c : P(X) \to P(X)$ on $X = \{1, 2, 3, 4\}$ which is defined by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{2\}, \{3\}, \{2,3,4\}\}, \\ \{1,2,3\}, & \text{if } A \in \{\{1\}, \{1,2\}, \{1,3\}\}, \\ \{2,3,4\}, & \text{if } A \in \{\{4\}, \{2,3\}, \{2,4\}, \{3,4\}\}. \\ X, & \text{otherwise.} \end{cases}$$

The family of $PO(X, c) = \{\phi, \{1\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}$.

The family of $g - PO(X, c) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}$

Let $f : (X, c) \to (X, c)$ be a function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 3 & \text{if } x = 4, \\ 4 & \text{if } x = 3. \end{cases}$$

It easy to see that f is g-pre-irresolute, but it is not pre-irresolute because $\{1, 4\}$ is preopen in (X, c) but $f^{-1}(\{1, 4\}) = \{1, 3\}$ is not preopen set in (X, c).

Example 3.6. Let $X = \{1, 2, 3\}$ and define a closure operator $c : P(X) \to P(X)$ on X by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{3\}\}, \\ \{1,2\}, & \text{if } A = \{1\}, \\ \{2,3\}, & \text{if } A = \{2\}. \\ X, & \text{otherwise.} \end{cases}$$

The family of open sets $= \{\phi, \{1, 2\}, X\}$.

$$PO(X, c) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\},$$
$$g - PO(X, c) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Let $f : (X, c) \to (X, c)$ be a function given by:

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if; } x = 3, \\ 3, & \text{if } x = 2. \end{cases}$$

Clearly f is pre-irresolute, but it is not g-pre-irresolute because $\{2\}$ is g-preopen in (X, c) but $f^{-1}(\{2\}) = \{3\}$ is not g-preopen set in (X, c).

Corollary 3.7. *If $f : (X, c_1) \to (Y, c_2)$ is a pre-irresolute function, then f is pre-continuous.*

Proof. It follows from the fact every open set is preopen. □

Remark 3.8. The converse of Corollary 3.7 is not true in general. We can show it by the following example:

Example 3.9. Let $X = \{1, 2, 3\}$ and let $c_1 : P(X) \to P(X)$ be a closure operator on X which is defined in Example 3.6. And also let $c_2 : P(X) \to P(X)$ be a closure operator on X which is defined by:

$$c_2(A) = \begin{cases} A, & \text{if } A = \phi, \\ \{1,3\}. & \text{if } A = \{1\}, \\ \{2,3\}. & \text{if } A = \{2\}, \\ X, & \text{otherwise.} \end{cases}$$

The family of all open sets $= \{\phi, X\}$.

$$\lambda O(X, c_2) = \{\phi, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\} = PO(X, c_2).$$

g-$PO(X, c_2) = P(X)$.

The identity function $f : (X, c_1) \to (X, c_2)$ is pre-continuous, but it is not pre-irresolute because $\{3\}$ is preopen in (X, c_2) but $f^{-1}(\{3\}) = \{3\}$ is not pre-open set in (X, c_1).

Remark 3.10. In the following the function f is g-pre-irresolute but not λ-continuous, (because, it is not pre-continuous).

Example 3.11. The identity function $f : (X, c_1) \to (X, c_2)$, where (X, c_1) is the space of Example 3.6 and (X, c_2) is a closure space such that $c_2 : P(X) \to P(X)$ is defined by $c_2(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{1\}, \{1,3\}\}, \\ \{1,3\}, & \text{if } A = \{3\}, \\ X, & \text{otherwise} \end{cases}$ is g-pre-continuous but not pre-continuous. Since $PO(X, c_1) = \{\phi, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$,

$$g - PO(X, c_1) = PO(X, c_1) \cup \{2\}.$$

The family of open sets of $(X, c_2) = \{\phi, \{2\}, \{2,3\}, X\}$.

Clearly $\{2\}$ is an open set in (X, c_2) but $f^{-1}(\{2\}) = \{2\}$ is not preopen set in (X, c_1).

Corollary 3.12. *If $f : (X, c_1) \to (Y, c_2)$ is a pre-continuous function, then f is g-pre-continuous.*

Proof. It follows from the fact every pre-open set is g-preopen. □

Corollary 3.13. *If $f:(X,c_1)\to(Y,c_2)$ is a g-pre-irresolute function, then f is g-pre-continuous.*

Proof. It follows from the fact every open set is g-preopen. □

Remark 3.14. 1. Since the function of Example 3.9 is pre-continuous. So by Corollary 3.12, f is g-precontinuous, but it is not g-pre-irresolute, because the set $\{3\}$ is g-preopen in (X,c_2) but $f^{-1}(\{3\})=\{3\}$ is not g-preopen in (X,c_1). This means that the converse of Corollary 3.13 is not true in general.

2. Since the function f in Example 3.11 is g-pre-irresolute but not λ-continuous, (because, it is not pre-continuous). However the function f of Example 3.9 is λ-continuous, but not g-pre-irresolute (because $f^{-1}(\{3\})=\{3\}$ is not g-preopen in (X,c_1) where $\{3\}$ is g-preopen in (X,c_2). This means that, the concepts of g-pre-irresoluteness and λ-continuity of functions in closure space are independent concepts.

Remark 3.15. The concepts of g-pre-irresolute functions and λ-continuous functions are independent. By Example 3.11, f is g-pre-irresolute, but f is not λ-continuous, because $\{2\}$ is open in (X,c_2) but $f^{-1}(\{2\})=\{2\}$ is not λ-open set in (X,c_1). By Example 3.9, the function f is λ-continuous, but not g-pre-irresolute, because $\{3\}$ is g-preopen in (X,c_2) but $f^{-1}(\{3\})=\{3\}$ is not g-preopen set in (X,c_1).

4. λ-irresolute functions in closure spaces

In this section, we introduce and study new type of function called λ-irresolute function. We discuss the relation among generalized-pre-irresolute functions λ-irresolute functions, generalized-pre-continuous functions and λ-continuous functions. And also by T_p-space, T_λ-spaces, $T_{p\lambda}$-spaces and T_{gp}-spaces,we find some relation between them.

Definition 4.1. A function $f:(X,c_1)\to(Y,c_2)$ is said to be λ-irresolute, if $f^{-1}(B)$ is a λ-open subset of (X,c_1) for every λ-open subset B of (Y,c_2).

Proposition 4.2. *A function $f:(X,c_1)\to(Y,c_2)$ is λ-irresolute if and only if the inverse image of every λ-closed subset of (Y,c_2) under f is λ -closed in (X,c_1).*

Proof. Obvious. □

Proposition 4.3. *Let $f:(X,c_1)\to(Y,c_2)$ and $g:(Y,c_2)\to(Z,c_3)$ be two λ-irresolute functions. then gof is also λ-irresolute.*

Proof. Obvious. □

Proposition 4.4. *Every λ-irresolute function is a λ-continuous function.*

Proof. It follows from the fact that every open set is a λ-open set. □

Remark 4.5. Since the function of Example 3.9 is λ-continuous but not λ-irresolute (because, the set $\{3\}$ is λ-open in (X, c_2). This implies that, the converse of Proposition 4.4 is not true in general.

By the following examples we will show that, the concepts of g-pre-irresoluteness and λ-irresoluteness of functions in closure space are independent concepts.

Example 4.6. The identity function $f : (X, c_1) \to (X, c_2)$, where (X, c_1) is a closure space such that the closure operator $c_1 : P(X) \to P(X)$ on $X = \{1, 2, 3, 4\}$ which is defined by:

$$c_1(A) = \begin{cases} \phi, & \text{if } A = \phi, \\ \{1,2\}, & \text{if } A = \{1\}, \\ \{2,3\}, & \text{if } A \in \{\{2\}, \{3\}\}, \\ \{2,4\}, & \text{if } A = \{4\}, \\ \{1,2,3\}, & \text{if } A \in \{\{1,2\}, \{1,3\}\}, \\ \{1,2,4\}, & \text{if } A = \{1,4\}, \\ \{2,3,4\}, & \text{if } A \in \{\{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\}, \\ X, & \text{otherwise.} \end{cases}$$

The family of all open sets= $\{\phi, \{1\}, X\}$. The family of $PO(X, c_1) = \{\phi, \{1\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, X\} = \lambda O(X, c_1)$.

The family of g-$PO(X, c_1) = PO(X, c_1) \cup \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and (X, c_2) is the same closure space of Example 3.5, is g-pre-continuous (because g-$PO(X, c_1)$ = g$-PO(X, c_2)$) but not λ-irresolute (because the set $\{1, 4\}$ is λ-open in (X, c_2) but $f^{-1}(\{1, 4\}) = \{1, 4\}$ is not λ-open in (X, c_1).

Example 4.7. Let $f : (X, c) \to (X, c)$ be a function given by:

$$f(x) = \begin{cases} 1, & \text{if } x = 3, \\ 2, & \text{if } x = 2, \\ 3, & \text{if } x = 1, \end{cases}$$

where (X, c) is the same closure space of Example such that $c : P(X) \to P(X)$ which defined by:

$$c(A) = \begin{cases} A, & \text{if } A \in \{\phi, \{3\}\}, \\ \{1,2\}, & \text{if } A = \{1\}, \\ \{2,3\}, & \text{if } A = \{2\}, \\ X, & \text{otherwise.} \end{cases}$$

Clearly f is λ-irresolute, but it is not g-pre-irresolute because $\{1\}$ is g-preopen in (X, c) but $f^{-1}(\{1\}) = \{3\}$ is not g-preopen set in (X, c).

Corollary 4.8. *If $f : (X, c_1) \to (Y, c_2)$ is a λ-irresolute function, then f is g-pre-continuous.*

Proof. It follows from Remark 4.5 and Remark 2.10. □

Remark 4.9. Since from Remark 4.5, we have the function f of Example 3.9 is λ-continuous, but not λ-irresolute. Then, in review of Remark 2.10 f is g-pre-continuous but not λ-irresolute. This means that, the converse of Corollary 4.8 is not true in general.

Remark 4.10. By Remark 2.10, Remark 3.4, Remark 3.8, Remark 3.14, Remark 3.15, Remark 4.5 and Remark 4.9, the interrelation among (g-pre-continuous, g-pre-irresolute, λ-continuous and λ-irresolute) functions is given by the following diagram:

Here, A B means A implies B and A B means A does not necessarily imply B .

Proposition 4.11. *Let (X, c_1) be a closure space, (Y, c_2) be a T_p-space. If $f : (X, c_1) \to (Y, c_2)$ is a g-pre-continuous function, then f is g-pre-irresolute.*

Proof. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is a T_p-space, by Proposition 2.11, B is open in (Y, c_2). Since f is g-pre-continuous, so $f^{-1}(B)$ is g-preopen in (X, c_1) . Therefore, f is g-pre-irresolute. □

Proposition 4.12. *Let (X, c_1) be a closure space,(Y, c_2) be a T_λ-space. If $f : (X, c_1) \to (Y, c_2)$ is a λ-continuous function, then f is λ-irresolute.*

Proof. Let B be a λ-open subset of (Y, c_2). Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2). by λ-continuity of f, we get $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute. □

Proof. Let B be a λ-open subset of (Y, c_2). Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2), by λ-continuity of f , we get $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute. □

Proposition 4.13. *Let (X, c_1) be a $T_{p\lambda}$-space and idempotent closure space and (Y, c_2) be a T_λ-space. If $f : (X, c_1) \to (Y, c_2)$ is a g-pre-continuous function, then f is λ-irresolute.*

Proof. Let B be a λ-open subset of (Y, c_2) . Since (Y, c_2) is a T_λ-space, B is open in (Y, c_2). by g-pre-continuity of f, we get $f^{-1}(B)$ is g-preopen in (X, c_1) . But (X, c_1) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, $f^{-1}(B)$ is λ-open. Therefore, f is λ-irresolute. □

Proposition 4.14. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function.*

1. If (X, c_1) is idempotent closure space and a $T_{p\lambda}$-space and f is g-pre-irresolute, then f is λ-irresolute.

2. If (Y, c_2) is idempotent closure space and a $T_{p\lambda}$-space and f is λ-irresolute, then f is g-pre-irresolute.

Proof. 1. Let B be a λ-open subset of (Y, c_2). Then, B is also g-preopen in (Y, c_2). Since f is g-pre-irresolute, so $f^{-1}(B)$ is g-preopen in (X, c_1). Since (X, c_1) is a $T_{p\lambda}$-space, by Proposition 2.18, $f^{-1}(B)$ is λ-open in (X, c_1). Therefore, f is λ-irresolute.

2. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is an idempotent $T_{p\lambda}$-space, by Proposition 2.18, B is λ-open in (Y, c_2). Since f is λ-irresolute, so $f^{-1}(B)$ is λ-open in (X, c_1). Consequently, $f^{-1}(B)$ is also g-preopen in (X, c_1). Therefore, f is g-pre-irresolute. □

Proposition 4.15. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function.*

1. *If (X, c_1) is an idempotent $T_{p\lambda}$-closure space and f is g-pre-irresolute, then f is λ-continuous.*
2. *If (Y, c_2) is a T_p-space and f is λ-continuous, then f is g-pre-irresolute.*

Proof. 1. Since f is g-pre-irresolute and (X, c_1) is a $T_{p\lambda}$-space, by part (1) of Preposition 4.14, f is λ-irresolute, and by Preposition 4.4, f is λ-continuous.

2. Let B be a g-preopen subset of (Y, c_2). Since (Y, c_2) is a T_p-space, by Proposition 2.11, B is open. Since f is λ-continuous, $f^{-1}(B)$ is λ-open in (X, c_1). It follows that $f^{-1}(B)$ is also g-preopen in (X, c_1). Therefore, f is g-pre-irresolute. □

Proposition 4.16. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a preopen g-pre-irresolute and surjective function. If (X, c_1) is a T_p-space, then (Y, c_2) is a T_{gp}-space.*

Proof. Let (X, c_1) be a T_p-space and let B be a g-preopen subset of (Y, c_2). Since f is g-pre-irresolute, $f^{-1}(B)$ is g-pre-open in (X, c_1). Since (X, c_1) is a T_p-space, so by Proposition 4.4, $f^{-1}(B)$ is open in (X, c_1). Since f is a preopen surjection, then $f\left(f^{-1}(B)\right) = B$ is preopen in (Y, c_2). Therefore, (Y, c_2) is a T_{gp}-space. □

The proof of the following results are similar to the proof of Proposition 4.16.

Proposition 4.17. *Let $f : (X, c_1) \to (Y, c_2)$ be a λ-open, pre-irresolute surjective function. If (X, c_1) is a T_p-space, then (Y, c_2) is a $T_{p\lambda}$-space.*

Proposition 4.18. *Let $f : (X, c_1) \to (Y, c_2)$ be an open, λ-irresolute surjective function. If (X, c_1) is a T_λ-space, then (Y, c_2) is a T_λ-space*

Proposition 4.19. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be two functions. Then, gof is g-pre-continuous if f is g-pre-irresolute and g is g-pre-continuous.*

Proof. Obvious. □

Proposition 4.20. *Let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be two functions. Then, gof is λ-continuous if f is λ-irresolute and g is λ-continuous.*

Proposition 4.21. *If $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ are two functions, then:*

1. *If (X, c_1) is an idempotent $T_{p\lambda}$-space, f is g-pre-irresolute and g is λ-irresolute, then gof is λ-irresolute.*
2. *If (Z, c_3) is an idempotent $T_{p\lambda}$-space, f is a g-pre-irresolute and g is λ-irresolute, then gof is g-pre-irresolute.*
3. *If (Y, c_2) is an idempotent $T_{p\lambda}$-space, f is λ-irresolute and g is g-pre-irresolute, then gof is both g-pre-irresolute and λ-irresolute.*

Proof. 1. Let B be a λ-open subset of (Z, c_3). Since g is λ-irresolute, $g^{-1}(B)$ is λ-open in (Y, c_2). Consequently, $g^{-1}(B)$ is also g-preopen in (Y, c_2). Since f is g-pre-irresolute, then $f^{-1}\left(g^{-1}(B)\right) = (gof)^{-1}(B)$ is g-preopen in (X, c_1). But (X, c_1) is an idempotent $T_{p\lambda}$-space, then, by Proposition 2.18, $(gof)^{-1}(B)$ is λ-open in (X, c_1). Therefore, gof is λ-irresolute.

2. Let B be a g-preopen subset of (Z, c_3). Since (Z, c_3) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, B is λ-open by. Since g is λ-irresolute, then $g^{-1}(B)$ is λ-open in (Y, c_2). Consequently, $g^{-1}(B)$ is also g-preopen in (Y, c_2). But f is g-pre-irresolute, hence $f^{-1}\left(g^{-1}(B)\right) = (gof)^{-1}(B)$ is g-preopen in (X, c_1). Therefore, gof is g-pre-irresolute.

3. If B is a λ-open (g-preopen) subset of (Z, c_3), then B is g-preopen in (Z, c_3). Since g is g-pre-irresolute, then $g^{-1}(B)$ is g-peropen in (Y, c_2). But, (Y, c_2) is an idempotent $T_{p\lambda}$-space, so by Proposition 2.18, $g^{-1}(B)$ is λ-open set in (Y, c_2). Since f is λ-irresolute, then $(gof)^{-1}(B) = f^{-1}\left(g^{-1}(B)\right)$ is a λ-open subset of (X, c_1). So, $(gof)^{-1}(B)$ is both λ-open and g-preopen. Therefore, gof is both λ-irresolute and g-pre-irresolute. □

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