Approximate methods for solving one-dimensional partial integro-differential equations of fractional order

Osama H. Mohammed
Department of Mathematics and Computer Applications
College of Science
Al-Nahrain University
Baghdad-Iraq
dr-usama79@yahoo.com

Ahmed K. Mohsin*
Department of Mathematics
College of Education
University of Mustansiriyah
Baghdad-Iraq
ahmed.almosawi83@gmail.com

Abstract. In this paper, an analytic methods which are so-called ADM and HAM are applied to obtain series solutions to a class of one dimensional partial-intgro differential equations of fractional order(FPIDEs). First, existence of a unique solution under certain condition is proved. Then, the ADM and HAM are performed to get a general solution for one dimensional partial-intgro differential equations. In the next step, the convergence analysis of the proposed methods is investigated. Finally, some examples are included to demonstrate the efficiency of the proposed methods.

Keywords: Caputo fractional order derivative, partial integro-differential equations, homotopy analysis method, adomian decomposition method.

1. Introduction

Fractional calculus is a generalization of integer order integral and derivative to an arbitrary real-valued order. This idea has first emerged at the end of 17^{th} century, and developed in the area of mathematics throughout 18^{th} and 19^{th} century in the works of Liouville, Riemann, Cauchy, Abel, Grünwald and many others [2].

Many problems can be treated by fractional integro-differential equations from different sciences applications. In fact, most mathematical problems are hard to solve analytically, and therefore finding an approximate solution, by investigating several numerical methods, would be very convenient. In recent years, several numerical methods have been applied to solve fractional differential equations (FDEs) and fractional Integro-differential equations (FIDEs).

∗. Corresponding author

In this paper, the semi-analytic methods which are so called HAM and ADM will be applied to solve some classes of fractional order partial integro differential equations.

In [5] study the numerical solution of nonlinear weakly singular partial integro-differential equation via operational matrices, he was analyzed and proposed an efficient matrix based on shifted Legender polynomials for the solution of non-linear volterra singular partial integro-differential equations (PIDEs).

In [6] has been studied the approximate controllability of a multi-valued fractional impulsive stochastic partial integro-differential equation with infinite delay in Hilbert spaces. In [1] was presented a finite difference scheme for non-linear integro- differential equation.

In this paper, the proposed methods will be applied for solving a class of partial integro-differential equations of fractional order given by the following form:

$$ {}^{C}D_t^{\alpha} u(x,t) = g(x,t) + \int_a^t k(x,s)F[u(x,s)]ds \tag{1} $$

subject to:

$$ u(x,0) = u_0(x), x \in [a,b] \tag{2} $$

and

$$ {}^{C}D_t^{\alpha} u(x,t) = g(x,t) + I_x^{\beta} k(x,s)F[u(x,s)] \tag{3} $$

subject to:

$$ u(x,0) = u_0(x), x \in [a,b]. \tag{4} $$

2. Preliminaries

In this section, we presents some necessary definitions and mathematical preliminaries of the fractional calculus theory that have been needed in the construction of this paper later on.

2.1 Fractional order derivatives and integrals

In this part we shall give some basic definitions and properties of the fractional order derivative and integrals [4].

Definition 2.1. *The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ is defined as follows:*

$$ I_x^{\beta} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \ x > 0, \, \alpha \in R^{+}. $$

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:*

$$ {}^{C}D_x^{\alpha}(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \dfrac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases}. $$

For $\alpha > 0$, we have [3]:

1. ${}^{C}D_x^{\alpha}\left(I_x^{\alpha} f(x)\right) = f(x)$.
2. ${}^{C}D_x^{-\alpha}\left({}^{C}D_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!}$.
3. ${}^{C}D_x^{\alpha}(c) = 0,\ c \in R$.

3. Existence and uniqueness theorems of the solutions of one-dimensional partial integro-differential equations of fractional order

The existence and uniqueness of problems (1)-(2) and (3)-(4) will be obtained in this section under certain conditions using Banach fixed point theorem.

3.1 The existence and uniqueness of the solution of problem (1)-(2)

In this subsection the existence and uniqueness of the solution of problems (1)-(2) will be investigated.

Lemma 3.1. *The function $u \in C([a,b] \times [0,T]) = X$ is a solution of problem (1)-(2) if and only if $u(x,t)$ is satisfying:*

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds. \tag{5} $$

Proof. Apply I_t^{α} on both sides of equation (1), yields:

$$ I_t^{\alpha C} D_t^{\alpha} u(x,t) = I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_0^t k(y,s) F[u(x,s)] ds $$

according to equation (2), we have:

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds $$

and hence the result is obtained. □

Theorem 3.1. *Let $T : X \longrightarrow X$ be defined as:*

$$ Tu = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y,s) F[u(y,s)] dy ds \tag{6} $$

such that k verified a Lipschitz condition with respect to $u(x,t)$ with a Lipschitz constant $L \geq 0$, and $\mid k(x,t) \mid \leq M$. Furthermore, if $\frac{ML(b-a)T^\alpha}{\Gamma(\alpha+1)} < 1$, then T has a unique solution.

Proof. Define the supermum norm, which will be needed later in the proof as:

$$\|u(x,t)\| = \sup_{\substack{x\in[a,b] \\ t\in[0,T]}} |u(x,t)| \,.$$

Now, to prove that T is a contractive mapping, we suppose that $u_1(x,t), u_2(x,t) \in X$, then:

$$\begin{aligned}&|Tu_1(x,t) - Tu_2(x,t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} |k(y,s)| \, |F[u_1(x,s)] - F[u_2(x,s)]| \, dyds.\end{aligned}$$

Hence:

$$\begin{aligned}\|Tu_1(x,t) - Tu_2(x,t)\| &\leq \frac{LM}{\Gamma(\alpha)} \|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x (t-s)^{\alpha-1} dyds \\ &\leq \frac{LM}{\Gamma(\alpha)} \|u_1(x,t) - u_2(x,t)\| \, \frac{(x-a)t^\alpha}{\alpha} \\ &\leq \frac{ML(b-a)T^\alpha}{\Gamma(\alpha+1)} \|u_1(x,t) - u_2(x,t)\| \,.\end{aligned}$$

since $\frac{ML(b-a)T^\alpha}{\Gamma(\alpha+1)} < 1$, then T is a contractive mapping therefore, the problem (1)-(2) has a unique solution. □

3.2 Existence and uniqueness theorems of the solutions of problem (3)-(4)

In this subsection the existence and uniqueness of the solution of problem (3)-(4) will be introduced.

Lemma 3.2. *The function $u \in X$ is a solution of problem* (3)-(4) *if and only if $u(x,t)$ is satisfying:*

$$\begin{aligned}u(x,t) &= u_0(x) + I_t^\alpha g(x,t) \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds\end{aligned} \tag{7}$$

Proof. Apply I_t^α on both sides of equation (3), yields:

$$I_t^{\alpha C} D_t^\alpha u(x,t) = I_t^\alpha g(x,t) + I_t^\alpha I_x^\beta k(x,s) F[u(x,s)].$$

According to equation (4), we have:

$$\begin{aligned} u(x,t) &= u_0(x) + I_t^\alpha g(x,t) \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds \end{aligned}$$

and hence the result is obtained. □

Theorem 3.2. *Let $T : X \longrightarrow X$ be defined as:*

$$\begin{aligned} Tu &= u_0(x) + I_t^\alpha g(x,t) \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y,s) F[u(y,s)] dyds \end{aligned} \tag{8}$$

such that k verified a Lipschitz condition with respect to $u(x,t)$ with a Lipschitz constant $L \geq 0$, and $\mid k(x,t) \mid \leq M$. Furthermore, if $\frac{MLT^\alpha (b-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$, then T has a unique solution.

Proof. Now, to prove that T is a contractive mapping, we suppose that $u_1(x,t)$, $u_2(x,t) \in X$, then:

$$\begin{aligned} &|Tu_1(x,t) - Tu_2(x,t)| \\ &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} |k(y,s)| \, |F[u_1(y,s)] - F[u_2(y,s)]| \, dyds. \end{aligned}$$

Hence:

$$\begin{aligned} &\|Tu_1(x,t) - Tu_2(x,t)\| \\ &\leq \frac{LM}{\Gamma(\alpha)\Gamma(\beta)} \|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} dyds \\ &\leq \frac{MLT^\alpha (b-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|u_1(x,t) - u_2(x,t)\| \end{aligned}$$

since $\frac{MLT^\alpha (b-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$, then T is a contractive mapping therefore, the problem (3)-(4) has a unique solution. □

4. ADM for solving one-dimensional partial integro-differential equations of fractional order

In this section the implementation of the ADM for solving one dimensional FPIDEs will be presented.

4.1 ADM for solving problem (1)-(2)

To apply the ADM for solving problem (1)-(2), first operating I_t^{α} on both sides of equation (1) to get:

$$u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_0^t k(x,s) F[u(x,s)] ds. \tag{9}$$

The solution will be considered to be:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{10}$$

Approximating the term $F[u(x,t)]$ as:

$$F[u(x,t)] = \sum_{n=0}^{\infty} A_n, \tag{11}$$

where:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{n=0}^{\infty} u_n \lambda^n \right) \right] \Bigg|_{\lambda=0}. \tag{12}$$

Substituting equations (10) and (11) into equation (5), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \left[\int_0^t k(x,s) \sum_{n=0}^{\infty} A_n \, ds \right].$$

Consequently, we can write:

$$u_0(x,t) = u_0(x) + I_t^{\alpha} g(x,t)$$

and

$$u_{n+1}(x,t) = I_t^{\alpha} \left[\int_0^t k(x,s) A_n \, ds \right], \; n \geq 1.$$

If we truncate the summation into equation (10) after m-terms, so we have the m^{th} approximate solution of problem (1)-(2)

$$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t).$$

4.2 ADM for solving problem (3)-(4)

In this subsection a similar manner that have been given in subsection (4.1) will be implemented in order to formulate a recurrence formula for finding the approximate solution of the problem (3)-(4) using ADM as follows:

$$u_0(x,t) = u_0(x) + I_t^{\alpha} g(x,t), \tag{13}$$

$$u_{n+1}(x,t) = I_t^{\alpha} I_x^{\beta} k(y,s) A_n,\ n \geq 1. \tag{14}$$

So, the m^{th} order approximate solution of the problem (3)-(4) is given by:

$$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t). \tag{15}$$

5. HAM for solving one-dimensional partial integro-differential equations of fractional order

In this section the implementation of the HAM for solving one dimensional fractional partial integro-differential equations given by problems and (1)-(2) and (3)-(4) will be presented.

5.1 HAM for solving problem (1)-(2)

Rewriting equation (1) as:

$$N[u(x,t)] = 0, \tag{16}$$

where:

$$N[u(x,t)] = {}^{C}D_t^{\alpha} u(x,t) - g(x,t) - \int_a^t k(x,s) F[u(x,s)] ds. \tag{17}$$

According to the HAM, we construct the so called zero-order deformation equation as:

$$(1-q)\pounds[\phi(x,t,q) - u_0(x)] = q\hbar H(x) N[\phi(x,t,q)], \tag{18}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(x) \neq 0$ is an auxiliary function $u_0(x)$ is an initial guess of $u(x,t)$ and £is an auxiliary linear operator defined by:

$$\pounds = {}^{C}D_t^{\alpha}. \tag{19}$$

Obviously, when $q = 0$ and 1, it holds $\phi(x,t,0) = u_0(x)$ and $\phi(x,t,1) = u(x,t)$, respectively. Expanding $\phi(x,t,q)$ in a Taylor series with respect to q, we have:

$$\phi(x,t,q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{20}$$

where:

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t,q)}{\partial q^m}\bigg|_{q=0}. \tag{21}$$

If the initial guess $u_0(x)$, the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$ are so properly chosen, then the series (15) converges when $q = 1$, so we have:

$$u(x,t) = u_0(x) + \sum_{m=1}^{\infty} u_m(x,t). \tag{22}$$

Define the vector:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}.$$

Differentiating equation (13) m-times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by $m!$, we have the so called m^{th} order deformation equation:

$$\pounds[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x) R_m(\vec{u}_{m-1}), \tag{23}$$

where:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t,q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{24}$$

Now, letting $\hbar = -1$ and $H(x) = 1$, then the solution of the m^{th} order deformation equations (23) becomes:

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) - I_t^{\alpha}[{}^{C}D_t^{\alpha} u_{m-1}(x,t) \\ &\quad - (1-\chi_m) g(x,t) - \int_0^t k(x,s) F[u(x,s)] ds] \end{aligned} \tag{25}$$

with an initial approximation $u_0(x,t) = u_0(x)$ and by means of the above iteration formula (25), we can obtain directly the other components in order one after one.

5.2 Convergence analysis

In this subsection, the convergence of the formula (25) to the exact solution $u(x,t)$, of problem (1)-(2) will be proved and before we study the convergence theorem we will consider $F[u(x,t)]$, in equation (1) to be $F[u(x,t)] = [u(x,t)]^p$, $p \geq 1$.

Theorem 5.1. *If the series $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent, where $u_m(x,t)$ is produced by:*

$${}^{C}D_t^{\alpha}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H R_m(\vec{u}_{m-1}) \tag{26}$$

where:

$$\begin{aligned} R_m(\vec{u}_{m-1}) &= {}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) \\ &\quad - \int_0^t k(x,s)[u(x,s)]^p ds, \ p \geq 1 \end{aligned} \tag{27}$$

and besides $\sum_{m=0}^{\infty} {}^{C}D_t^{\alpha}\left[u_m(x,t)\right]$ *also converges, then it is the exact solution of the problem* (1)-(2).

Proof. Suppose that $\sum_{m=0}^{\infty} u_m(x,t)$ converges uniformly to $u(x,t)$, then it is clear that:

$$\lim_{m\to\infty} u_m(x,t) = 0, \text{ for all } x \text{ and } t \in R^{+} \tag{28}$$

since ${}^{C}D_t^{\alpha}$ is a linear operator, we have:

$$\begin{aligned}
&\sum_{m=1}^{n} {}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] \\
&= \sum_{m=1}^{n}\left[{}^{C}D_t^{\alpha} u_m(x,t) - \chi_m {}^{C}D_t^{\alpha} u_{m-1}(x,t)\right] \\
&= {}^{C}D_t^{\alpha} u_1(x,t) + \left({}^{C}D_t^{\alpha} u_2(x,t) - {}^{C}D_t^{\alpha} u_1(x,t)\right) + \cdots \\
&+ \left({}^{C}D_t^{\alpha} u_n(x,t) - {}^{C}D_t^{\alpha} u_{n-1}(x,t)\right) \\
&= {}^{C}D_t^{\alpha} u_n(x,t)].
\end{aligned} \tag{29}$$

Then from equation (28) and (29), we have:

$$\begin{aligned}
\sum_{m=1}^{n} {}^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] &= \lim_{n\to\infty} {}^{C}D_t^{\alpha} u_n(x,t) \\
&= {}^{C}D_t^{\alpha}\left[\lim_{n\to\infty} u_n(x,t)\right] = 0.
\end{aligned}$$

Hence:

$$\hbar H \sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0$$

since $\hbar$ and $H \neq 0$, then yields:

$$\sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0 \tag{30}$$

and since:

$$R_m(\vec{u}_{m-1}) = {}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) - \int_0^t k(x,s)[u_{m-1}(x,s)]^p ds$$

so, we have:

$$
\begin{aligned}
0 &= \sum_{m=1}^{\infty}\left[{}^{C}D_t^{\alpha}u_{m-1}(x,t)-(1-\chi_m)g(x,t)-\int_0^t k(x,s)[u_{m-1}(x,s)]^p ds\right]\\
&= {}^{C}D_t^{\alpha}\sum_{m=1}^{\infty}u_{m-1}(x,t)-g(x,t)\\
&-\sum_{m=1}^{\infty}\left[\int_0^t k(x,s)\left[\sum_{r_1=1}^{m-1}u_{m-1-r_1}(x,s)\sum_{r_2=0}^{r_1}u_{r_1-r_2}(x,s)\right.\right.\\
&\left.\left.\sum_{r_3=0}^{r_2}u_{r_2-r_3}(x,s)\cdots\sum_{r_{p-2}=0}^{r_{p-3}}u_{r_{p-3}-r_{p-2}}(x,s)\sum_{r_{p-1}=0}^{r_{p-2}}u_{r_{p-2}-r_{p-1}}(x,s)\right]\right]ds.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
0 &= {}^{C}D_t^{\alpha}\sum_{m=1}^{\infty}u_{m-1}(x,t)-g(x,t)-\int_0^t k(x,s)\left(\sum_{r_{p-1}=0}^{\infty}u_{m-1-r_1}\sum_{r_{p-2}=r_{p-1}}^{\infty}u_{r_1-r_2}\right.\\
&\left.\sum_{r_{p-3}=r_{p-2}}^{\infty}u_{r_2-r_3}\cdots\sum_{r_2=r_3}^{\infty}u_{r_{p-3}-r_{p-2}}\sum_{r_1=r_2}^{\infty}u_{r_{p-2}-r_{p-1}}\sum_{m=r_1}^{\infty}u_{m-r_1}\right)(x,s)\,ds\\
0 &= {}^{C}D_t^{\alpha}\sum_{m=0}^{\infty}u_m(x,t)-g(x,t)-\int_0^t k(x,s)\left(\sum_{i_1=0}^{\infty}u_{i_1}(x,s)\sum_{i_2=0}^{\infty}u_{i_2}(x,s)\right.\\
&\left.\sum_{i_3=0}^{\infty}u_{i_3}(x,s)\cdots\sum_{i_{p-1}=0}^{\infty}u_{i_{p-1}}(x,s)\sum_{i_p=0}^{\infty}u_{i_p}(x,s)\right)ds
\end{aligned}
$$

so, from equation (1), with $F[u(x,t)]=[u(x,t)]^p$, $p\geq 1$, we obtain:

$$0={}^{C}D_t^{\alpha}u(x,t)-g(x,t)-\int_0^t k(x,s)[u(x,s)]^p ds.$$

Moreover $\sum_{m=0}^{\infty}u_m(x,t)$ also satisfies the initial condition:

$$\sum_{m=0}^{\infty}u_m(x,0)=u_0(x,0)=u_0(x).$$

Therefore, we conclude that it's an exact solution of problem (1)-(2). □

6. HAM for solving problem (3)-(4)

The HAM can be performed for solving problem (3)-(4) in a similar manner that have been given in subsection (5.1) and therefore, we have the following recurrence formula:

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) \\ &\quad - I_t^{\alpha}\left[{}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) - k(x,s) F[u(x,s)]\right] \end{aligned} \tag{31}$$

and by means of equation (31), we can obtain the other components in order one after one.

7. Convergence analysis

In this subsection, the convergence of the formula (31) to the exact solution $u(x,t)$, of the problem (3)-(4), will be proved, similarly $F[u(x,t)]$ in equation (3) will be considered as $F[u(x,s)] = [u(x,t)]^p,\ p \geq 1$.

Theorem 7.1. *If the series $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent, where $u_m(x,t)$ is produced by:*

$$^{C}D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar H R_m(\vec{u}_{m-1}) \tag{32}$$

where:

$$R_m(\vec{u}_{m-1}) = {}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m) g(x,t) - I_x^{\beta} k(x,s)[u_{m-1}(x,s)]^p,\ p \geq 1$$

and besides $\sum_{m=0}^{\infty} {}^{C}D_t^{\alpha}\left[u_m(x,t)\right]$ also converges, then it is the exact solution of problem (3)-(4).

Proof. The proof of Theorem 7.1 will be in similar manner to the proof of Theorem 5.1. □

8. Applications

In this section, some illustrative examples are given in order to confirm the applicability and accuracy of the HAM and ADM, for solving non- linear one dimensional partial integro-differential equation of fractional order.

Example 8.1. Consider the following linear FPIDEs:

$$^{C}D_t^{3/4} u(x,t) = g(x,t) + \int_0^t (x-s) u(x,s) ds \tag{33}$$

subject to:

$$u_0(x,t) = 0 \tag{34}$$

where $g(x,t) = \frac{xt^{0.25}}{\Gamma(1.25)} - \frac{x^2t^2}{2} + \frac{xt^3}{3}$, and the exact solution of problem (33)-(34) is $u(x,t) = xt$.

Following Figures 1-2 represent a comparison between the approximate solution of problem (33)-(34) using HAM, ADM up to 4-terms and the exact solution.

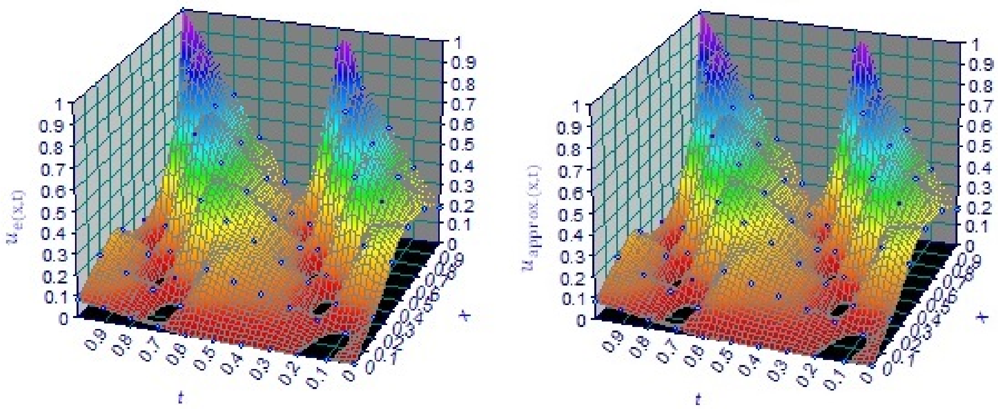

Figure 1: Comparison between the approximate solution of problem (33)-(34) using HAM up to 4-terms and the exact solution.

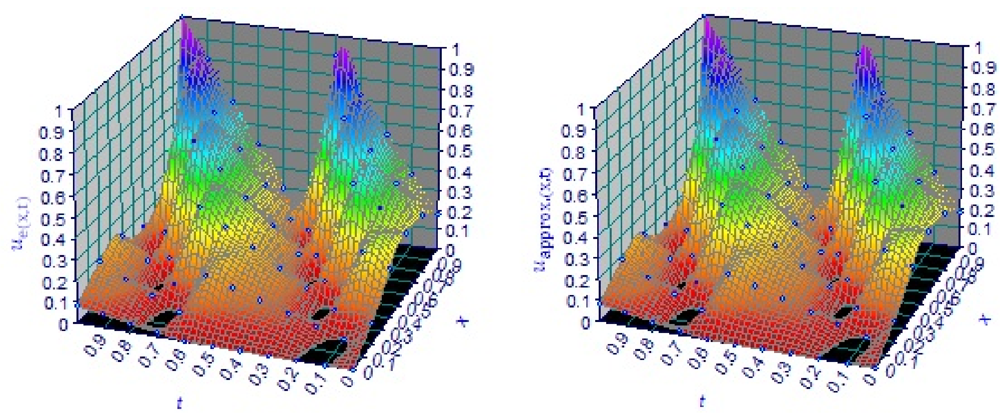

Figure 2: Comparison between the approximate solution of problem (33)-(34) using ADM up to 4-terms and the exact solution.

Example 8.2. Consider the following nonlinear FPIDEs:

$$ {}^{C}D_t^{3/4}u(x,t) = g(x,t) + \int_0^t (x-s)[u(x,s)]^2 ds \tag{35}$$

subject to:

$$ u_0(x,t) = 0 \tag{36}$$

where $g(x,t) = \frac{\Gamma(3)xt^{1.25}}{\Gamma(2.25)} - \frac{x^3t^5}{5} + \frac{x^2t^6}{6}$, and the exact solution of problem (35)-(36) is $u(x,t) = xt^2$.

Following Figures 3-4 represent a comparison between the approximate solution of problem (35)-(36) using HAM, ADM up to 4-terms and the exact solution.

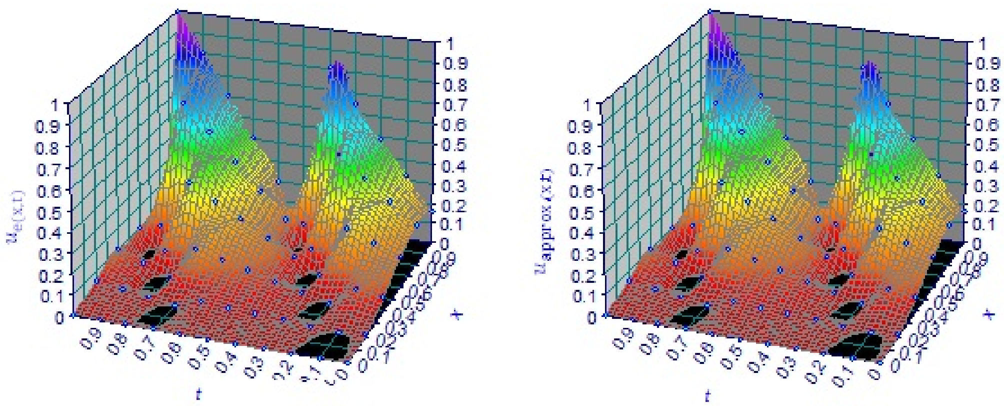

Figure 3: Comparison between the approximate solution of problem (35) -(36) using HAM up to 4-terms and the exact solution.

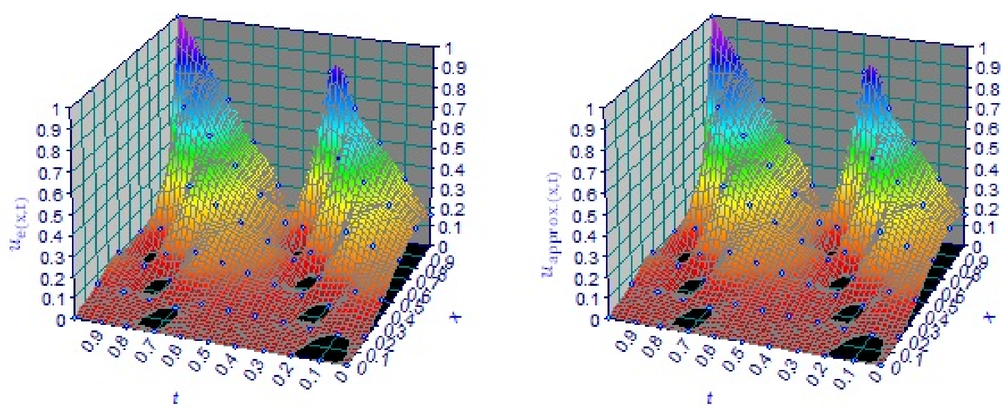

Figure 4: Comparison between the approximate solution of problem (35)-(36) using ADM up to 4-terms and the exact solution.

Example 8.3. Consider the following linear FPIDEs:

$$^{C}D_t^{1/2}u(x,t) = g(x,t) + I_x^{1/3}(x-t)u(x,t) \tag{37}$$

subject to:

$$u_0(x,t) = 0 \tag{38}$$

where $g(x,t) = \frac{\Gamma(3)xt^{3/2}}{\Gamma(5/2)} - \frac{\Gamma(3)x^{7/3}t^2}{\Gamma(10/3)} + \frac{\Gamma(2)x^{4/3}t^3}{\Gamma(7/3)}$, and the exact solution of problem (37)-(38) is $u(x,t) = xt^2$.

Following Figures 5-6 represent a comparison between the approximate solution of problem (37)-(38) using HAM, ADM up to 4-terms and the exact solution.

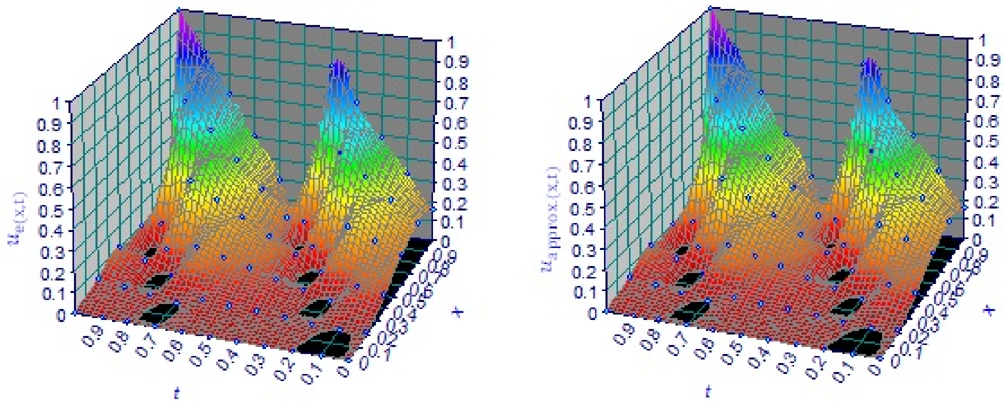

Figure 5: Comparison between the approximate solution of problem (37)-(38) using HAM up to 4-terms and the exact solution.

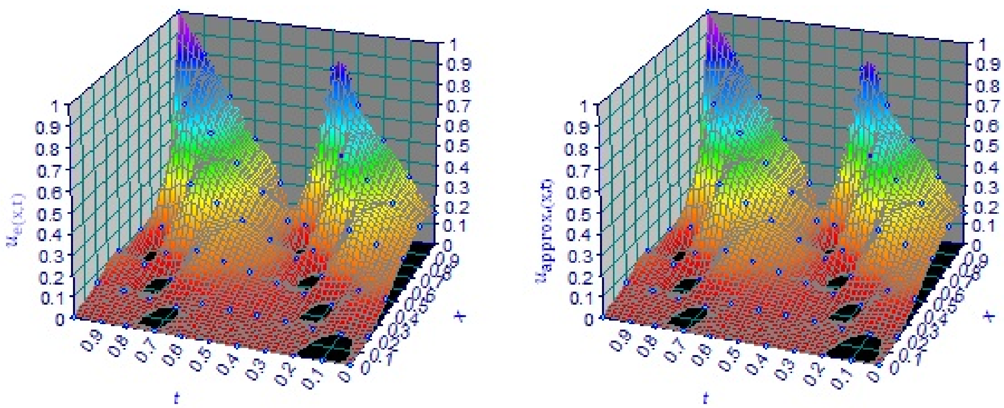

Figure 6: Comparison between the approximate solution of problem (37)-(38)) using ADM up to 4-terms and the exact solution.

Example 8.4. Consider the following nonlinear FPIDEs:

$$^{C}D_t^{1/2}u(x,t) = g(x,t) + I_x^{1/3}(x-t)[u(x,t)]^2 \tag{39}$$

subject to:

$$u_0(x,t) = 0 \tag{40}$$

where

$$g(x,t) = \frac{\Gamma(2)xt^{1/2}}{\Gamma(3/2)} - \frac{\Gamma(4)x^{10/3}t^2}{\Gamma(13/3)} + \frac{\Gamma(3)x^{7/3}t^3}{\Gamma(10/3)},$$

and the exact solution of problem (39)-(40) is $u(x,t) = xt$.

Following Figures 7-8 represent a comparison between the approximate solution of problem (39)-(40) using HAM, ADM) up to 4-terms and the exact solution.

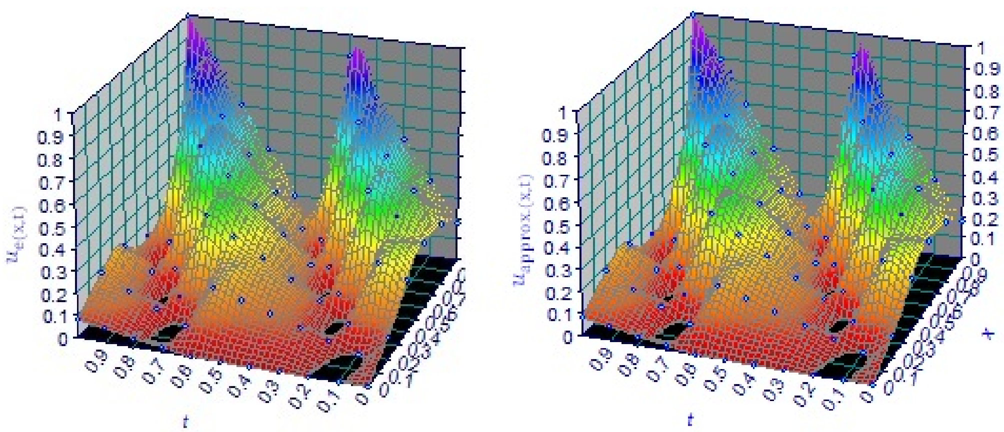

Figure 7: Comparison between the approximate solution of problem (39)-(40) using HAM up to 4-terms and the exact solution.

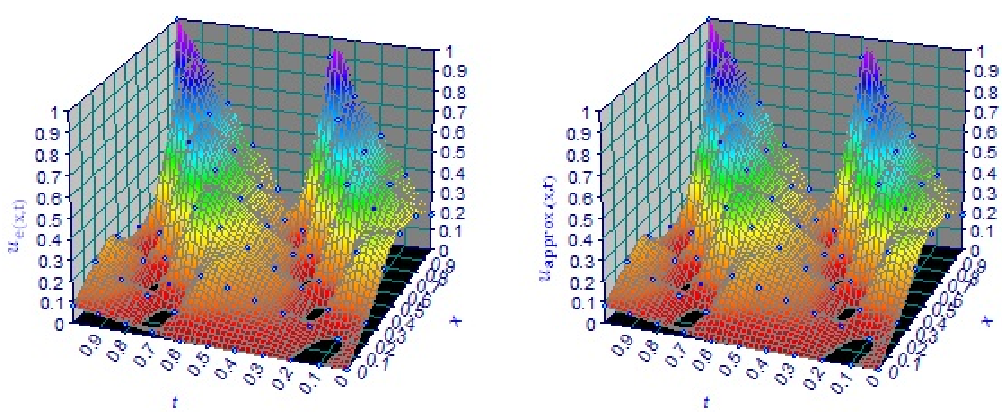

Figure 8: Comparison between the approximate solution of problem (39)-(40) using ADM up to 4-terms and the exact solution.

9. Conclusions

In this paper we have been discussed an existence and uniqueness of some classes of one dimensional partial integro-differential equations of fractional order. Then a computational methods such as ADM and HAM is employed for solving these class of equations.

The results of the numerical examples illustrate the efficiency of the present schemes for solving such kinds of equations.

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