Wrapped shanker distribution

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Abstract. By wrapping Shanker distribution on the real line around a circle, we drive a new circular distribution named wrapped Shanker distribution. The expressions for the densities functions, trigonometric moments, and some related parameters for the new distribution are also obtained.
Keywords: Shanker distribution, circular statistics, wrapped distribution, trigonometric moments.

1. Introduction

Directional data are used in many scientific fields, where the measurements are direction (see, for example, [6],[7]). Direction measurements could be in two or three dimensions, and in many scientific fields the two-dimensional data are known as circular data, since we can represent them as points on the circumference of a unit circle centered at the origin, such that, these points are angles with respect to the starting point and the direction of rotation (i.e., clockwise or anti-clockwise). Obtaining the best statistical inference from any data set depends on selecting the optimal probability distribution for the data set, and for circular data there are different methods to generate circular models from known probability distribution. Many authors generate circular distribution by wrapping a linear distribution around the unit circle, for some of these wrapped probability distributions (see, for example, [1], [2] and [3]).

For the wrapped Shanker distribution, we derive explicit forms for the probability density function, and the cumulative distribution function in Section 2. In Section 3 we obtain the maximum likelihood estimator for the distribution parameter λ. The characteristic function and the trigonometric moments and

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some related parameters are given in Section 4 and Section 5 respectively. Also, we discuss the location and dispersion of the wrapped Shanker distribution in Section 6. We calculate the skewness and kurtosis of the wrapped Shanker distribution in Section 7. While in Section 8 some simulations are done. Finally, some conclusions in Section 9.

2. A wrapped Shanker distribution

A circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle $\{(\cos\ \theta, \sin\ \theta)\}|\theta \in [0, 2\pi)\}$ (see, for example,[8],[9]). The probability density function $g(\theta)$ for a circular random variable θ in a continuous circular distribution must satisfies the following properties:

(1) $g(\theta) \geq 0$ for all θ,

(2) $\int_0^{2\pi} g(\theta)d\theta = 1$,

(3) $g(\theta) = g(\theta + 2\pi m)$ for $m \in \mathbb{Z}$ (i.e., $g(\theta)$ is periodic function).

For any linear random variable X on the real line with probability density function $f(x)$ we may define a circular random variable $\theta = X(mod\ 2\pi)$ with probability density function

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2\pi m), \qquad 0 \leq \theta < 2\pi. \tag{1}$$

Shanker distribution is one parameter lifetime distribution which is a mixture of exponential (λ) and gamma $(2, \lambda)$ distributions introduced by [10]. Let X have a Shanker distribution with probability density function and cumulative distribution function defined as follows:

$$f(x; \lambda) = \frac{\lambda^2}{\lambda^2 + 1}(\lambda + x)e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0, \tag{2}$$

and

$$F(x; \lambda) = 1 - \frac{\lambda^2 + x\lambda + 1}{\lambda^2 + 1} e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0. \tag{3}$$

Also, Shanker distribution has a characteristic function defined as

$$\varphi_X(t) = \left(1 - \frac{\lambda i t}{\lambda^2 + 1}\right)\left(1 - \frac{it}{\lambda}\right)^{-2} \qquad ; i = \sqrt{-1}. \tag{4}$$

Then we can transform X to a circular random variable $\theta = X(mod\ 2\pi)$, and the probability density function $g(\theta;\lambda)$ can be derived by (1) as follows

$$
\begin{aligned}
g(\theta;\lambda) &= \sum_{m=0}^{\infty} f(\theta+2m\pi) \\
&= \sum_{m=0}^{\infty} \frac{\lambda^2}{\lambda^2+1}(\lambda+\theta+2m\pi)e^{-(\theta+2m\pi)\lambda} \\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\sum_{m=0}^{\infty}(\lambda+\theta)e^{-2m\pi\lambda}+\sum_{m=0}^{\infty}2m\pi e^{-2m\pi\lambda}\right] \qquad (5)\\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[(\lambda+\theta)\sum_{m=0}^{\infty}e^{-2m\pi\lambda}+2\pi\sum_{m=0}^{\infty}me^{-2m\pi\lambda}\right] \\
&= \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\frac{\lambda+\theta}{1-e^{-2\pi\lambda}}+\frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right]; \qquad \theta\in[0,2\pi),\ \lambda>0.
\end{aligned}
$$

The cumulative distribution function $G(\theta;\lambda)$ for $\theta \in [0, 2\pi)$ and $\lambda > 0$ is given by

$$
\begin{aligned}
G(\theta;\lambda) &= \sum_{m=0}^{\infty}\left[F(\theta+2m\pi)-F(2m\pi)\right] \\
&= \sum_{m=0}^{\infty}\left[\left(1-\frac{\lambda^2+\lambda(\theta+2m\pi)+1}{\lambda^2+1}e^{-(\theta+2m\pi)\lambda}\right)\right. \\
&\qquad\qquad \left.-\left(1-\frac{\lambda^2+\lambda(2m\pi)+1}{\lambda^2+1}e^{-(2m\pi)\lambda}\right)\right] \qquad (6)\\
&= \sum_{m=0}^{\infty}\left[\frac{(\lambda^2+1+2m\pi\lambda)}{\lambda^2+1}e^{-2m\pi\lambda}-\frac{(\lambda^2+1+\theta\lambda+2m\pi\lambda)}{\lambda^2+1}e^{-\theta\lambda-2m\pi\lambda}\right] \\
&= \frac{1}{1-e^{-2\pi\lambda}}\left[1-e^{-\theta\lambda}-\frac{\theta\lambda e^{-\theta\lambda}}{\lambda^2+1}\right]+\frac{2\pi\lambda(1-e^{-\theta\lambda})e^{-2\pi\lambda}}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}.
\end{aligned}
$$

Now we are ready to defined the wrapped Shanker distribution as follows

Definition 2.1. *A random variable θ on the unit circle is said to have a wrapped Shanker distribution with parameter $\lambda > 0$, denoted by $WS(\lambda)$, if the probability density function is given by*

$$
g(\theta;\lambda) = \frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\left[\frac{\lambda+\theta}{1-e^{-2\pi\lambda}}+\frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right]; \qquad \theta\in[0,2\pi).
$$

Figure 1 shows the probability density function (PDF) of the wrapped Shanker distribution for different values of λ. While, Figure 2 shows the cumulative distribution function of the wrapped Shanker distribution with different values of λ .

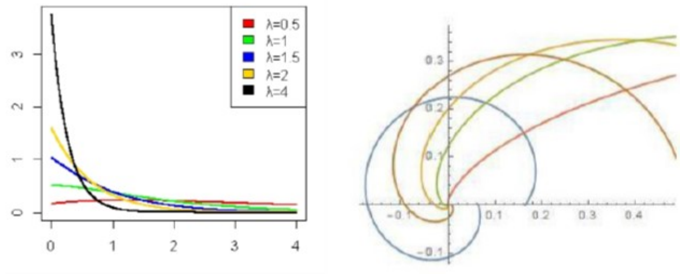

Figure 1: The Linear and Circular representation for the PDF of wrapped Shanker distribution for different values of λ

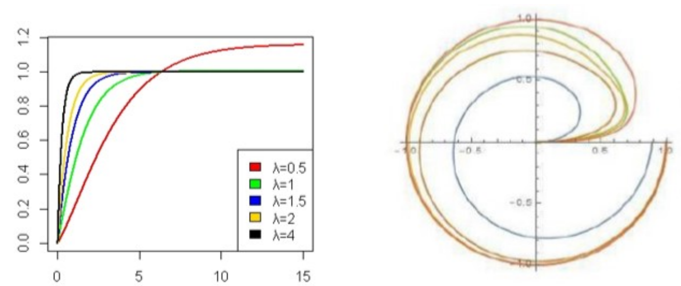

Figure 2: The Linear and Circular representation for the CDF of wrapped Shanker distribution for different values of λ

3. Maximum likelihood estimator

To estimate the distribution parameter λ we use the maximum likelihood estimation technique, the likelihood function $L(\theta_1, \theta_2, \ldots, \theta_n; \lambda)$ is given by

$$\begin{aligned} L(\theta_1, \theta_2, \ldots, \theta_n; \lambda) &= \prod_{i=1}^{n} g(\theta_i; \lambda) \\ &= \prod_{i=1}^{n} \frac{\lambda^2 \ e^{-\theta_i \lambda}}{\lambda^2 + 1} \left[\frac{\lambda + \theta_i}{1 - e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{(1 - e^{-2\pi\lambda})^2} \right] \\ &= \frac{\lambda^{2n} e^{-\lambda \sum_{i=1}^{n} \theta_i}}{(\lambda^2 + 1)^n (1 - e^{-2\pi\lambda})^{2n}} \prod_{i=1}^{n} [\lambda + \theta_i + (2\pi - \lambda - \theta_i) e^{-2\pi\lambda}]. \end{aligned} \tag{7}$$

Then the log likelihood function is

$$\begin{aligned} \ln L(\theta_1, \theta_2, \ldots, \theta_n; \lambda) &= 2n \ln(\lambda) \\ &\quad - \lambda \sum_{i=1}^{n} \theta_i - n \ln(\lambda^2 + 1) - 2n \ln(1 - e^{-2\pi\lambda}) \end{aligned} \tag{8}$$

$$+\sum_{i=1}^{n}\ln[\lambda+\theta_i+(2\pi-\lambda-\theta_i)e^{-2\pi\lambda}],$$

the partial derivative of the log likelihood function with respect to λ is

$$\begin{aligned}\frac{d\ \ln L(\theta_1,\theta_2,\ldots,\theta_n;\lambda)}{d\lambda} &= \frac{2n}{\lambda}-\sum_{i=1}^{n}\theta_i-\frac{2n\lambda}{\lambda^2+1}-\frac{4n\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\\ &\quad+\sum_{i=1}^{n}\left[\frac{1-(1-2\pi(\lambda+\theta_i)+4\pi^2)e^{-2\pi\lambda}}{\lambda+\theta_i+(2\pi-\lambda-\theta_i)e^{-2\pi\lambda}}\right].\end{aligned}\tag{9}$$

Equating the partial derivative (9) to zero and using some numerical techniques we can find a solution for the estimator of the parameter λ.

4. Characteristic function

The continuous circular wrapped Shanker distribution can be described by its characteristic function as follows

$$\begin{aligned}\varphi_\theta(t) &= E\left(e^{it\theta}\right)=\int_0^{2\pi}e^{it\theta}g(\theta;\lambda)d\theta\\ &= \int_0^{2\pi}e^{it\theta}\left[\left(\frac{\lambda^2\ e^{-\theta\lambda}}{\lambda^2+1}\right)\left(\frac{\lambda+\theta}{1-e^{-2\pi\lambda}}+\frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}\right)\right]d\theta\\ &= \frac{\lambda^2}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}\int_0^{2\pi}e^{(it-\lambda)\theta}\left((\lambda+\theta)(1-e^{-2\pi\lambda})+2\pi e^{-2\pi\lambda}\right)d\theta\\ &= \frac{\lambda^2}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2}\left[\left(\lambda(1-e^{-2\pi\lambda})+2\pi e^{-2\pi\lambda}\right)\int_0^{2\pi}e^{(it-\lambda)\theta}d\theta\right.\\ &\qquad\left.+(1-e^{-2\pi\lambda})\int_0^{2\pi}\theta e^{(it-\lambda)\theta}d\theta\right]\\ &= \frac{\lambda^2(1-e^{-2\pi\lambda})(e^{2\pi(it-\lambda)}-1)(\lambda(it-\lambda)-1)+2\pi\lambda^2(it-\lambda)e^{2\pi(it-\lambda)}}{(\lambda^2+1)(1-e^{-2\pi\lambda})^2(it-\lambda)^2}.\end{aligned}\tag{10}$$

However, we know that θ is a periodic random variable (i.e., $g(\theta;\lambda)=g(\theta+2\pi;\lambda)$), then the characteristic function $\varphi_\theta(t)$ must be defined only at integer values.

5. Trigonometric moments and some related parameters

[6] showed that the p^{th} trigonometric moment for a wrapped circular distribution is equal to the value of the characteristic function of the unwrapped random variable at the integer value p. Then the p^{th} trigonometric moment for wrapped

Shanker distribution is given by

$$\begin{aligned}\varphi_\theta(p) = \varphi_X(p) &= \left(1 - \frac{\lambda i p}{\lambda^2+1}\right)\left(1 - \frac{ip}{\lambda}\right)^{-2} \\ &= \frac{\lambda^2}{\lambda^2+1}((\lambda^2+1) - ip\lambda)(\lambda - ip)^{-2} \; ; i = \sqrt{-1}, \; p = \mp 1, \mp 2, \ldots\end{aligned} \tag{11}$$

For $a, b, r \in \Re$, we know that $(a - ib)^{-r} = (a^2 + b^2)^{\frac{-r}{2}} e^{ir \arctan{(b/a)}}$. Then we may write

$$((\lambda^2+1) - ip\lambda) = ((\lambda^2+1)^2 + (p\lambda)^2)^{\frac{1}{2}} e^{-i \; \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)}, \tag{12}$$

$$(\lambda - ip)^{-2} = (\lambda^2 + p^2)^{-1} e^{2i \; \arctan\left(\frac{p}{\lambda}\right)}. \tag{13}$$

Therefore, by substituting formulas (9) and (10) in equation (8) the p^{th} trigonometric moment for the wrapped Shanker distribution $\varphi_\theta(p)$ can be written as $\varphi_\theta(p) = \rho_p e^{i\mu_p}$; $\quad i = \sqrt{-1}, \; p = \mp 1, \mp 2, \ldots$, where

$$\rho_p = \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)}, \tag{14}$$

$$\mu_p = 2 \; \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right). \tag{15}$$

According to Euler's formula, there is a relation between the trigonometric functions and the complex exponential function, such that, for any real number z, we have $e^{iz} = \cos \; z + i \sin \; z$. Then we can defined the p^{th} trigonometric moment of the wrapped Shanker distribution $\varphi_\theta(p)$ in terms of the p^{th} cosine and sine moments given by $\alpha_p = E(\cos \; p\theta)$ and $\beta_p = E(\sin \; p\theta)$ as $\varphi_\theta(p) = \alpha_p + i\beta_p$. By the definition of the p^{th} trigonometric moment, we have $\varphi_\theta(p) = \rho_p e^{i\mu_p}$, $\alpha_p + i\beta_p = \rho_p(\cos \; \mu_p + i \sin \; \mu_p)$, then the non-central trigonometric moments of the wrapped Shanker distribution are given by

$$\begin{aligned}\alpha_p &= \rho_p \cos \; \mu_p \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \cos\left[2 \; \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right],\end{aligned} \tag{16}$$

and

$$\begin{aligned}\beta_p &= \rho_p \sin \; \mu_p \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \sin\left[2 \; \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right].\end{aligned} \tag{17}$$

Now, the central trigonometric moments of the wrapped Shanker distribution are given by

$$\begin{aligned}\bar{\alpha}_p &= \rho_p \cos(\mu_p - p\mu_1) \\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2 + p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)} \cos\left[2 \; \arctan\left(\frac{p}{\lambda}\right) - \arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\right.\end{aligned} \tag{18}$$

$$-2p\ \arctan\left(\frac{1}{\lambda}\right)+p\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\Bigg],$$

and

$$\begin{aligned}\bar{\beta}_p &= \rho_p\sin(\mu_p-p\mu_1)\\ &= \frac{\lambda^2\sqrt{(\lambda^2+1)^2+p^2\lambda^2}}{(\lambda^2+1)(\lambda^2+p^2)}\sin\Bigg[2\arctan\left(\frac{p}{\lambda}\right)-\arctan\left(\frac{p\lambda}{\lambda^2+1}\right)\\ &\quad -2p\arctan\left(\frac{1}{\lambda}\right)+p\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\Bigg].\end{aligned} \tag{19}$$

6. Location and dispersion

The first trigonometric moment of the wrapped Shanker distribution given by $\varphi_\theta(1)=\rho_1 e^{i\mu_1}$ contains important information about the distribution. The mean direction and concentration of θ are measured by the mean resultant length ρ, and the mean direction μ given by

$$\rho=\rho_1=\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}, \tag{20}$$

$$\mu=\mu_1=2\arctan\left(\frac{1}{\lambda}\right)-\arctan\left(\frac{\lambda}{\lambda^2+1}\right). \tag{21}$$

The length ρ lies between 0 and 1, and measures the dispersion around the mean. The closer ρ is to 1 the more concentration towards μ. In a circular distribution, the circular variance is calculated by $V=1-\rho$, then the circular variance for the wrapped Shanker distribution is given by

$$V=1-\rho=1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}, \tag{22}$$

While, the circular standard deviation is calculated by $\sigma=\sqrt{-2\ln\rho}$, then the circular standard deviation for the wrapped Shanker distribution is given by

$$\sigma=\sqrt{-2\ln\rho}=\sqrt{-2\ln\left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)}. \tag{23}$$

7. Skewness and kurtosis

For a circular model the skewness coefficient is calculated by $\zeta_1=\bar{\beta}_2 V^{-3/2}$, and the kurtosis coefficient is calculated by $\zeta_2=\left(\bar{\alpha}_2-(1-V)^4\right)V^{-2}$. Then for the wrapped Shanker distribution the skewness coefficient and the kurtosis coefficient are given respectively by

$$\zeta_1 = \bar{\beta}_2 V^{-3/2}$$

$$(24) \quad = \left(1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^{\frac{-3}{2}} \left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+4\lambda^2}}{(\lambda^2+1)(\lambda^2+4)} \sin\left[2\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{2\lambda}{\lambda^2+1}\right) - 4\ \arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\right]\right),$$

and

$$\zeta_2 = \left(\bar{\alpha_2} - (1-V)^4\right) V^{-2}$$

$$(25) \quad = \left(1-\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^{-2} \left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+4\lambda^2}}{(\lambda^2+1)(\lambda^2+4)} \cos\left[2\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{2\lambda}{\lambda^2+1}\right) - 4\arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{\lambda}{\lambda^2+1}\right)\right] - \left(\frac{\lambda^2\sqrt{(\lambda^2+1)^2+\lambda^2}}{(\lambda^2+1)^2}\right)^4\right).$$

The following Table 1 shows different parameters of the wrapped Shanker distribution for various values of λ, which are also used in Figure 1 and Figure 2.

Parameter	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 4$
μ	1.83379	1.10715	0.74360	0.54679	0.25887
α_1	-0.05600	0.25000	0.56122	0.73600	0.93466
α_2	-0.02976	0.04000	0.21822	0.40000	0.78177
β_1	0.20800	0.50000	0.51615	0.44800	0.24751
β_2	0.06920	0.28000	0.43865	0.50000	0.41412
ρ	0.21541	0.55902	0.76249	0.86163	0.96688
V	0.78459	0.44098	0.23751	0.138373	0.03312
σ	1.75227	1.07850	0.73644	0.54577	0.25955
$\bar{\alpha}_1$	0.21541	0.55902	0.76249	0.86163	0.96688
$\bar{\alpha}_2$	-0.00901	0.20000	0.45534	0.62786	0.88375
$\bar{\beta}_1$	0	0	0	0	0
$\bar{\beta}_2$	-0.07479	-0.20000	-0.18082	-0.12566	-0.02677
ζ_1	-0.10762	-0.68296	-1.56217	-2.44118	-4.44089
ζ_2	-0.01813	0.52628	2.07978	4.00602	8.93007

Table 1: Values of different parameters of Wrapped Shanker Distribution for various values of λ

8. Simulation

We used simulation technique to study the behaviour of wrapped Shanker distribution for different values of λ and different sample size n. Figure 3, Figure 4, and Figure 5 show the distribution for a sample size $n = 30, 50, 100$ and $\lambda = 0.5, 1, 2$ respectively.

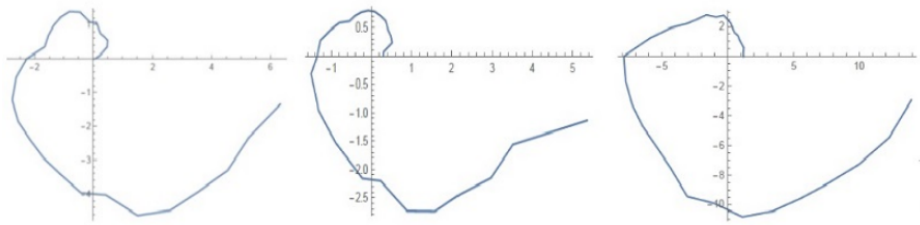

Figure 3: Wrapped Shanker distribution for $n = 30$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	1.1524	0.2222	0.7778	99.3702	-0.1392	0.1171
$\lambda = 1$	1.2737	0.5812	0.4188	59.6901	-0.9314	0.9769
$\lambda = 2$	2.0672	0.3181	0.6819	86.7154	-0.5026	0.4019

Table 2: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 30$.

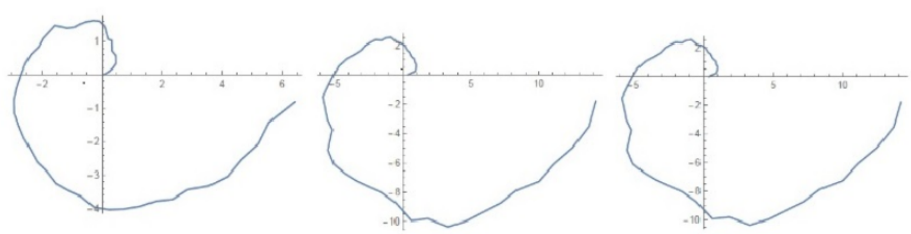

Figure 4: Wrapped Shanker distribution for $n = 50$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	2.3342	0.1728	0.8272	107.3629	0.0725	-0.1802
$\lambda = 1$	2.3334	0.2840	0.7160	90.9095	0.1224	0.4682
$\lambda = 2$	1.5500	0.2421	0.7579	96.4978	-0.0188	0.3177

Table 3: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 50$.

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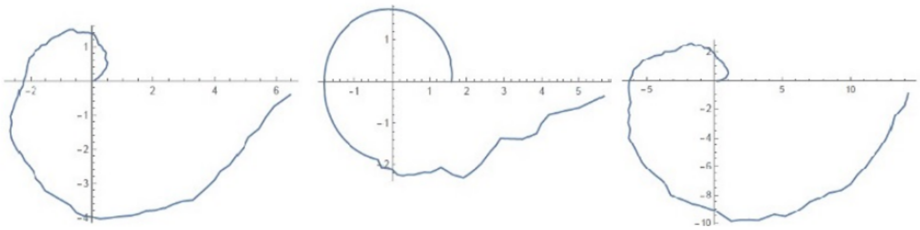

Figure 5: Wrapped Shanker distribution for $n = 100$, $\lambda = 0.5$, $\lambda = 1$, and $\lambda = 2$ respectively

Parameter	μ	ρ	V	σ	ζ_1	ζ_2
$\lambda = 0.5$	2.0016	0.1794	0.8206	106.2023	-0.1689	-0.0341
$\lambda = 1$	1.9764	0.8301	0.1699	34.9679	-2.3751	7.7983
$\lambda = 2$	1.4219	0.3088	0.6912	87.8387	-0.1648	0.2604

Table 4: Simulation values of different parameters of Wrapped Shanker Distribution for various values of λ for sample size $n = 100$.

9. Conclusions

We introduced a new circular distribution, namely, wrapped Shanker distribution. The explicit expressions for the probability density function and the cumulative density function were derived for the new distribution. Also, we obtained the characteristic function, and studied the trigonometric moments and some related parameters for the new distribution. Finally, we discussed the location, dispersion, skewness, and kurtosis of the new distribution.

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