Adjoints of some weighted composition operators on the Fock space

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Abstract. In this paper, we find the adjoints of weighted composition operators $C_{\psi,\varphi}$ on the Fock space, when $\psi = K_c$ for $c \in \mathbb{C}$. Then, we show that $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C^*_{\psi,\varphi} + C_{\psi,\varphi}$ commute, when $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and $|a| = 1$.
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1. Introduction

For f an entire function on $\mathbb{C}$, the Fock space $\mathcal{F}^2$ is the set

$$\left\{ f : \|f\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty \right\},$$

where dA is the Lebesgue measure on $\mathbb{C}$. The Fock space is a reproducing kernel Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dA(z)$$

and reproducing kernel function $K_w(z) = e^{\overline{w}z}$, for any $w \in \mathbb{C}$. Not that for any $w \in \mathbb{C}$, $\|K_w\| = e^{|w|^2/2}$. It is well known that the set $\{e_m : m \geq 0\}$ is an orthonormal basis for $\mathcal{F}^2$. For entire functions ψ and φ and $f \in \mathcal{F}^2$, composition operator C_φ and weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ are given by $C_\varphi(f) = f \circ \varphi$ and $C_{\psi,\varphi}(f) = \psi f \circ \varphi$, respectively. Moreover, we define a multiplication operator $M_\psi : \mathcal{F}^2 \to \mathcal{F}^2$ by $M_\psi(f) = \psi f$, for all $f \in \mathcal{F}^2$.

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Recently many authors have worked on the weighted composition operators on the Fock space (see [1], [2], [3], [5], [6], [7] and [4]). Carswell et al. in [1] showed that C_φ is bounded on the Fock space if and only if $\varphi(z) = az+b$, where $|a| \leq 1$ and if $|a| = 1$, then $b = 0$. Ueki in [4] characterized boundedness and compactness of $C_{\psi,\varphi}$ on the Fock space. Next in [3], Le found the easier criteria for boundedness and compactness of weighted composition operators. Note that there are some interesting papers like [1] which were written on another Fock space (see [8]), but their results hold for $\mathcal{F}^2$ by the same idea. Then we use them frequently in this paper.

Suppose that H is a Hilbert space. For bounded operators A and B on H, we use the notation $[A, B] = AB - BA$ for the commutator of A and B.

Finding adjoint of an operator is one of an important tool to investigate their properties. In the second section, we obtain the adjoints of weighted composition operators $C_{\psi,\varphi}$, when $\psi = K_c$ for $c \in \mathbb{C}$. Next in Section 3, we show that $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi} + C_{\psi,\varphi}] = 0$, where $\varphi(z) = az + b$ and $|a| = 1$.

2. Adjoints of some weighted composition operators

In this section, we find the adjoint of weighted composition operator $C_{\psi,\varphi}$, where $\psi = K_c$ for $c \in \mathbb{C}$. Le in [3, Theorem 2.2] showed that if $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then $\varphi(z) = \varphi(0) + \lambda z$ with $|\lambda| \leq 1$. First, we assume that $|\lambda| < 1$ and find the adjoint of $C_{\psi,\varphi}$ in Theorem 2.3. Then, we suppose that $|\lambda| = 1$ and obtain the adjoint of $C_{\psi,\varphi}$ in Theorem 2.5.

Lemma 2.1. *Suppose that $\varphi(z) = a_1z+b_1$ and $\psi(z) = a_2z+b_2$, where $|a_1| \leq 1$, $|a_2| \leq 1$ and $b_1, b_2 \in \mathbb{C}$. Let C_φ and C_ψ be bounded on $\mathcal{F}^2$. Then*

$$C^*_\varphi C_\psi = M_{e^{\overline{b_1}z}} C_{a_2\overline{a_1}z+b_2}. \tag{1}$$

Proof. By [1, Lemma 2], we see that

$$\begin{aligned} C^*_\varphi C_\psi &= M_{K_{b_1}} C_{\overline{a_1}z} C_{a_2z+b_2} = M_{K_{b_1}} C_{(a_2z+b_2)\circ(\overline{a_1}z)} \\ &= M_{K_{b_1}} C_{a_2\overline{a_1}z+b_2}. \end{aligned}$$

□

In the following proposition, we show that if $\varphi(z) = az + b$, where $|a| < 1$ and ψ is a kernel function K_c, then $C_{\psi,\varphi}$ can be written by the product of the adjoint of a composition operator and another composition operator.

Proposition 2.2. *Let $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$, when $|a| < 1$ and $b, c \in \mathbb{C}$. Then there is a positive integer n such that*

$$C_{\psi,\varphi} = C^*_{\frac{\overline{a}(n+1)}{n}z+c} C_{\frac{n}{n+1}z+b}.$$

Proof. Since $|a| < 1$, there is a positive integer n such that $|a + \frac{a}{n}| < 1$. Let $a_1 = \frac{\overline{a}(n+1)}{n}$ and $a_2 = \frac{n}{n+1}$, $b_1 = c$ and $b_2 = b$. Then by Lemma 2.1, $C_{\psi,\varphi} = C^*_{a_1z+b_1} C_{a_2z+b_2}$. □

In the next theorem, we find the adjoint of weighted composition operator $C_{k_c,az+b}$, where $|a|<1$ and $b,c\in\mathbb{C}$.

Theorem 2.3. *Let $\psi(z)=K_c(z)$ and $\varphi(z)=az+b$, when $|a|<1$ and $b,c\in\mathbb{C}$. Then $C^*_{\psi,\varphi}=C_{e^{\bar{b}z},\bar{a}z+c}$.*

Proof. By Proposition 2.2, there is a positive integer n such that

$$C_{\psi,\varphi}=C^*_{\frac{\bar{a}(n+1)}{n}z+c}C_{\frac{n}{n+1}z+b}.$$

Then

$$C^*_{\psi,\varphi}=C^*_{\frac{n}{n+1}z+b}C_{\frac{\bar{a}(n+1)}{n}z+c}.$$

From Equation (1), we have $C^*_{\psi,\varphi}=M_{e^{\bar{b}z}}C_{\overline{a}z+c}$ and so the result follows. □

Lemma 2.4. *Suppose that $C_{\psi,\varphi}$ is a bounded weighted composition operator on $\mathcal{F}^2$. Then*

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}.$$

Proof. Let f be an arbitrary function in $\mathcal{F}^2$. We see that

$$\begin{aligned}\langle C^*_{\psi,\varphi}K_w,f\rangle &= \langle K_w,C_{\psi,\varphi}f\rangle=\langle K_w,\psi f\circ\varphi\rangle\\ &= \overline{\psi(w)f(\varphi(w))}=\overline{\psi(w)}\langle K_{\varphi(w)},f\rangle.\end{aligned}$$

Hence $C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}$. □

We know that for $\varphi(z)=az+b$ such that $|a|=1$, if $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then $\psi(z)=\psi(0)e^{-a\bar{b}z}$ (see [3, Proposition 2.1]). In the following theorem, we find the adjoints of this case of weighted composition operators.

Theorem 2.5. *Let $\varphi(z)=az+b$ and $\psi(z)=\psi(0)e^{-a\bar{b}z}$, where $|a|=1$ and $b\in\mathbb{C}$. Then $C^*_{\psi,\varphi}=C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}$.*

Proof. By Lemma 2.4, we can see that

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}=\overline{\psi(0)}e^{-\bar{a}b\overline{w}}K_{aw+b}=\overline{\psi(0)}e^{-\bar{a}b\overline{w}}e^{\overline{(aw+b)}z}$$

and

$$C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}K_w=\overline{\psi(0)}e^{\bar{b}z}e^{\overline{w}(\frac{1}{a}z-\frac{b}{a})}=\overline{\psi(0)}e^{\bar{b}z}e^{\frac{\overline{w}}{a}z-\frac{b}{a}\overline{w}}=\overline{\psi(0)}e^{-\bar{a}b\overline{w}}e^{\overline{(aw+b)}z}.$$

Since the span of the kernel functions is dense in $\mathcal{F}^2$, the result follows. □

Example 2.6. (a) Let $\varphi(z)=\frac{z}{2}+i$ and $\psi(z)=e^{iz}$. Then, Theorem 2.3 implies that $C^*_{\psi,\varphi}=C_{e^{-iz},\frac{z}{2}+i}$.

(b) Let $\varphi(z)=iz-\frac{\sqrt{2}}{2}i$ and $\psi(z)=e^{\frac{\sqrt{2}}{2}z}$. We infer from Theorem 2.5 that $C^*_{\psi,\varphi}=C_{e^{\frac{\sqrt{2}}{2}iz},\bar{i}z+\frac{\sqrt{2}}{2}}$.

3. Weighted composition operators for which $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C^*_{\psi,\varphi}+C_{\psi,\varphi}$ commute

In this section, we show that $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$, where $\varphi(z)=az+b$ and $|a|=1$ and $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$.

Theorem 3.1. *Let $\varphi(z)=az+b$ and $|a|=1$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$.*

Proof. From [3, Proposition 2.1], $\psi(z)=\psi(0)e^{-a\bar{b}z}$. By Theorem 2.5, we see that

$$
\begin{aligned}
&[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]\\
&=C^*_{\psi,\varphi}C_{\psi,\varphi}C^*_{\psi,\varphi}+C^*_{\psi,\varphi}C_{\psi,\varphi}C_{\psi,\varphi}-C^*_{\psi,\varphi}C^*_{\psi,\varphi}C_{\psi,\varphi}-C_{\psi,\varphi}C^*_{\psi,\varphi}C_{\psi,\varphi}\\
&=C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&-C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&-C_{\psi(0)e^{-a\bar{b}z},az+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&=\overline{\psi(0)}\psi(0)M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+\overline{\psi(0)}\psi(0)M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}C_{\psi(0)e^{-a\bar{b}z},az+b}\\
&-\overline{\psi(0)}\psi(0)C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}\\
&-\overline{\psi(0)}\psi(0)C_{\psi(0)e^{-a\bar{b}z},az+b}M_{e^{\bar{b}z}}M_{e^{-a\bar{b}(\frac{1}{a}z-\frac{b}{a})}}C_{a(\frac{1}{a}z-\frac{b}{a})+b}\\
&=|\psi(0)|^2e^{|b|^2}C_zC_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}\\
&+|\psi(0)|^2e^{|b|^2}C_zC_{\psi(0)e^{-a\bar{b}z},az+b}\\
&-|\psi(0)|^2e^{|b|^2}C_{\overline{\psi(0)}e^{\bar{b}z},\frac{1}{a}z-\frac{b}{a}}C_z\\
&-|\psi(0)|^2e^{|b|^2}C_{\psi(0)e^{-a\bar{b}z},az+b}C_z
\end{aligned}
$$

and so obviously $[C^*_{\psi,\varphi}C_{\psi,\varphi}, C^*_{\psi,\varphi}+C_{\psi,\varphi}]=0$. □

References

[1] B. J. Carswell, B. D. MacCluer, A. Schuster, *Composition operators on the Fock space*, Acta Sci. Math. (Szeged), 69 (2003), 871-887.

[2] M. Fatehi, *Weighted composition operators on the Fock space*, Operators and Matrices, 13 (2019), 461-469.

[3] T. Le, *Normal and isometric weighted composition operators on the Fock space*, Bull. London Math. Soc., 46 (2014), 847-856.

[4] S. Ueki, *Weighted composition operator on the Fock space*, Proc. Amer. Math. Soc., 135 (2007), 1405-1410.

[5] L. Zhao, *Unitary weighted composition operators on the Fock space of* $\mathbb{C}^n$, Complex Anal. Oper. Theory, 8 (2014), 581-590.

[6] L. Zhao, *Invertible weighted composition operators on the Fock space of* $\mathbb{C}^n$, J. Funct Spaces, 2015. Art. ID 250358.

[7] L. Zhao, C. Pang, *A class of weighted composition operators on the Fock space*, Journal of Mathematical Research with Applications, 35 (2015), 303-310.

[8] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

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