Note on a relative Hilali conjecture

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Abstract. In this paper, we focus on the relative Hilali conjecture proposed by T. Yamaguchi and S. Yokura, that for a continuous map between two simply connected elliptic spaces $f : X \to Y$, dim Ker $\Pi_*(f)_{\mathbb{Q}} \leq$ dim Ker $H_*(f;\mathbb{Q}) + 1$. Our aim is to prove this conjecture for fibrations whose fibre has at most two-oddly generators. Also we show it in the case $H \to G \to G/H$, where G is a compact connected Lie group and H is a closed sub-Lie group of G.

Keywords: rational homotopy theory, elliptic space, Sullivan minimal model, cohomology, Halperin conjecture, Hilali conjecture.

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1. Introduction

In rational homotopy theory there are a lot of conjectures dealing with the question of how large the cohomology algebra of a space has to be under certain conditions. An example of those conjectures is Hilali's [7], which is based on the size of the rationnally elliptic spaces i.e., simply connected spaces satisfying $\dim \pi_*(X)_{\mathbb{Q}}$ and $\dim H^*(X;\mathbb{Q})$ both finite.

Conjecture 1 (Hilali)**.** *Let X be a simply connected rationnally elliptic space, then*

$$\text{(H)} \qquad dim\ \pi_*(X)_{\mathbb{Q}} \leq\ dim\ H^*(X;\mathbb{Q}).$$

Recall that the computation of Betti numbers of a simply connected elliptic space is a NP-hard (see, [6]). However, little is known about rational cohomology of function spaces. In particular, in 2018, T. Yamaguchi and S. Yokura proposed a relative version of the conjecture (H) (see, [17]).

Conjecture 2 (Yamaguchi-Yokura)**.** *Let $f : X \to Y$ be a continuous map between two elliptic spaces, then*

$$\text{(RH)} \qquad dim\ Ker\ \pi_*(f)_{\mathbb{Q}} \leq dim\ Ker\ H_*(f;\mathbb{Q}) + 1,$$

where

$$Ker\ \pi_*(f)_{\mathbb{Q}} := \oplus_{i\geq 1} Ker\ (\pi_i(f)_{\mathbb{Q}} : \pi_i(X)_{\mathbb{Q}} \to \pi_i(Y)_{\mathbb{Q}})\ and$$
$$Ker\ H_*(f;\mathbb{Q}) := \oplus_{i\geq 0} Ker\ (H_i(f;\mathbb{Q}) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})).$$

Note that when $Y = \{*\}$, we get

$$Ker\ \pi_*(f)_{\mathbb{Q}} = \pi_*(X)_{\mathbb{Q}} \qquad and \qquad Ker\ H_*(f;\mathbb{Q}) \oplus \mathbb{Q} = H_*(X;\mathbb{Q}).$$

So by taking their dimension, we obtain the conjecture (H). This proves that the conjecture (RH) is a generalization of the conjecture (H).

Our goal in this paper is to prove the conjecture (RH) in some new cases.
Our main results are the following:

Theorem 3. *The conjecture (RH) holds for every fibration with fiber F such that rank $\pi_{odd}(F)_{\mathbb{Q}} \leq 2$.*

Theorem 4. *Let G be a compact connected Lie group and H be a closed subgroup of G, then the fibration $H \to G \xrightarrow{f} G/H$ satisfies the conjecture (RH).*

Now we briefly summarise the content of this paper. In section 2 we recall the necessary definitions and preliminaries concerning elliptic spaces and some of their properties. We go on to discuss the model of a rational fibration and the main technical tool that we use. Section 3 is devoted to the proof of our main results. In the begining we prove Theorem 3 using standard familiar tools from rational homotopy theory, namely the Sullivan minimal model, the homotopy Euler characteristic, the formal dimension, Halperin conjecture... . So that, we prove Theorem 4 using Sullivan minimal model of Lie group and of homogeneous spaces. Finally, we use the conjecture (H) as a condition to prove the conjecture (RH) for fibration over product of spheres. We conclude this section by an example and suggestion for further work.

2. Preliminaries

We begin by recalling the definition of Sullivan minimal model of a simply connected space. Further details can be found in the references [4] and [5]. All spaces are assumed to be simply connected. A minimal algebra is a free graded commutative algebra ΛV, for some finite type graded vector space V, togother with a differential d of degree $+1$ that is decomposable, i.e., satisfies $d : V^i \to \left(\Lambda^{\geq 2} V\right)^{i+1}$.

We assume that the minimal algebra is simply connected, i.e., that the vector space V has no generators in degree lower than 2. If $\{v_1, ..., v_n\}$ is a graded basis for V, then we write ΛV as $\Lambda(v_1, ..., v_n)$. A basis can always be chosen so that $dv_1 = 0$ and $dv_i \in \Lambda(v_1, ..., v_{i-1})$ for $i \geq 2$.

Every simply connected space with rational cohomology of finite type has a corresponding Sullivan minimal model, which is a minimal algebra that encodes the rational homotopy type of the space. In particular, if $(\Lambda V, d)$ is the Sullivan minimal model of X, there are isomorphism's:

$$\begin{aligned} H^*(\Lambda V, d) &\cong H^*(X; \mathbb{Q}) \text{ as graded commutative algebras,} \\ Hom(V, \mathbb{Q}) &\cong \Pi_*(X)_{\mathbb{Q}} \text{ as graded vector spaces.} \end{aligned}$$

Although our results are stated and proved in purely algebraic terms, they do admit topological interpretations via this correspondance.

Therefore, we can also characterize an elliptic space in terms of its Sullivan minimal model. A space X with Sullivan minimal model $(\Lambda V, d)$ is elliptic if V and $H^*(\Lambda V, d)$ are both finite dimensional. There is a remarkable sub-class of elliptic spaces called pure spaces.

Definition 5 (Pure space/Pure Sullivan minimal model). *An elliptic Sullivan minimal model $(\Lambda V, d)$ is called pure if $dV^{even} = 0$ and $dV^{odd} \subset \Lambda V^{even}$. Also, a simply connected elliptic space X is called pure if its Sullivan minimal model is so.*

An elliptic space has a variety of very nice properties. Let us briefly show them up. By the formal dimension of an elliptic space X we denote

$$fd(X) = \max\left\{k \in \mathbb{N} \text{ such that } H^k(X; \mathbb{Q}) \neq 0\right\}.$$

Also, we know from (Theorem 32.6 of [4]), that when $v_1, ..., v_n$ are the elements of a homogeneous basis of V

$$fd\,(X) = \sum_{|v_i| \text{ odd}} |v_i| - \sum_{|v_i| \text{ even}} (|v_i| - 1)\,,$$

where $|v|$ denotes the degree of v. Moreover, define the homotopy Euler characteristic by

$$\chi_\pi\,(X) = \sum_k (-1)^k \dim\, \pi_k(X)_{\mathbb{Q}};$$

and the cohomology Euler characteristic by

$$\chi_c\,(X) = \sum_k (-1)^k \dim\, H^k\,(X;\mathbb{Q})\,.$$

It is well known that:

Theorem 6 (Theorem 32.10 of [4])**.** *If X is a simply connected elliptic space, then*

$$\chi_\pi\,(X) \leq 0 \text{ and } \chi_c\,(X) \geq 0.$$

Moreover, the following conditions are equivalent:

$$\chi_\pi\,(X) = 0 \text{ and } \chi_c\,(X) > 0.$$

Then the cohomology Euler characteristic of an elliptic space is always non-negative. Now, we may take the following definition:

Definition 7 (F_0-space)**.** *An elliptic space X is said to be an F_0-space if $\chi_c(X) > 0$.*

Examples 8. There are many examples of such spaces:

- Finite products of even dimensional spheres,
- Finite products of complex projective spaces,
- Homogeneous spaces G/H, where H is a closed sub-group of maximal rank of a compact connected Lie group G.

Therefore, we remark that in terms of Sullivan minimal model, the conjecture (H) can be rewritten as follows:

Conjecture 9. *If $(\Lambda V, d)$ is a simply connected elliptic Sullivan minimal model, then*

$$\text{(H)} \qquad dim\ V \ \leq\ dim\ H^*(\Lambda V, d)$$

This conjecture has been established in various special cases, but in general it remains open. It is known for a pure elliptic spaces [7], or hyperelliptic spaces [2], or two-stage spaces [1] and for the other cases see ([3], [8], [9], [10], [13], [14], [18]).

Let us see another version of the conjecture (RH) in the context of fibration.

Let $F \to X \xrightarrow{f} Y$ be a fibration. Recall that the KS-model for f is a short exact sequence:

$$(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$$

of DGA, in which $j : (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D)$ is the inclusion and $q : (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, d_F)$ is the projection onto the quotient of $(\Lambda W \otimes \Lambda V, D)$ by the ideal generated by $\Lambda^+ W$. The differential D satisfies: $D(w) = d_Y(w)$ for $w \in W$ and $D(v) - d_F(v) \in \Lambda^+ W.(\Lambda W \otimes \Lambda V)$ for $v \in V$.

Here $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively (Proposition 15.5 of [4]). The DGA $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model for the total space X but is not in general minimal. As in [17], we denote by $H^*(j, D_1) : W \to H^*(W \oplus V, D_1)$, where D_1 is the linear part of D and $H^*(j) : H^*(\Lambda W, d_Y) \to H^*(\Lambda W \otimes \Lambda V, D)$, thus

$$\text{Coker } H^*(j, D_1) \cong \text{Hom}(\text{Ker } \pi_*(f)_{\mathbb{Q}}, \mathbb{Q})$$
$$\text{Coker } H^*(j) \cong \text{Hom}(\text{Ker } H_*(f; \mathbb{Q}), \mathbb{Q}).$$

Remark 10. Note that Coker $H^*(j, D_1)$ is a subspace of V and the equality holds when D is decomposable.

Hence the conjecture (RH) can also be rewritten as:

Conjecture 11. *If* $(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$ *is the KS-model for a rational fibration of elliptic spaces, then*

$$\text{(RH)} \qquad \textit{dim Coker } H^*(j, D_1) \leq \textit{dim Coker } H^*(j) + 1$$

Next, we give an example where the conjecture (RH) is true.

Example 12. Let us consider the fibration

$$\mathbb{S}^3 \times \mathbb{S}^4 \to X \xrightarrow{f} \mathbb{CP}^2$$

given by the KS-model

$$(\Lambda (x, y), d) \xrightarrow{j} (\Lambda (x, y) \otimes \Lambda (u, v, w), D) \to (\Lambda (u, v, w), d_F),$$

where $|x| = 2$, $|y| = 5$, $|u| = 3$, $|v| = 4$, $|w| = 7$ and the non-zero differentials are given by: $Dy = x^3$, $Du = x^2$ and $Dw = v^2 + vx^2$. Hence, we have dim Coker $H^*(j, D_1) \leq 3$ and $\mathbb{Q}\{[v], [vx]\} \subset$ Coker $H^*(j)$, then

$$\text{dim Coker } H^*(j) + 1 \geq 3 \geq \text{ dim Coker } H^*(j, D_1).$$

Thus, f satisfies the conjecture (RH).

Until now, this conjecture is affirmed for spherical fibration, for T.N.C.Z fibration whose fibre satisfies the conjecture (H) (see, [17]) and for certain reasonable cases (see, [18]).

A fibration $F \xrightarrow{i} X \to Y$ is called totally non cohomologous to zero (abbreviated T.N.C.Z) if the induced homomorphism $H^*(i) : H^*(X;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is surjective. It is equivalent to require that the Serre spectral sequence collapses at E_2-term. In this case, there is an isomorphism: $H^*(X;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(Y;\mathbb{Q})$ of $H^*(Y;\mathbb{Q})$-modules. The following conjecture plays a central role in rational homotopy theory.

Conjecture 13 (Halperin)**.** *Every rational fibration with fibre an F_0-space is T.N.C.Z.*

Examples 14. This conjecture holds for large classes of F_0-spaces:

• Homogeneous spaces of the form G/H, where G is a compact connected Lie group and H is a closed subgroup of maximal rank (see, [15]),

• If the cohomolgy of the fibre has at most 3 generators (see, e.g., [11], [16]).

We end this section by some notations and conventions. In general, we use V or W to denote a positively graded rational vector space of finite type. The cohomology of a DGA (A, d) is denoted $H^*(A, d)$ or just $H^*(A)$ and let $[x] \in H^*(A, d)$ stand for the cohomolgy class of the cocycle $x \in A$.

As an overriding hypothesis, we assume that all spaces appearing in this paper are *rational simply connected elliptic spaces.*

3. Proof of our results

In this section, we give a proof of our results using essentially the algebraic form of the conjecture (RH), the theory of Sullivan minimal model and other invariants in rational homotopy theory.

We are now ready to proceed with the proof of Theorem 3.

Proof of Theorem. 3 Let $F \to X \to Y$ be a fibration with rank $\Pi_{odd}(F) \leq 2$, then the KS-model of this fibration is given as

$$(\Lambda W, d_Y) \xrightarrow{j} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, d_F)$$

with $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively. Further, as stated in the preliminaries $Hom(W, \mathbb{Q}) \cong \pi_*(F)$, the condition rank $\Pi_{odd}(F) \leq 2$ is equivalent to dim $W^{odd} \leq 2$. Our aim is to prove that the dimension of Coker $H^*(j)$ is strictly greater than the dimension of Coker $H^*(j, D_1)$. In the remainder of this proof, we use essentially x_1, x_2 for the generators of even degree and y_1, y_2 for the generators of odd degree, then we consider the following cases:

Case 1. $(\Lambda V, d_F) = (\Lambda(y_1), 0)$, so F has the rational homotopy type of $\mathbb{S}^{|y_1|}$, hence our primary fibration becomes

$$\mathbb{S}^{|y_1|} \to X \to Y,$$

then the conjecture (RH) holds from Theorem 1.1 of [17].

Case 2. $(\Lambda V, d_F) = (\Lambda(x_1, y_1), d_F)$ or $(\Lambda V, d_F) = (\Lambda(x_1, x_2, y_1, y_2), d_F)$, this implies that $\chi_\pi(F) = 0$ and since $(\Lambda V, d_F)$ is elliptic, it follows by Theorem 6 that F is an F_0-space. It is well known that this fibration satisfies Halperin conjecture (Theorem 3 of [16]). Then by Theorem 1.5 of [17] the conjecture (RH) holds.

Case 3. $(\Lambda V, d_F) = (\Lambda(y_1, y_2), 0)$, regarding the total space, since X is elliptic we have $[\mu_X] \neq 0$, where $[\mu_X]$ denotes the fundamental class of X. Thus we have immediately by degree arguments $[\mu_X] \in \text{Coker } H^*(j)$. Finally, we deduce that

$$\begin{aligned} \dim \text{ Coker } H^*(j) + 1 \geq 2 &= \dim V \\ &\geq \dim \text{ Coker } H^*(j, D_1) \qquad \text{(Remark 10)}. \end{aligned}$$

Case 4. $(\Lambda V, d_F) = (\Lambda(x_1, y_1, y_2), d_F)$, if there is at least an element $\omega \in \Lambda^1(x_1, y_1, y_2)$ such that $D\omega = 0$, then $[\omega]$ and $[\mu_X]$ in Coker $H^*(j)$. This implies that

$$\begin{aligned} \dim \text{ Coker } H^*(j) + 1 \geq 3 &= \dim V \\ &\geq \dim \text{ Coker } H^*(j, D_1) \qquad \text{(Remark 10)}. \end{aligned}$$

Now, we assume that Dx_1, Dy_1 and Dy_2 are non-zero. We distinguish two possibilities:

The first when $|y_1| < |x_1|$. For degree reasons and by ellipticity argument we write $d_F(x_1) = d_F(y_1) = 0$ and $d_F(y_2) = x_1^m$ for some natural $m \geq 2$. Hence, we can write the Sullivan minimal model of the fiber F as $(\Lambda(y_1) \otimes \Lambda(x_1, y_2), d_F)$. So, we form the following short exact sequence

$$(\Lambda W \otimes \Lambda(y_1), d) \to (\Lambda W \otimes \Lambda(y_1, x_1, y_2), D) \to \left(\Lambda(x_1, y_2), \overline{d_F}\right),$$

where $d(w) = d_Y(w)$ for $w \in W$, $d(y_1) \in \Lambda W$ and $\left(\Lambda(x_1, y_2), \overline{d_F}\right)$ be the quotient obtained by factoring out the DG ideal generated by the generator y_1. Hence as recalled above (see Examples 14) this fibration satisfies Halperin conjecture, thus

$$(1)\quad H^*(\Lambda W \otimes \Lambda(y_1, x_1, y_2), D) \cong H^*(\Lambda W \otimes \Lambda(y_1), d) \otimes H^*\left(\Lambda(x_1, y_2), \overline{d_F}\right).$$

On the other hand, the inclusion j will be defined as the composite of:

$$(\Lambda W, d_Y) \hookrightarrow (\Lambda W \otimes \Lambda(y_1), d) \hookrightarrow (\Lambda W \otimes \Lambda(y_1, x_1, y_2), D).$$

Further, on passing to cohomology and from (1), we get

$$H^*(j): H^*(\Lambda W, d_Y) \to H^*(\Lambda W \otimes \Lambda(y_1), d) \hookrightarrow H^*(\Lambda W \otimes \Lambda(y_1), d) \otimes H^*(\Lambda(x_1, y_2), \overline{d_F}).$$

Besides, it is easy to see that $[x_1] \in H^*(\Lambda(x_1, y_2), \overline{d_F})$ and $fd(X) > fd(F)$, these imply that in particular $[x_1]$ and $[\mu_X] \in$ Coker $H^*(j)$. So

$$\dim \text{ Coker } H^*(j) + 1 \geq \dim \text{ Coker} H^*(j, D_1).$$

Second, we suppose $|x_1| < |y_1| \leq |y_2|$. Then we may write $Dx_1 = \alpha$ for $\alpha \in \Lambda W$. If $[\alpha] = 0$, i.e., there is γ in ΛW such that $D\gamma = d_Y(\gamma) = \alpha$, we obtain $[x_1 - \gamma]$ is non-zero in Coker $H^*(j)$. Hence, as above we have

$$[x_1 - \gamma] \text{ and } [\mu_X] \in \text{Coker } H^*(j).$$

We now consider $[\alpha] \neq 0$ in $H^*(Y; \mathbb{Q})$. Therefore, it is easy to verify that

$$fd(X) > 3|x_1|. \tag{2}$$

To complete this proof we will focus on $d_F(y_1)$, for this we show the following lemmas:

Lemma 15. *The conjecture (RH) holds if $d_F(y_1) = x_1^n$ for some $n \geq 2$.*

Proof. The condition on the differential implies that $Dy_1 = x_1^n + \sum_{p=0}^{n-1} a_p x_1^p$ with $a_p \in \Lambda W$ for $0 \leq p \leq n-1$. Then, applying D again

$$0 = nD(x_1)x_1^{n-1} + D(a_{n-1})x_1^{n-1} + (n-1)a_{n-1}D(x_1)x_1^{n-2} + D(\sum_{p=0}^{n-2} a_p x_1^p).$$

We obtain $x_1 + \frac{1}{n}a_{n-1}$ is non-exact D-cycle. So from (2), we have

$$\left[x_1 + \frac{1}{n}a_{n-1}\right] \text{ and } [\mu_X] \quad \text{in Coker } H^*(j).$$

We see immediately from Remark 10 dim Coker $H^*(j) + 1 \geq 3 = \dim V \geq$ dim Coker $H^*(j, D_1)$. □

Lemma 16. *The conjecture* (*RH*) *holds also when $d_F(y_1) = 0$.*

Proof. Under our assumptions ($d_F(y_1) = 0$ and F is elliptic), we may write

$$Dy_1 = \sum_{p=0}^{r} a_p x_1^p \text{ and } Dy_2 = x_1^m + \sum_{k=0}^{m-1} b_k x_1^k + \sum_{l=0}^{m-2} c_l x_1^l y_1$$

with a_p, b_k and $c_l \in \Lambda W$ for $0 \leq p \leq r$, $0 \leq k \leq m-1$ and $0 \leq l \leq m-2$.

1. Let us deal with $r \geq 2$, applying D on Dy_1 we obtain in particular $D(a_{r-1} + ra_r x_1) = 0$. Since $|a_{r-1} + ra_r x_1| < |y_1| \leq |y_2|$, we have $[a_{r-1} + ra_r x_1]$ is non-zero in $H^*(X, \mathbb{Q})$. For degree reasons we have

$$[a_{r-1} + ra_r x_1] \text{ and } [\mu_X] \text{ in Coker } H^*(j).$$

2. When $r = 0$, this means that $Dy_1 = a_0$ with $a_0 \in \Lambda W$. We can apply again the same process on Dy_2 to deduce that $[b_{m-1} + mx_1]$ is non-zero in $H^*(X, \mathbb{Q})$. From (2) and Remark 10, we conclude that

$$\dim \text{ Coker } H^*(j) + 1 \geq \dim \text{ Coker } H^*(j, D_1).$$

3. For $r = 1$, we write $Dy_1 = a_1 x + a_0$ with a_0 and a_1 in ΛW. Our key observation is: since $|\alpha|$ is odd, there exists $i \geq 2$ such that $\alpha^i = 0$ and $\alpha^k \neq 0$ for $1 \leq k \leq i-1$, thus $D\left(\alpha^{i-1} x_1\right) = 0$.

3-1. Next, we can consider $\left[\alpha^{i-1} x_1\right] = 0$, i.e., there exist β and β' in ΛW so that $D(\beta y_1 + \beta') = \alpha^{i-1} x_1$, then we compute $D\left(\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'\right) = 0$.

- If $i = 2$, we have $D\left(x_1^2 - 2\beta y_1 - 2\beta'\right) = 0$. It is possible there is ω in $\Lambda W \otimes \Lambda(x_1, y_1)$ so that $D(y_2 + \omega) = x_1^2 - 2\beta y_1 - 2\beta'$ (otherwise the result follows from (2)).

  A direct computation shows that $D(\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3) = 0$. We claim that $[\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3]$ is non-zero in $H^*(X, \mathbb{Q})$. Indeed, suppose there are γ and γ' in $\Lambda W \otimes \Lambda(x_1, y_1)$ for which $D(\gamma y_2 + \gamma') = \alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3$. This implies $D\gamma = \alpha$, which contradicts the fact that $[\alpha]$ is supposed non-zero in $H^*(Y, \mathbb{Q})$. Using (2) again, we conclude that

$$\left[\alpha y_2 + \alpha\omega + 2\beta x_1 y_1 + 2\beta' x_1 - x_1^3\right] \text{ and } [\mu_X] \text{ in Coker } H^*(j).$$

- If $i \geq 3$, we see that $D(\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta') = 0$, then we have $[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'] \neq 0$. Indeed, suppose there is φ in $\Lambda W \otimes \Lambda V$ such that $D(\varphi) = \alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'$. This implies that $D\left(\varphi - \frac{1}{3}\alpha^{i-3} x_1^3\right) = -2\beta y_1 - 2\beta'$. It is automatic that $D(\beta y_1 + \beta') = 0$, which is a contradiction. Also, we have $D(\alpha^{i-1} x_1^2) = 0$, then we distinguish two cases:

  $[\alpha^{i-1} x_1^2] \neq 0$, thus $[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta']$ and $[\alpha^{i-1} x_1^2]$ in Coker $H^*(j)$, as required.

  $[\alpha^{i-1} x_1^2] = 0$, i.e., there is ψ in $\Lambda W \otimes \Lambda V$ so that $D(\psi) = \alpha^{i-1} x_1^2$. By the same manner we verify that $\psi - \frac{1}{3}\alpha^{i-2} x_1^3$ is non-exact D-cycle. Consequently, we have

$$\left[\alpha^{i-2} x_1^2 - 2\beta y_1 - 2\beta'\right] \text{ and } \left[\psi - \frac{1}{3}\alpha^{i-2} x_1^3\right] \text{ in Coker } H^*(j).$$

3-2. To finish this case, we assume that $\left[\alpha^{i-1}x_1\right]$ is non-zero. We have $D\left(\alpha^{i-1}x_1^2\right)=0$, then if $\left[\alpha^{i-1}x_1^2\right]=0$ (otherwise, we obtain $\left[\alpha^{i-1}x_1\right]$ and $[\alpha^{i-1}x_1^2]$ in Coker $H^*(j)$), we use a similar argument as above to prove that $\left[\phi-\frac{1}{3}\alpha^{i-2}x_1^3\right]$ is non-zero with ϕ in $\Lambda W\otimes\Lambda^+(x_1,y_1,y_2)$ such that $D\phi=\alpha^{i-1}x_1^2$ for $i\geq 3$. For $i=2$, we have from (2) $|\mu_X|>|\alpha x_1|$. Then $[\mu_X]$ and $[\alpha x_1]$ are in Coker $H^*(j)$, which implies that dim Coker $H^*(j)+1\geq$ dim Coker $H^*(j,D_1)$ (Remark 10).

□

Now, we go on to establish the second theorem, for this purpose let us recall the definition of homogeneous spaces.

Let G be a compact connected Lie group and H a closed sub-group of G, the quotient G/H is called homogeneous spaces. These spaces form a very well-studied and interesting class of manifolds. They appear abundantly in geometry and topology. It is a classical result that the quotient G/H is an elliptic spaces. We refer the reader to [4], page 218-221 and [5], page 26 for more details.

Proof of Theorem. 4 We denote the Sullivan minimal model of the group G by $(\Lambda W,0)=(\Lambda(y_1,...,y_q),0)$ and the Sullivan minimal model of the sub-Lie group H by $(\Lambda V,0)=(\Lambda(x_1,...,x_p),0)$, where $|x_k|$ and $|y_l|$ are odd for $1\leq k\leq p$ and $1\leq l\leq q$ with $q\geq p$ (Example 2.39 of [5]). Hence a Sullivan model for the homogeneous spaces G/H is given by (Theorem 2.71 of [5])

$$\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right).$$

Here, the notation "$s^{-1}V$" stands for the suspension of V, i.e., the degrees of V are shifted by $+1$. We denote by $i:H\to G$ the canonical inclusion, it induces a map $Bi:BH\to BG$ between the classifying spaces.

The differential d in $\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)$ is defined by $d=0$ in $s^{-1}V$ and $d(y_l)=H^*(Bi)(s^{-1}y_l)$ for $1\leq l\leq q$, where

$$H^*(Bi):H^*(BG)\to H^*(BH).$$

Now, we can consider the KS-model of f:

$$M(f):\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)\to(\Lambda W,0)$$

sending the elements of $s^{-1}V$ to zero. On passing to cohomology we obtain

$$H^*(M(f)):H^*\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)\to H^*(\Lambda W,0)\cong\Lambda(y_1,...,y_q).$$

Since $\left(\Lambda\left(s^{-1}V\right)\otimes\Lambda W,d\right)$ is a pure elliptic space and from degree consideration, it follows immediately

$$\dim\text{ Im }H^*(M(f))\leq 2^{q-p}.$$

So, we obtain

$$\begin{aligned}\dim \text{ Coker } H^*\left(M\left(f\right)\right)+1 &= \dim \ H^*\left(G\right)-\dim \text{ Im } H^*\left(M\left(f\right)\right)+1\\ &\geq 2^q-2^{q-p}+1\\ &\geq p,\end{aligned}$$

which completes the proof. □

In the remainder of this section, we focus on the case in which the base is a product of spheres, in particular we prove the following:

Proposition 17. *Let $F \to X \xrightarrow{f} Y$ be a fibration, suppose that:*

(i) X satisfies the conjecture (H),

(ii) Y has the rational homotopy type of $\mathbb{S}^n$ or $\mathbb{S}^{2n}\times\mathbb{S}^{2m+k}$ for $k=0$ or 1,

then f satisfies the conjecture (RH).

Proof. Let

$$(\Lambda W, d_Y)\xrightarrow{j}(\Lambda W\otimes\Lambda V, D)\to(\Lambda V, d_F)$$

be the KS-model of f with $(\Lambda W, d_Y)$ and $(\Lambda V, d_F)$ are the Sullivan minimal models for Y and F respectively . With the same notation as above, we must show that

$$\dim \text{ Coker } H^*(j)+1\geq \dim \text{ Coker } H^*(j,D_1).$$

Let us begin by $Y\simeq_{\mathbb{Q}}\mathbb{S}^{2n}$, hence its Sullivan minimal model is

$$(\Lambda W, d_Y)=\left(\Lambda\left(x_1,y_1\right),d\right),$$

where degrees and differential are described by: $|x_1|=2n$, $|y_1|=4n-1$ and $dx_1=0$, $dy_1=x_1^2$. Here " $\simeq_{\mathbb{Q}}$ " means having the same rational homotopy type.

Therefore, we have

$$\dim \text{ Im } H^*(j)\leq 2$$

this implies that

$$\begin{aligned}\dim \text{ Coker } H^*(j)+1 &= \dim \ H^*(\Lambda W\otimes\Lambda V,D)-\ \dim \text{ Im } H^*(j)+1\\ &\geq \dim \ H^*(\Lambda W\otimes\Lambda V,D)-1\\ &\geq \dim V\ +\ \dim W-1 \quad \text{(conjecture (H))}\\ &\geq \dim V+1\\ &\geq \dim \text{ Coker } H^*(j,D_1).\end{aligned}$$

Now, if $Y\simeq_{\mathbb{Q}}\mathbb{S}^{2n+1}$, the Sullivan minimal model of Y takes the form $(\Lambda\left(y\right),0)$ with $|y|=2n+1$. Further, we argue exactly as above to deduce our result in this case.

Next, consider the product $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n} \times \mathbb{S}^{2m}$, then we have

$$(\Lambda W, d_Y) = (\Lambda(x_1, y_1) \otimes (x_2, y_2), d),$$

with $|x_1| = 2n$, $|x_2| = 2m$, $|y_1| = 4n-1$ and $|y_2| = 4m-1$. The differential d is defined as follows: $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$ and $dy_2 = x_2^2$. Consequently

$$\dim \operatorname{Im} H^*(j) \leq 4$$

and hence

$$\begin{aligned} \dim \operatorname{Coker} H^*(j) + 1 &\geq \dim H^*(\Lambda W \otimes \Lambda V, D) - 3 \\ &\geq \dim V + \dim W - 3 \qquad \text{(conjecture (H))} \\ &\geq \dim V + 1. \end{aligned}$$

Finally, if $Y \simeq_{\mathbb{Q}} \mathbb{S}^{2n} \times \mathbb{S}^{2m+1}$ it suffices to write the Sullivan minimal model of Y, and by the same way as above we can conclude the result. □

Example 18. Note that under (i), the condition above (ii) is sufficient but not necessary. Indeed, consider the non-trivial fibration $F \to X \xrightarrow{f} \mathbb{S}^3 \times \mathbb{S}^5$ given by the following KS-model

$$(\Lambda(x,y), 0) \to \left(\Lambda(x,y) \otimes \Lambda\left(a, b, a', b', c\right), D\right) \to \left(\Lambda\left(a, b, a', b', c\right), \bar{D}\right)$$

with deg $x = 3$, deg $y = 5$, deg $a = 2$, deg $b = 3$, deg $a' = 6$, deg $b' = 11$, deg $c = 7$, $Dx = Dy = Da = Da' = 0$, $Db = a^2$, $Db' = a'^2$ and $Dc = aa' + xy$. A careful check reveals that D defines a differential. It is easy to see that *dim* Coker $H^*(j, D_1) = 5$, and since $\mathbb{Q}\{[x], [y], [a], [a'], [ax], [ay], [a'x], [a'y]\} \subset H^*(\Lambda(x,y) \otimes \Lambda(a, b, a', b', c); \mathbb{Q})$, so the total space satisfies the conjecture (H).

In addition to that, we have $\mathbb{Q}\{[a], [a'], [ax], [ay], [a'x], [a'y]\} \subset \operatorname{Coker} H^*(j)$, so

$$\dim \operatorname{Coker} H^*(j) + 1 \geq 7 \geq \dim \operatorname{Coker} H^*(j, D_1).$$

Thus, f satisfies the conjecture (RH).

We conclude this section with a question arising from Proposition 17 in which we proved the conjecture (RH) for a particular base. It would be interesting to have a generalization of this proposition. We suggest the following as a specific question in this area.

Question 19. *Let $F \to X \to Y$ be a fibration of simply connected elliptic spaces. If X satisfies the conjecture (H), does the fibration satisfy the conjecture (RH)?*

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