Scale elasticity in the presence of undesirable and nondiscretionary factors: an application to bank branches

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Abstract. The concept of returns to scale and scale elasticity has been frequently studied in the framework of data envelopment analysis. Returns to scale is a qualitative characterization to the frontier points of the technology set and its estimation using tools such as data envelopment analysis (DEA) has attracted considerable attention among researchers. In order to quantify this characterization, we can use scale elasticity that is a quantitative measure of response of the outputs to the change of the inputs of the frontier points in a production set. In the current paper, we propose a method to calculate scale elasticity in banking sector when there are non-discretionary and undesirable factors in the process.

Keywords: Data envelopment analysis, scale elasticity, undesirable outputs, nondiscretionary factors, efficiency

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1. Introduction

Modern performance measurement techniques began with the pioneering work of Farell (1957), who was the first that introduced the relative efficiency assessment of a set of homogenous decision making units (DMUs) with multiple incommensurate inputs and single output in an empirical production possibility set. An extension to the case of multiple inputs and multiple outputs has been done by Charnes, Cooper and Rhodes (1978). They termed their mathematical programing formulation as data envelopment analysis (DEA). The initial work of Charnes et al. (1978) is given in constant return to scale environment and it has been extended to variable return to scale case by Banker, Charnes and Cooper (1984). In the last three decades, theoretical and applied contributions of DEA have been numerous. (See for instances Azadeh and Moradi (2014), Pjevcevic et al. (2018), Pitchipoo et al. (2018) and Kordrostami et al. (2019)).

One of the most frequently studied topics in the context of DEA is the estimation of returns to scale that is a qualitative characterization of boundary points of the technology set. The notion of returns to scale is well-established in DEA and DEA has been proven as an excellent tool to characterize the nature of returns to scale.

In order to quantify this characterization, we can easily use scale elasticity that is a useful characterization of production frontier. Scale elasticity is a measure of response of the outputs to the change of the inputs to each point of the technology frontier. When the production function has an analytical form, scale elasticity would also has an analytical form. However, empirical production possibility set in DEA is a polyhedral set and its boundary points make the production function that is generally a piecewise linear frontier. So, there is a need to provide a procedure to characterize the scale properties of the DEA frontier in various situations. Literature on the response of outputs to the change of inputs is divided in to two groups: qualitative and quantitative methods. In what follows, we briefly review some of the works on this subject.

Scale elasticity is a quantitative measure to reflect the response of outputs to the changes of inputs. The calculation of this measure in DEA framework has been studied by Førsund and Hjalmarsson (2004). Their contributions in their paper were the application of DEA to calculate the scale elasticity and the development of formulas for SE for radial projections of inefficient observations. Krivonozhko (2004) have studied the constructions of economic functions and proposed a DEA-based procedure to calculate marginal rate of substitution using parametric optimization methods. Førsand et al. (2007) have presented two methods of obtaining numerical values of SE by direct and indirect approaches. In direct approach, they evaluate numerical scale elasticity at frontier points of DEA technology. In the indirect approach, they have used efficiency scores and dual variables for radial projections of inefficient points to the frontier. Podinovski et al. (2009) suggested a simple approach to derive formulae for scale elasticity in the variable returns to scale technology. Krivonozhko et al.

(2014) proposed and substantiated a direct method for the returns to scale measurement in the non-radial DEA models. Krivonozhko et al. (2017) have proposed a general approach in order to measure returns to scale elasticity at projection points in the non-radial DEA models and in the radial DEA models.

In real applications, we confront cases in which both undesirable products and nondiscretionary factors are present in the production process. Scale properties of the DEA frontier in such a cases are interesting and important and we have to modify the existing approaches in such a way that to be applicable in such cases. As an example of such a production process, we consider performance evaluation in banking sector. Proportionate to the volume of the banking operations, the bank branches are classified in to different degrees and the branches in high degree do business in an environment that is more favorable than the branches in low degree, hence, the degree of the branches is used as an environmental variable that in not at the discretion of the decision maker. Moreover, as some of DEA applications on banking sector, overdue debts in bank branches are considered as an undesirable output.

As this example shows, we simultaneously have both undesirable outputs and nondiscretionary factors and the calculation of scale elasticity in such case is an interesting subject. Matin and Mirjaberi (2015) have studied the problem of scale elasticity computation in the presence of exogenously fixed inputs. In the current paper, we reformulate the scale elasticity calculation in the presence of both undesirable outputs and nondiscretionary factors. Directional distance function is used to evaluate the relative efficiency of the DMUs and to calculate the quantitative measure of scale elasticity of the efficient DMUs.

The rest of the paper is organized as follows: in the next section a directional distance model is formulated when there are both undesirable outputs and nondiscretionary factors. In Section 3, we propose a procedure to calculate scale elasticity in such a production processes. A real application in banking sector is given in Section 4. Conclusions appear in Section 5.

2. Undesirable outputs and non-discretionary factors

In this section, we consider a production process within which both undesirable and nondiscretionary factors play important role in the process. To incorporate undesirable outputs and nondiscretionary factors in to the performance analysis, we modify the traditional BCC model of Banker, Charnes and Cooper (1984). One type of nondiscretionary factors is environmental factors and from experience we know that these factors are not at the discretion of the decision maker (see Ruggiero (1996, 1998, 2004)). Clearly, the environmental factors z affect the transformation of inputs to outputs. So, in the model we will propose, the effect of the environment in which the DMUs are doing business, have to be considered.

Suppose the production process consists of J DMUs and each $DMU_j : j = 1, ..., J$ consumes M inputs $x_j = (x_{1j}, ..., x_{Mj} \geq 0)$, to generate the desirable outputs $v_j = (v_{1j}, ..., v_{Sj} \geq 0)$ and the undesirable outputs $w_j = (w_{1j}, ..., w_{Nj} \geq 0)$, given the nondiscretionary inputs $z_j = (z_{1j}, ..., z_{Kj} \geq 0)$. To operationalize the DEA technique in this process, the production set is defined as:

$$T\left(z\right) = \left\{\left(x, v, w\right) : x \text{ can produce} \left(v, w\right) \text{in given } z\right\}.$$

To include the environmental effect to the performance analysis, it suffices to note that to evaluate the relative efficiency of a specific DMU, it is necessary to compare it only to those DMUs with at least as harsh an environment as the DMU faces. Axioms such as inclusion, convexity, weak disposability of outputs, monotonicity and minimal extrapolation are assumed to construct the technology set. The unique technology set under these assumptions is as follows:

$$\begin{aligned} \bar{T} = \{(x, v, w) : &\sum_{j=1}^{J} \theta_j \gamma_j v_j \geq v, \\ &\sum_{j=1}^{J} \theta_j \gamma_j w_j = w, \\ &\sum_{j=1}^{J} x_j \gamma j \leq x, \\ &\sum_{j=1}^{J} \gamma_j = 1, \\ \gamma_j = 0 \text{ if } z_j > z; \text{ else } &\gamma_j \geq 0, j = 1, ..., J\ \} \end{aligned} \tag{1}$$

Clearly, due to the existence of $\theta_j \gamma_j$ in the representation of technology set (1),$\bar{T}$ is nonlinear and using the manner of Kuosmanen (2005) it can be transformed to the following linear form:

$$\begin{aligned} T = \{(x, v, w) : &\sum_{j=1}^{J} \lambda_j v_j \geq v, \\ &\sum_{j=1}^{J} \lambda_j w_j = w, \\ &\sum_{j=1}^{J} \left(\lambda_j + \gamma_j\right) x_j \leq x, \\ &\sum_{j=1}^{J} \left(\lambda_j + \gamma_j\right) = 1, \\ \lambda_j + \gamma_j = 0 \text{ if } z_j > z; \text{ else } &\lambda_j + \gamma_j \geq 0, j = 1, ..., J\ \} \end{aligned} \tag{2}$$

We can now apply the performance measurement techniques to evaluate the DMUs. For example if we want to increase the desirable outputs and simultaneously decrease inputs and undesirable outputs, we can use the directional distance function as follows:

$$
(3)\qquad
\begin{aligned}
& e_0^* = \text{Max } e \\
\text{s.t.} & \\
& \sum_{j=1}^{J} (\lambda + \gamma_j)\, x_{mj} \geq (1-e)\, x_{mo}, \;\; m = 1, ..., M \\
& \sum_{j=1}^{J} \lambda_j w_{nj} = (1-e)\, w_{no}, \qquad n = 1, ..., N \\
& \sum_{j=1}^{J} \lambda_j v_{sj} \geq (1+e)\, v_{so}, \qquad s = 1, ..., S \\
& \sum_{j=1}^{J} (\lambda_j + \gamma_j) = 1, \\
& \lambda_j + \gamma_j = 0 \text{ if } z_{kj} > z_{ko}, \;\; k = 1, ..., K \\
& \text{ else } \lambda_j + \gamma_j \geq 0; \;\text{ else } \lambda_j + \gamma_j \geq 0; \;\; j = 1, ..., J
\end{aligned}
$$

Note that the direction we have used in model (3) is $(d_x, d_v, d_w) = (x_o, v_o, w_o)$.

Definition 1. DMU_o is said to be efficient if and only if $e_0^* = 0$.
If $DMU_o = (x_o, v_o, w_o)$ prevails as inefficient, then its efficient projection on the technology frontier is as follows:

$$
(4)\qquad
\begin{aligned}
\bar{x}_o &= (1 - e_o^*)\, x_o - s^{(x)} \\
\bar{x}_o &= (1 - e_o^*)\, w_o \\
\bar{v}_o &= (1 - e_o^*)\, v_o - s^{(v)}
\end{aligned}
$$

In which $s^{(x)}$ and $s^{(v)}$ are respectively the optimal slack variables corresponding to the first and third constraints of (3). In model (3), the objective function and the first four constraints are linear. However due to the existence of the conditional constraint, model (3) is not a complete linear programing problem and we cannot normally use the dual rule of linear programing problems to write the dual formulation of model (3). In what follows, we provide a pseudo-dual formulation to model (3):

$$
\begin{aligned}
& Min \Lambda - [\sum_{s=1}^{S} \delta_s^{(v)} v_{so} - \sum_{n=1}^{N} \delta_s^{(w)} w_{no} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mo}] \\
\text{s.t.} & \\
(5)\qquad & \sum_{s=1}^{S} \delta_s^{(v)} v_{sj} - \sum_{n=1}^{N} \delta_s^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0,
\end{aligned}
$$

$$\sum_{s=1}^{S} \delta_s^{(v)} v_{so} - \sum_{n=1}^{N} \delta_s^{(w)} w_{no} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mo}$$
$$d_j = 1 \text{ if } z_j > z_o \text{ else } d_j = 0 \;\; j = 1, ..., J$$
$$\delta_s^{(v)}, \delta_m^{(x)} \geq 0, \text{ for all s and m}$$

in which $\epsilon > 0$ is sufficiently small. The binary variable d_j in the first constraint indicates that if DMU_j must be included in the reference set of DMU_o or not. If $d_j = 1$, the j-th constraint $\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} - \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_{mo}^{(x)} - \Lambda \leq 0$ is not binding and this guarantees that is not included in the reference set of . However, if we have, permits to be included in the reference set of if needed. If we ignore the conditional constraint in (5), the remaining is a linear programming problem, however it can easily be solved using optimization softwares such as GAMS or MATLAB in its current form.

3. Scale elasticity in the presence of undesirable and ND factors

As we stated before, scale elasticity in quantitative form, or returns to scale in qualitative form are the response of the outputs to the change of inputs in frontier points of the technology set. In the current section, we propose a procedure to calculate the scale elasticity of a frontier DMU in the presence of both nondiscretionary inputs and undesirable outputs.

Suppose there are J DMUs to be evaluated. $DMU_j : j = 1, ..., J$ uses the discretionary inputs $x_j = (x_{1j}, ..., x_{Mj}) \geq 0$ and nondiscretionary inputs $z_j = (z_{1j}, ..., z_{Kj}) \geq 0$ to produce the desirable outputs $v_j = (v_{1j}, ..., v_{Sj}) \geq 0$ and undesirable outputs $w_j = (w_{1j}, ..., w_{Nj}) \geq 0$. The response function $\beta(\alpha)$, for each $\alpha > 0$, can be calculated by the following linear programming problem:

$$\beta(\alpha) = Max \;\; \rho$$

s.t.

$$\sum_{j=1}^{J} (\lambda_j + \gamma_j) x_{mj} \leq \alpha x_{mo}, \quad m = 1, ..., M \tag{6}$$
$$\sum_{j=1}^{J} \lambda_j w_{nj} = \rho w_{no}, \qquad n = 1, ..., N$$
$$\sum_{j=1}^{J} \lambda_j v_{sj} \geq \rho v_{so}, \qquad s = 1, ..., S$$
$$\sum_{j=1}^{J} (\lambda_j + \gamma_j) = 1,$$

$$\lambda_j + \gamma_j = 0 \text{ if } z_{kj} > z_{ko} : \quad k = 1, ..., K \text{ else } \lambda_j + \gamma_j \geq 0;$$
$$\text{else } \lambda_j + \gamma_j \geq 0; j = 1, ..., J,$$
$$\lambda_j, \gamma_j, \alpha, \rho \geq 0.$$

In model (6) the input bundle x_o is scaled by $\alpha > 0$ and the response of this change is calculated by ρ . It should be pointed out that $\beta(\alpha)$ is the maximum proportion of the output (v_o, w_o) for a given vector αx_o in all environments no better than z_o.

Definition 2. Scale elasticity (denoted by $\epsilon(x_o, v_o, w_o, z_o)$) is a function of inputs and outputs and at any frontier point $(\alpha x_o, \beta(\alpha) v_o, \beta(\alpha) w_o, z_o)$ we have:

$$\epsilon(x_o, v_o, w_o, z_o) = \frac{\beta'(\alpha)}{\frac{\beta(\alpha)}{\alpha}} = \frac{\alpha \beta'(\alpha)}{\beta(\alpha)} \tag{7}$$

in which $\beta'(\alpha)$ is the marginal productivity and $\frac{\beta(\alpha)}{\alpha}$ is the average productivity. (See Podinovski et al. (2009)). Note that z_o is unchanged and this means that we do not want to change the environment in which DMU_o is doing business. In other word, scaling the inputs and outputs up and down will not change the performance environment of DMU_o . To interpret $\epsilon(x_o, v_o, w_o, z_o)$, we can say that if the quantity of the input x_o is increased by $\alpha > 1$, then, the maximum quantity of the outputs (v_o, w_o) will increase by $\epsilon(x_o, v_o, w_o, z_o) \times \alpha$, while remaining in the technology set in same environment. Clearly, if $(x_o, v_o, w_o, z_o) = (\alpha x_o, \beta(\alpha) v_o, \beta(\alpha) w_o, z_o)$, then $\alpha = \beta(\alpha) = 1$ and $\epsilon(x_o, v_o, w_o, z_o) = \beta'(1)$. (Note that (x_o, v_o, w_o, z_o) is an efficient point.) In DEA technology set, the production frontier is a piecewise linear function and in the extreme efficient points, the function $\beta(\alpha)$ is not differentiable, but, its one-sided derivatives are available. So, we can define the right and left scales elasticity as follows:

$$\begin{aligned} \epsilon_+(x_o, v_o, w_o, z_o) &= \beta'_+(1) \\ \epsilon_-(x_o, v_o, w_o, z_o) &= \beta'_-(1) \end{aligned} \tag{8}$$

Now suppose $DMU_o : (x_o, v_o, w_o, z_o)$ is inefficient. To calculate the scale elasticity to this unit, we first need to project it to the efficient frontier and then, the scale elasticity of the projected point is considered as a scale elasticity to this inefficient unit. Clearly, different projection points lead to different scale elasticities. However, in this study, we will use the directional model (3) to evaluate the relative efficiency of a specific DMU.

Cleary, we do not have an analytical form to $\beta'_+(\alpha)$ and $\beta'_-(\alpha)$. In order to calculate the numerical values to $\beta'_+(1)$ and $\beta'_-(1)$, we can easily use the following theorem.

Theorem 1. *Suppose Λ^{Max} and Λ^{Min} are respectively the optimal values to the following programs* (9) *and* (10):

$$\begin{aligned}
&\Lambda^{Max} = Max\ \Lambda \\
\text{s.t.}\quad & \\
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mo} + \Lambda = 1 \\
&\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0 \\
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mj} + \Lambda + \epsilon d_j \geq 0 \\
&\sum_{s=1}^{S} \delta_s^{(v)} v_{so} + \sum_{n=1}^{N} \delta_n^{(w)} w_{no} = 1 \\
&d_j \text{ if } z_j > z_o \text{ else } d_j = 0 \ \ j = 1, ..., J \\
&\delta_s^{(v)}, \delta_m^{(x)} \geq 0, \ \text{for all s and m}
\end{aligned} \tag{9}$$

$$\Lambda^{Min} = Min\ \Lambda \tag{10}$$

Subject to the same conditions as in program (9).
Then, $\beta'_+ (1) = 1 - \Lambda^{Max}$ and $\beta'_- (1) = 1 - \Lambda^{Min}$

Proof. The function $\beta(\alpha)$ defined in (6) can be viewed as the function $\varphi(b)$ of the vector $b = (\alpha x_o, 0, 0, 1) \in R^{2m+n+s+2}$ on the right-hand side of (6). As Podinovski et al. (2009) stated, $\beta'_+ (1)$ is the directional derivative of the function $\varphi(b)$ taken at $\hat{b} = (x_o, 0, 0, 1)$ in the direction $\hat{d} = (x_o, 0, 0, 0) \in R^{2m+n+s+2}$ that is $\beta'_+ (1) = \hat{\varphi}\left(\hat{b}; \hat{d}\right)$. Similarly, we have $\beta'_- (1) = -\hat{\varphi}\left(\hat{b}; -\hat{d}\right)$. As we stated before, model (6) is not complete linear programing problem and we cannot use dual role for it. The following pseudo-dual formulation is given to model (6) with $\alpha = 1$:

$$\begin{aligned}
&\beta(1) = Min \sum_{m=1}^{M} \delta_m^{(x)} x_{mo} + \Lambda \\
\text{s.t.}\quad & \\
&\sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} - \sum_{m=1}^{M} \delta_m^{(x)} x_{mj} - \Lambda + \epsilon d_j \leq 0 \\
&\sum_{m=1}^{M} \delta_m^{(x)} x_{mj} + \Lambda \geq 0
\end{aligned}$$

$$(11) \qquad \sum_{s=1}^{S} \delta_s^{(v)} v_{sj} + \sum_{n=1}^{N} \delta_n^{(w)} w_{nj} = 1$$
$$d_j \text{ if } z_j > z_o \text{ else } d_j = 0 \;\; j = 1, \ldots, J$$
$$\delta_s^{(v)}, \delta_m^{(x)} \geq 0, \text{ for all s and m.}$$

Since we assume a non-zero vector is a feasible direction at b,then the directional derivative of φ at $\hat{b}$ exist and $\varphi\prime\left(\hat{b}, \hat{d}\right) = Min\{w\hat{d}|w \in \Omega\}$ and $-\varphi\prime\left(\hat{b}, -\hat{d}\right) = -Max\{w\hat{d}|w \in \Omega\}$, in which Ω is the set of all optimal solutions to model (11). This leads to programs (9) and (10) which include the constraints of (11) and the first constraint of (9) that is equal to the objective function of (11). □

4. An application to bank branches

To illustrate the real applicability of the proposed methodology of calculating scale elasticity, we conduct an empirical study on Iranian banking sector in 2015. Banking in Iran is one of the most prosperous professions and a trusted industry on the Iranian stock exchange. The main function of the Iranian banks is the attracting deposits and then the dispatching of the deposits to the customers. A bank branch is seen as a transformer of resources in to services and products and clearly, this transformation is affected by a lot of variables that are controllable and non-controllable. The systematic view of this process is illustrated in Figure 1.

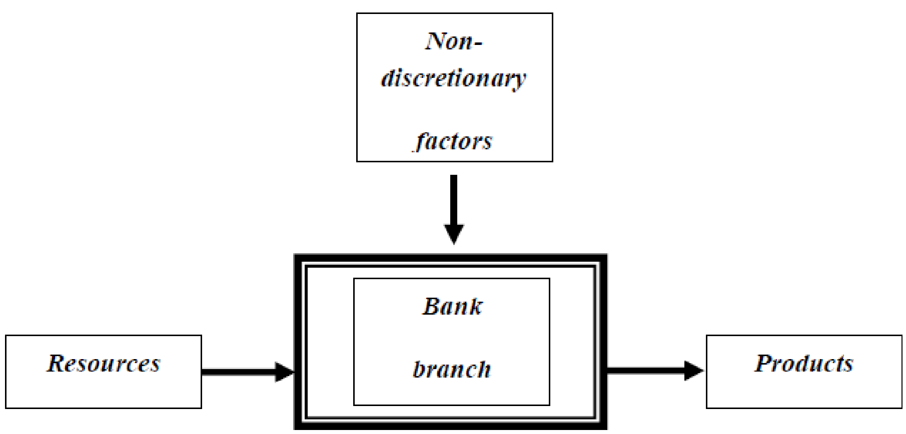

Figure 1: Systematic view of a bank branch

As Khoshandam et al. (2014) stated, bank branches in Iran are divided in to seven degrees and in this study we have focused on the branches in degree 2 and 3. It should be pointed out that the environment that the branches with degree 2 do business are more favorable than the environment of the branches with degree 3. So, the degree of the branches is considered as an environmental variable.

Data on 35 sample branches in 2015 is selected. To evaluate the relative efficiencies of the branches, we have used seven variables from data set as inputs and outputs. We have considered two discretionary inputs, four outputs and one non-discretionary input.

Inputs include: number of check accounts (x_1) and operational costs (including staff costs)(x_2) and outputs include deposits (v_1), loans (v_2), profits (v_3) and overdue debts (w_1) . Note that (w_1) (overdue debts) is the single undesirable output. As we said before, the single nondiscretionary variable that is used in our study is the degree of the branches.

In what follows, a brief description of the variables are given:

-Number of check account: A checking account is a deposit account held at bank branches that allows withdrawals and deposits.

-Operational costs: Operational costs are the expenses which are related to the operations of business, or to the operation of a device, component and piece of equipment or facility. Moreover, staff costs are included in operational costs.

-Deposits: Bank deposits consist of money placed into banking institutions for safekeeping. These deposits are made to deposits accounts such as saving accounts, checking accounts and money market accounts.

-Loans: A loan is a debt provided by a bank branch to a customer at an interest rate and evidenced by a promissory note which specifies, among other things, the principle amount of money borrowed, the interest rate the lender is charging, and date of repayment.

- Profits: Profit is an amount of money that the branches gain when they do their business.

- Overdue debts: A debt is a creditor's right to be paid by a debtor. A debt becomes an overdue debt once it is due and outstanding, provided that it is certain, liquidated, due and payable.

Table 1 shows summary statistics for the inputs and outputs of these 35 branches.

Table 1: Summary statistics for 35 bank branches

	Number of check account	Operational costs	Deposits	Loans	Profits	Overdue debts
Mean	2357.229	30.97536	59825.7	126208.8	8908.08	18438.73
STDEV	607.9711	1085246	210861.3	98499.38	3577.111	10542.38
Minimum	1500	12	250000	75	2500	12000
Maximum	3500	50	1000000	300000	1500	3500

The monetary variables are stated in current Iranian one-hundred million Rial.

The Pearson correlation coefficients of the variables (given in Table 2) show that all of the variables are almost independent and we cannot ignore none of the variables.

Table 2: Pearson correlation coefficients of the variables

	x_1	x_2	v_1	v_2	v_3	w_1
x_1	1	0.295	-	-	-	-
x_2	-	1	-	-	-	-
v_1	-	-	1	0.229	-0.501	-0.005
v_2	-	-	-	1	0.160	0369
v_3	-	-	-	-	1	0.030
w_1	-	-	-	-	-	1

To evaluate the relative efficiencies of the branches, the directional model (3) is applied to this data set. The results are listed in second column of Table 3. Of 35 branches, 28 branches are extreme efficient. So, we can directly calculate their scale elasticities. However, to other seven inefficient branches, we first constructed their efficient projection on the frontier and then, the scale elasticity to these branches are calculated. The right and left scale elasticities are calculated to these branches and the results are shown in the third and fourth columns of Table 3. Among 28 efficient branches, 19 branches have zero right scale elasticities and other nine branches have positive scale elasticities. However, all efficient branches have positive left scale elasticities. Considering this fact that the efficient projection of the inefficient branches are non-extreme points of the frontier, the right and left scale elasticity of these branches are exist, as we expected.

We have also calculated the response function $\beta\,(\alpha)$ to six different values of α : $\alpha = 1.05$, $\alpha = 1.1$, $\alpha = 1.2$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.8$. The results are listed in the last six columns of Table 3.

An important point to be noted is that $\epsilon_+ = 0$ means that if we scale up the inputs, no changes will result in the outputs. This is also true when $\epsilon_- = 0$. This clearly is shown in columns 5, 6 and 7 of Table 3. All response functions for $\alpha > 1$ are feasible and this means that scaling up the inputs will result a feasible point in the production set. However, when we come to calculate the response of outputs to $\alpha < 1$, we have infeasibility in some cases.

Now, we analyze a sample branch: Consider, for example, DMU_1. This branch is efficient in our directional model (3) and its right and left scale elasticities are respectively $\epsilon_+ = 0$ and $\epsilon_- = 5.2$. Clearly, with $\epsilon_+ = 0$, no changes in outputs is allowable when $\alpha > 1$ and hence, we have $\beta\,(\alpha) = 1$ for $\alpha = 1.05, 1.1, 1.2$.

However, for $\alpha < 1$, we have two infeasibility and one feasible response. Scaling down the inputs by 0.90 and 0.85, lead to infeasible points outside of the production technology set. However, the response for $\alpha = 0.95$ is $\beta(0.95) =$

0.7317. This means that if we scale down the inputs to %95, the outputs should be contracted by 0.7317, while remaining in the production set. The response function for $\alpha = 0.9$ and $\alpha = 0.8$ is infeasible.

Now consider the efficient branch 12 with $\epsilon_+ = 0.4375$ and $\epsilon_- = 3.2$. All responses to this unit are feasible. If the inputs are scaled up to $\alpha = 1.05$, then, the outputs are also scaled up to $\beta = 1.0219$. When $\alpha = 0.8$, the response is 0.3598. These values to α along with the response values β show that for $\alpha > 1$, the increase in inputs are faster than the increase in outputs.

Table 3: The results efficiency score and scale elasticity

Branches	Efficiency	ϵ_+	ϵ_-	$\beta\,(1.05)$	$\beta\,(1.1)$	$\beta\,(1.2)$	$\beta\,(0.95)$	$\beta\,(0.9)$	$\beta\,(0.8)$
1	0	0	5.2	1.0000	1.0000	1.0000	0.7317	Infeasible	Infeasible
2	0	0	0.4056	1.0000	1.0000	1.0000	0.9847	0.9694	0.9388
3	0.0591	0.2924	1.683	1.0071	1.0141	1.0281	0.743	0.4874	Infeasible
4	0	0.0471	24.4	1.0024	1.0047	1.0094	Infeasible	Infeasible	Infeasible
5	0	0	0.2195	1.0000	1.0000	1.0000	0.9890	0.9780	0.9561
6	0	0	7.3	1.0000	1.0000	1.0000	0.6367	0.2734	Infeasible
7	0	0	Undefined	1.0000	1.0000	1.0000	Infeasible	Infeasible	Infeasible
8	0.0202	0.0006	0.081	1.0000	1.0000	1.0000	0.9863	0.9691	0.8815
9	0	0	1.086	1.0000	1.0000	1.0000	0.9565	0.9138	0.8277
10	0	0.2577	15.1	1.0153	1.0306	1.0611	Infeasible	Infeasible	Infeasible
11	0	0	2	1.0000	1.0000	1.0000	0.8978	0.7956	Infeasible
12	0	0.4375	3.2	1.0219	1.0437	1.0875	0.8400	0.6799	0.3598
13	0	0	0.058	1.0000	1.0000	1.0000	0.9971	0.9942	0.9875
14	0	0	0.7489	1.0000	1.0000	1.0000	0.9626	0.9251	0.8502
15	0	0	Undefined	1.0000	1.0000	1.0000	0.9734	0.9468	0.8936
16	0	0.1116	2	1.0056	1.0112	1.0223	Infeasible	Infeasible	Infeasible
17	0.1526	0.3746	0.5833	1.0188	1.0376	1.0628	0.9708	0.9413	0.4565
18	0	0	1.18	1.0000	1.0000	1.0000	0.9410	0.8820	0.7640
19	0.4628	1.949	5	1.0441	1.0519	1.0675	Infeasible	Infeasible	Infeasible
20	0.0212	0	0.1611	1.0000	1.0000	1.0000	0.9754	0.8458	Infeasible
21	0.1813	0.341	0.5916	1.0141	1.0257	1.0361	0.97	0.7722	Infeasible
22	0	0	3.7	1.0000	1.0000	1.0000	0.8126	0.6253	0.2506
23	0	0.7021	6.4	1.0351	1.0675	1.1141	0.6810	0.3619	Infeasible
24	0	0	0.8078	1.0000	1.0000	1.0000	0.9596	0.9192	0.8148
25	0	0.4201	Undefined	1.0340	1.0659	1.0968	Infeasible	Infeasible	Infeasible
26	0	0	0.0443	1.0000	1.0000	1.0000	0.9978	0.9956	0.9911
27	0	0	0.441	1.0000	1.0000	1.0000	0.9780	0.9559	0.9118
28	0	0	5.6	1.0000	1.0000	1.0000	0.7212	0.3906	Infeasible
29	0.3355	0.2216	0.6474	1.0108	1.0215	1.0319	0.9676	0.9092	0.2787
30	0	0	5.3	1.0000	1.0000	1.0000	0.7326	0.4653	Infeasible
31	0	0.3566	Undefined	1.0178	1.0357	1.0713	Infeasible	Infeasible	Infeasible
32	0	0.265	1.888	1.0132	1.0265	1.0530	0.9056	0.7700	Infeasible
33	0	0.0749	24.6	1.0037	1.0075	1.0150	Infeasible	Infeasible	Infeasible
34	0	0	3.1	1.0000	1.0000	1.0000	0.8452	0.6903	0.3806
35	0	0	9.8	1.0000	1.0000	1.0000	Infeasible	Infeasible	Infeasible

5. Conclusions

One of the most frequently studied problems in the context of DEA, is determining the nature of returns to scale to the points on the production function. Returns to scale is a qualitative characterization of frontier points of production technology set and we can use scale elasticity as a quantification of scale properties for boundary points of the technology set. Calculating the scale elasticity of efficient DMUs in DEA framework in the presence of undesirable outputs and nondiscretionary factors is the problem that we have studied in the current paper. A DEA-based procedure is proposed to calculate the scale elasticity of the DMUs. Then, the applicability of the proposed approach has been illustrated by a real case on banking industry in Iran. The advantages of the proposed procedure is to introduce a DEA-based method to investigate the scale properties of frontier points in the presence of both undesirable outputs and non-discretionary factors.

It is possible to use scale elasticities to determine optimal scale size in DEA technologies. This is our future research.

References

[1] A. Azadeh, B. Moradi, *Simulation optimization of facility layout design problem with safety and ergonomics factors,* International Journal of Industrial Engineering: Theory, Applications and Practice, 21 (2014), 209-230.

[2] R.D. Banker, A. Charnes, W.W. Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Management Science, 30 (1984), 1078-1092.

[3] A. Charnes, W.W. Cooper, E. Rhodes, *Measuring the efficiency of decision making units*, European Journal of Operational Research, 1978.

[4] M.J. Farell, *The measurement of productive efficiency*, Journal of the Royal Statistical Society, 120 (1957), 253-290.

[5] F.R. Førsund, L. Hjalmarsson, *Calculating scale elasticity in DEA models*, Journal of the Operational Research Society, 55 (2004), 1023-1038.

[6] F.R. Førsund, L. Hjalmarsson, V.E. Krivonozhko, O.B Utkin, *Calculation of scale elasticity in DEA models: direct and indirect approaches*, Journal of Productivity Analysis, 28 (2007), 45-56.

[7] R.K. Kazemi Matin, M. Mirjaberi, *Computation of scale elasticity in presence of exogenously fixed inputs*, Measurement, 73 (2015), 275-283.

[8] L. Khoshandam, A. Amirteimoori, R. Kazemi Matin, *Marginal rates of substitution in the presence of non-discretionary factors: a data envelopment analysis approach*, Measurement, 58 (2014), 409-415.

[9] S. Kordrostami, M. Jahani, A. Amirteimoori, *An alternative DEA approach for evaluating the system efficiency in the presence of dual-role factors*, International Journal of Industrial Engineering: Theory, Applications and Practice, 26 (2019), 17-29.

[10] T. Kousmanen, *Weak disposability in nonparametric production analysis with undesirable outputs*, Am. J. Agr. Econ, 87 (2005), 1077-1082.

[11] V.E. Krivonozhko, A.V. Volodin, I.A. Sablin, M. Patrin, *Constructions of economic functions and calculation of marginal rates in DEA using parametric optimization methods*, Journal of the Operational Research Society, 55 (2004), 1049–1058.

[12] V.E. Krivonozhko, F.R. Førsund, A.V. Lychev, *Measurement of returns to scale using non-radial DEA models*, European Journal of Operational Research, 232 (2014), 664-670.

[13] V.E. Krivonozhko, F.R. Førsund, A.V. Lychev, *Measurement of returns to scale in radial DEA models*, Computational Mathematics and Mathematical Physics, 57 (2017), 83-93.

[14] P. Pitchipoo, P. Venkumar, S. Rajakarunakaran, R. Ragavan, *Decision model for supplier evaluation and selection in process industry: A hybrid DEA approach*, International Journal of Industrial Engineering: Theory, Applications and Practice, 25 (2018), 186-199.

[15] D. Pjevcevic, B. Dimitrijevic, I.V. Bisevac, K. Vukadinovic, *Combining data envelopment analysis and analytic hierarchy process for efficient Inland port services: case study of the port Dunav Pancevo,* International Journal of Industrial Engineering: Theory, Applications and Practice, 25 (2018), 45-59.

[16] V.V. Podinovski, F.R. Førsund, V.E. Krivonozhko, *A simple derivation of scale elasticity in data envelopment analysis*, European Journal of Operational Research, 197 (2009), 149–153.

[17] J. Ruggiero, *On the measurement of technical efficiency in public sector*, European Journal of Operational Research, 90 (1996), 553–565.

[18] J. Ruggiero, *Theory and methodology ND inputs in DEA,* Eur. J. Oper. Res., 111 (1998), 461–469.

[19] J. Ruggiero, *Performance evaluation when non-discretionary factors correlate with technical efficiency*, European Journal of Operational Research, 159 (2004), 250-257.

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