Closed multiplication modules

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Abstract. In this paper, all rings are commutative with identity, all modules are unitary, S denotes a ring and W denotes an S-module. A submodule U of an S-module W is said to be closed in W, if U has no proper essential extensions in W. In this paper we introduce the notion of multiplication module concerning closed submodule as a new generalization of multiplication module namely closed multiplication module. Some basic properties of this notion are given. Some related modules with this concept are investigated and studied too

Keywords: multiplication module, closed multiplication module, CL-duo module.

1. Introduction and preliminaries

Throughout, all rings are commutative with identity, all modules are unitary. A module W in which each submodule U of W has the form IW for some an ideal I in S is called a multiplication module. A submodule U of an S-module W is said to be closed in W, if U has no proper essential extensions in W. Some authors studied various types of generalizations of the multiplication module as small multiplication module in [1] and pure multiplication module in [4], and fully invariant multiplication module [18]. In this paper, we shall introduce a new generalization of the multiplication module concerning closed submodule, where an S-module W is called closed multiplication if each closed submodule U of W is equal to IW for some ideal I of S, and hence every multiplication module is c-multiplication. The converse is true under the class semisimple module. We studied some basic properties of this notion. Also, we investigate some connections between it and other related concepts where we proved that every simple closed module is closed multiplication but not conversely such as the Z_6 as

Z-module. Then we proved that under the class semisimple module both of the concepts closed multiplication and simple closed are coincided. Recall that "a submodule U of W is called stable (fully invariant), if U contains $f(U)$ for each S-homomorphism $f : U \to W(f : W \to W)$, and an S-module M is called fully stable (duo) if each submodule of W is stable (fully invariant)" [12]. It is clear that every fully stable module is duo but the converse may not be true. Recall that "a module W is called CL-duo if for each closed submodule U of W is fully invariant"[2]. Also we proved every closed multiplication is CL-duo module. After that, we proved that any closed submodule U of closed multiplication module W is stable. Finally, recall that "a submodule U of an S-module W is called dual stable, if $U \subseteq kerf$ for each S-homomorphism $f : W \to W/U$ [1]. We proved that every closed submodule of c-multiplication module is dual stable. We used $U \leq_c W$ to denote a closed submodule U of W and $End(W)$ to denote for the set all endomorphisms of W.

2. Closed multiplication module

We introduced the notion of a closed multiplication module as a new generalization of the multiplication module.

Definition 2.1. *An S-module W is called closed multiplication (shortly, c-multiplication) if each closed submodule U of W is equal to IW for some ideal I of S. A ring S is called c-multiplication if S is an c-multiplication S-module.*

Eamples and Remarks 2.1.

(1) Every multiplication S-module is c-multiplication. The converse may be not true in general, for example: each of the Z-modules Z_{P^∞} and Q are c-multiplication since the only closed submodules are (0) and Z_{P^∞} of Z_{P^∞} and (0) and Q are the only closed submodules of Q but it is well-known that each of them is not multiplication module.

(2) Every cyclic module over a commutative ring S is c-multiplication S-module.

(3) Let $W = Z_8 \oplus Z_2$ as Z-module, M is not c-multiplication Z-module since the submodule $U =< (\bar{2}, \bar{1}) >= \{(\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1}), (\bar{0}, \bar{0})\}$ is closed submodule of W, but $U \subseteq IW$ for each ideal I of Z.

(4) It is clear that every simple module is c-multiplication, but not conversely, see examples in part(1).

(5) Let S be a ring and W be a module. Then W is c-multiplication S-module if and only if W is c-multiplication $\bar{S}$-module, where $\bar{S} = S/annW$.
Proof. It easy since each submodule is closed of S-module W if and only if it is closed of $\bar{S}$-module.

(6) One can easy to see if U is a proper closed submodule of c-multiplication module then $U = (U : W)W$. Where $(U : W)= \{ r \in S : W \subseteq U\}$ which is called residual ideal.

(7) It is known every commutative ring with identity is multiplication ring and hence c-multiplication.

(8) Every simple closed S-module is c-multiplication where a module W is called simple closed if W has the only two closed submodules are 0 and W [12]. But the converse may be not true in general, for example: Z_6 as Z-module is c-multiplication but it is not simple closed.

Proposition 2.1. *Let $f : W \to W'$ be a monomorphism and W is c-multiplication module. Then $f(W)$ is c-multiplication.*

Proof. Let $U \leq_c f(W)$. Then we can show that $f^{-1}(U) \leq_c W$. Assume $f^{-1}(U) = F$. Since f is monomorphism, $f(f^{-1}(U)) = f(F)$ is closed submodule of $f(W)$. But one sees easily that $U = f(f^{-1}(U))$, so $U \leq_c f(W)$, and hence $f(F) = U$. It follows that $f^{-1}(U) = f^{-1}(f(F)) = F$. Therefore $f^{-1}(U) \leq_c W$, so that $f^{-1}(U) = IW$ for some $I \leq S$ and hence $f(f^{-1}(U)) = f(IW) = If(W) = IW$. Thus $U = If(W)$ and so $f(W)$ is c-multiplication module. □

Proposition 2.2. *For c-multiplication module W over a ring S, the following assertions hold:*

(*i*) *Every closed submodule U of W is a fully invariant submodule of W,*

(*ii*) *If U is a submodule of W such that $U \cap IW = IU$ for every ideal I in S, then U is a c-multiplication module.*

Proof. (i) Let $L \leq_c W$, $f : W \to W$ there exists an ideal I in S such that $L = IW$, consider $f(L) = f(IW) \subseteq If(W) = IW = L$ and hence L is fully invariant submodule of W.

(ii) Let $V \leq_c U$. Since W is a c-multiplication module, there exists an ideal I of S such that $V = IW$. By assumption, $U \cap W = IU$. Therefore, $U \cap V = IU$ and so $V = IU$ hence U is c-multiplication. □

Proposition 2.3. *Every direct summand of a c-multiplication module is c-multiplication.*

Proof. Let U be a direct summand of W and $L \leq_c U$. Then $W = U \oplus V$ for some submodule V of W, $L \oplus V \leq U \oplus V = W$. But $L \leq_c U$ and $V \leq_c V$, so $L \oplus V \leq_c W$. Then $L \oplus V = IW = I(U \oplus V) = IU \oplus V$ for some ideal I of S. Hence $L = IU$, so U is c-multiplication module. □

Proposition 2.4. *Let W be a divisible module over integral domain S. Then W is c-multiplication if and only if W is simple closed.*

Proof. Let $U \leq_c W$ and $U \neq 0$. Since W is c-multiplication, so there exists an ideal I of S such that $U = IW$. But W is divisible module over integral domain hence $IW = W$ and hence $U = W$. Thus W is simple closed. Conversely, It is clear by Remark 2.1(8). □

Recall that, an S-module W is called essentially second if for all an ideal I in S, either $IW \leq_e W$ or $IW = (0)$[13].

Proposition 2.5. *Every second c-multiplication S-module is simple closed.*

Proof. Let $U \leq_c W$. Since W is c-multiplication, then $U = IW$ for some $I \leq S$. As W is an essentially second module, so either $IW \leq_e W$ or $IW = (0)$; that is either $U \leq_e W$ or $U = (0)$. Since $U \leq_e W$, then $U = W$ or $U = 0$. Therefore W is simple closed. □

In the following, we have both concepts c-multiplication and multiplication modules are equivalent under the class semisimple module.

Proposition 2.6. *Let W be a semisimple S-module. Then W is c-multiplication if and only if W is multiplication module.*

Proof. Since every submodule of W is a direct summand, hence it is closed. Thus the result holds. □

Since each module over a semisimple ring S is semisimple S-module, then we have the following:

Corollary 2.1. *Let W be a module over semisimple ring S. Then W is c-multiplication iff W is multiplication.*

Recall that "an S-module W is called extending (or CS-module), if every submodule of W is essential in a direct summand" [3]. "An S-module W is called quasi-injective if for each monomorphism $f : U \to W$, where U is a submodule of W, and for each S-homomorphism $g : U \to W$ there exists an S-homomorphism $h : W \to W$ such that $h \circ f = g$ [11].

Proposition 2.7. *Let W be a quasi-injective (or extending) module. If W is c-multiplication S-module, then every closed submodule of W is stable.*

Proof. Let $U \leq_c W$. Since W is quasi-injective (or extending) module, then U is direct summand of W. Let $f : U \to W$ be a homomorphism. There exists $g : W \to W$ defined by $g(x) = f(x)$ for all $x \in U$ and $g(x) = 0$ otherwise, W is c-multiplication so $U = IW$ for some ideal I in S, hence $f(U) = g(U) = g(IW) = Ig(W) \subseteq IW = U$, and so $f(U) \subseteq U$. Thus U is stable. □

Recall that, an S-module W is called CL-duo if for each closed submodule U of W is fully invariant[2].

Proposition 2.8. *Every c-multiplication S-module is CL-duo.*

Proof. Let $U \leq_c W$ and W be a c-multiplication module, so $U = IW$ for some $I \leq S$, $f : W \to W$. $f(U) = f(IW) = If(W) \subseteq IW = U$. Thus W is CL-duo. □

Recall that "an S-module W is called a projective module if for each epimorphism $f : A \to B$ and a homomorphism $g : W \to B$, there exists a homomorphism $h : W \to A$ satisfies $f \circ h = g$, where A and B are any S-modules" [7]. The converse of Proposition 2.8 is true under the class projective module.

Proposition 2.9. *Let W be a projective S-module. Then W is CL-duo if and only if W is c-multiplication.*

Proof. To prove this we can do analogous the same proof way in [18, Theorem 3.12]. □

Since S is projective S-module, then we obtain the following:

Corollary 2.2. *Let S be a ring. Then S is CL-duo if and only if it is c-multiplication.*

Remarks 2.1. The direct sum of two c-multiplication modules may be not c-multiplication module. For example: Consider $W = Z \oplus Z$ as Z-module. It is clear that Z is multiplication and hence it is c-multiplication but $W = Z \oplus Z$ is not c-multiplication because let $U = Z \oplus (0) \leq_c W$, but there is no ideal I in S holds $IW = U$.

Recall that "an S-module W is said to be fully invariant multiplication if each fully invariant submodule U of W there exists an ideal I of S such that $U = IW$[18].

Proposition 2.10. *Let W be a CL-duo module. If W is fully invariant multiplication S-module, then W is c-multiplication.*

Proof. Let $U \leq_c W$. Since W is CL-duo, U is fully invariant in W. But W is fully invariant multiplication, then $U = IW$ for some $I \leq S$. Thus W is c-multiplication. □

Proposition 2.11. *Let $T = W \oplus W$ and W is a c-multiplication S-module. If T is a distributive module, then T is c-multiplication.*

Proof. Let $U \leq_c T$. Since T is distributive, then $U = (U \cap W) \oplus (U \cap W)$. But $U \leq_c T$, so $U \cap \leq_c W$ and hence $U \cap W = IW$ for some ideal I in S. Thus $U = IW \oplus IW = I(W \oplus W) = IT$. Therefore T is c-multiplication module. □

Proposition 2.12. *Let $T = W \oplus W$ be a CL-duo module. If W is a c-multiplication S-module, then T is c-multiplication.*

Proof. Let $U \leq_c T$. Since T is CL-duo, so U is a fully invariant submodule of T, and so $U = (U \cap W) \oplus (U \cap W)$ by [16]. But $U \cap W$ is closed submodule of W, so $U \cap W = IW$. Hence $U = IW \oplus IW = I(W \oplus W) = IT$, then T is c-multiplication. □

Proposition 2.13. *Let W be a c-multiplication S-module such that $W = W_1 \oplus W_2$. Then the following assertions hold:*

(i) $f(W_1) \subseteq W_1$ *for any homomorphism* $f : W_1 \to W$.

(ii) $Hom(W_1, W_2) = Hom(W_2, W_1) = 0$.

Proof. (i) There exists an endomorphism g of the module M such that $g(x + y) = f(x)$ for all elements $x \in W_1$ and $y \in W_2$. By proposition 2.3, all submodules of the c-multiplication module W are fully invariant submodules of W. Therefore, $f(W_1) = g(W_1) \subseteq W_1$.

(ii) Let $f \in Hom(W_1, W_2)$, then by (i), $f(W_1) \subseteq W_1 \cap W_2 = 0$. □

Proposition 2.14. *Let W_1 be a c-multiplication S_1-module and W_2 be a c-multiplication S_2-module where S_1 and S_2 are commutative rings with identities). Let $W = W_1 \oplus W_2$ be a distributive S-module where $S = S_1 \oplus S_2$. Then W is c-multiplication iff W_1 and W_2 are c-multiplication modules.*

Proof. Let $U \leq_c W$, then $U = (U \cap W_1) \oplus (U \cap W_2)$. $U \cap W_1 \leq_c W_1$ and $U \cap W_2 \leq_c W_2$, so $U \cap W_1 = IW_1$ and $U \cap W_2 = JW_2$ where $I \leq S_1$ and $J \leq S_2$. As $(I \oplus J).(W_1 \oplus W_2) = IW_1 \oplus JW_2 = (U \cap W_1) \oplus (U \cap W_2) = U$; that is $U = (I \oplus J)(W)$.

Conversely, Let $U \leq_c W_1$, so $U \oplus W_2 \leq_c W_1 \oplus W_2$. Then there exists an ideal $I \oplus J$ in $S_1 \oplus S_2$ such that $U \oplus W_2 = IW_1 \oplus JW_2$. Thus $U = IW_1$, then W_1 is c-multiplication. Similarly, W_2 is c- multiplication. □

Remarks 2.2. Let W be an S-module and $U \leq W$. If W/U is c-multiplication, then W is not necessary c-multiplication module. For example, consider $W = Z \oplus Z$ and $U = 2Z \oplus 3Z$ clear that $W/U = (Z \oplus Z)/(2Z \oplus 3Z) \cong Z_2 \oplus Z_3 \cong Z_6$ which is c-multiplication but $W = Z \oplus Z$ is not c-multiplication by Remark 2.1.

Recall that "an S-module W is called antihopfian if $W/U \cong W$ for all a proper submodule U of W[8]". The following proposition state that the endomorphism of any antihopfian module is c-multiplication ring.

Proposition 2.15. *Let W be an antihopfian S-module. Then the endomorphism ring of W is c-multiplication.*

Proof. Let $End(W)$ is the endomorphism ring of c-multiplication module W. Since W is antihopfian so $End(W)$ is an integral domain by [14] and hence it is a commutative ring with identity so by Remark 2.1(7). $End(W)$ is a c-multiplication ring. □

Recall that "a module W is called a prime if $ann(x) = ann(y)$ for each nonzero element x and y in W. Equivalently, a module W is prime if for all nonzero submodule K of W, $ann(K) = ann(W)$ "[5]. Recall that "a module W of commutative ring S is called a scalar module if for all $f \in End(W)$, $f \neq 0$, there exists $0 \neq r \in S$ such that $f(m) = mr$ for each $m \in W$" [17]. Now we ready to prove the following:

Proposition 2.16. *Let W be a scalar S-module with $ann(W)$ is prime ideal in S. Then $End(W)$ is c-multiplication ring.*

Proof. By [15, Lemma 6.2], we have $End(W) \cong S/(ann(W))$. But W is scalar module and $ann(W)$ is prime, then $S/(ann(W))$ is integral domain hence $End(W)$ is commutative ring with identity so by Remark 2.1(7), it is c-multiplication ring. □

Since $ann(W)$ is prime for each prime module. Thus we can get the following corollary:

Corollary 2.3. *Let W be a prime scalar S-module. Then $End(W)$ is a c-multiplication ring.*

Recall that "a module W is called faithful if $ann(W) = 0$".

Proposition 2.17. *Let W be a faithful scalar S-module. Then S is c-multiplication ring if and only if W is c-multiplication module.*

Proof. Since W is scalar S-module, then $End(W) \cong S/(ann(W))$. But M is faithful so $ann(W) = 0$, hence $End(W) \cong S$ and hence S is c-multiplication ring if and only if W is c-multiplication module. □

Recall that "a submodule U of an S-module W is called characteristic if, for all automorphism β of W, $\beta(U) = U$"[9]. Under c-multiplication module we shall see every closed submodule is characteristic as the next proposition:

Proposition 2.18. *Let W be a c-multiplication S-module. Then every closed submodule of W is characteristic.*

Proof. Let U be a closed submodule of a c-multiplication module W, so there exists an ideal I of S such that $U = IW$. Let β be an automorphism that is β is epimorphism and so $\beta(U) = \beta(IW) = I\beta(W) = IW = U$. Thus U is characteristic submodule. □

Inaam in [10] introduced purely duo modules (briefly P-duo module), which is "a module in which every pure submodule is fully invariant". Atani in [4] introduced the concept pure multiplication where "an S-module W is called a pure multiplication provided for each proper pure submodule U of W, $U = IW$ for some ideal I of S". Also Th. Ghawi in [9] studied pure multiplication modules.

Proposition 2.19. *Let W be a semisimple S-module. Then The following statements are equivalent:*

(i) *W is c-multiplication;*

(ii) *W is multiplication;*

(iii) *W is CL-duo;*

(iv) *W is duo;*

(v) *W is p-duo;*

(vi) *W is fully stable;*

(vii) *W is pure multiplication;*

Proof. (i)$\Leftrightarrow$(ii) It is clear by Proposition 2.6.

(ii) $\Leftrightarrow$(iii) $\Leftrightarrow$(iv) $\Leftrightarrow$(v) $\Leftrightarrow$ (vi) follows by [2, Theorem(5.1)].

(i)$\Leftrightarrow$ (vii) Since W is semisimple S-module by hypothesis so every submodule of W is direct summand and so every submodule of W is closed and pure. Thus W is c-multiplication if and only if W is pure multiplication module. □

Recall that "a submodule U of an S-module W is called dual stable, if $U \subseteq kerf$ for each S-homomorphism $f : W \to W/U$ [1]".

Proposition 2.20. *Every closed submodule c-multiplication S-module is dual stable.*

Proof. Let $U \leq_c W$, since W is c-multiplication, hence $U = IW$ for some ideal I in S. Let $f : W \to W/U$ be an homomorphism, so $f(U) = f(IW) = If(W) \subseteq I(W/U) = IW/U = U/U = 0_{W/U}$, and so $U \subseteq kerf$. Thus U is dual stable □

Proposition 2.21. *Let S be a commutative ring and W be a c-multiplication S-module such that $W = IW$ for some ideal I in S, then $U = IU$ for each closed submodule U of W.*

Proof. Let $U \leq_c W$. Since W is c-multiplication, hence U=JW for some ideal J in S. By assumption, $W = IW$. Hence $U = JW = JIW$. But $JI = IJ$, since S is a commutative ring. Therefore $U = IJW = IU$. □

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