Solution of random ordinary differential equations using Laplace variational iteration method

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Abstract. In this article, we shall introduce the Laplace transformation method in connection with the vriational iteration method to study and solve random ordinary differential equations. The sequence of approximated closed form iterated solutions is derived based on the general Lagrange multiplier evaluated using the well-known convolution theorem of Laplace transformation method. Two illustrative examples are considered, linear and nonlinear random ordinary differential equations. The obtained results of the closed form solution have very height accuracy in comparison with the exact solution, if exist, or have a very height convergence between the iterated solutions.
Keywords: variational iteration method, Laplace transformation, Random ordinary differential equations.

1. Introduction

Random ordinary differential equations (RODE's) are considered to be an ordinary differential equation (ODE) that include a stochastic process in their vector field. It is notable that, there is a great amount of researchers and articals con-

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cerned with random differential equations that appeared in recent years, such as [1]-[5], [13]-[15], [18] and [19]. Also, since many random differential equations have no known analytical solution, so it is necessary to derive numerical methods to generate approximations to the exact solutions.

J. He in 1999 described and used a new kind of analytical technique for solving nonlinear problems called the variational iteration method (VIM) to give an approximate solution of the problem [6]-[9]. E. Hesameddini and H. Latifzadeh in 2009 [10] suggests an alternative approach to the derivation of the variational iteration formulations using the Laplace transform. S. A. Khuri and A. Sayfy in 2012 [16] introduced the Laplace variational numerical scheme based on the VIM and the Laplace transform for evaluation the approximate solution of certain classes of linear and nonlinear differential equations. Also G. C. Wu in 2012 [20] present a Laplace transform approach in the determination of the Lagrange multiplier when the VIM is applied to time fractional heat diffusion equation. G. C. Wu and D. Baleanu in 2013 [21] proposed a modification of the VIM by means of the Laplace transformation method to solve the fractional order differential equations. E. M. Hilal and T. M. Elzaki in 2014 [11] give in his study a good strategy for solving some linear and nonlinear partial differential equations in engineering and physics fields by combining Laplace transform and the modified VIM.

In this paper, we will combine between the Laplace transforms and the VIM for solving linear and nonlinear RODEs. This approach give closed form solution of the considered RODE, which are so difficult to solve analytically and also give a very high accurate results in comparison with other numerical or approximate methods for solving such type of equations.

2. Fundamental and basic concepts

In this section, some primitive and basic concepts that are necessary for this work will be introduced. These concepts include basic definitions of stochastic calculus and some basic properties of Laplace transform.

Definition 2.1 ([17]). *A probability space is a triplet* $(\Omega, \mathcal{F}, P)$*, where* Ω *is a sample space (set of all possible outcomes of random increment),* $\mathcal{F}$ *is class of all subset of* Ω *and* P *is a probability measure whose domain is* Ω *and counter domain is the interval [0,1].*

Definition 2.2 ([17]). *A random variable is a real valued function* $X(\omega), \omega \in \Omega$*, which is measurable with respect to the probability measure* P*.*

In the theory stochastic calculus and related fields, a stochastic or random process is a mathematical object which is usually defined as a family of random variables that depends also on the space and/or time variables. These processes are widely used as mathematical modeling of systems and phenomena's that appear to vary in a random manner.

Definition 2.3 ([17]). *A stochastic process is a family of random variables $X_t(\omega)$ (or briefly X_t) of two variables. Let $t \in [t_0, T] \subset [0, \infty), \omega \in \Omega$ on a common probability space $(\Omega, \mathcal{F}, P)$, which assumes real values and is P measurable as a function of ω for each fixed t. The parameter t is interpreted as time. $X_t(.)$ represents a random variable on the above probability space Ω, while $X.(\omega)$ is called a sample path or trajectory of the stochastic process.*

Among the different types of stochastic processes, which will be used in this paper, is the so called Wiener process or Brownian motion, as it is given in the next definition.

Definition 2.4 ([17]). *A stochastic process W_t, for all $t \in [0, \infty)$, is said to be a Wiener process or Brownian motion, if:*

1. $P(\omega \in \Omega \mid W_0(\omega) = 0) = 1$.
2. *For* $0 < t_0 < t_1 < \ldots < t_n$, *the increments* $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ *are independent.*
3. *For an arbitrary* t *and* $h > 0$, $W_{t+h} - W_t$ *has a Normal distribution with mean 0 and variance* h.

The Laplace transform is an integral transform that changes a real variable function $f(t)$ into a function $F(s)$, which is defined as follows [11]:

$$F(s) \equiv \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

where, in general, s is a real variable. Also, after a certain problem is transformed into an algebraic equation and then solved for $F(s)$, we need to transform $F(s)$ back into $f(t)$. The solution of the original problem which is called the inverse Laplace transform of $F(s)$ is denoted by $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

Among the most important properties of the Laplace transform, that will be used later with the VIM, is the convolution of two functions f_1 and f_2. Let the functions $f_1(t)$ and $f_2(t)$ be defined for all $t \geq 0$, then the convolution of the functions f_1 and f_2, denoted by $(f_1 * f_2)(t)$, and is defined by the integral, [12]:

$$(f_1 * f_2)(t) = \int_0^t f_1(t - \xi) f_2(\xi)\, d\xi.$$

Now, let $\mathscr{L}\{f_1(t)\} = F_1(s)$, $\mathscr{L}\{f_2(t)\} = F_2(s)$, then:

$$\mathscr{L}\left\{\int_0^t f_1(t - \xi) f_2(\xi)\, d\xi\right\} = F_1(s) F_2(s).$$

Conversely:

$$\mathscr{L}^{-1}\{F_1(s) F_2(s)\} = \int_0^t f_1(t - \xi) f_2(\xi)\, d\xi.$$

Also, the convolution is commutative, that is:

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\xi) f_2(\xi)\, d\xi = \int_0^t f_2(t-\xi) f_1(\xi)\, d\xi = (f_2 * f_1)(t).$$

In addition, the most two important properties of the Laplace transform for solving differential equations is that it transform the differential operator $\frac{df}{dt}$ into an algebraic operator $sF(s) - f(0)$.

3. Solution of RODEs using Laplace VIM

To illustrate the idea of the Laplace VIM for solving RODE's, the following general RODE will be considered in operator form:

$$L[x(t;\omega)] + N[x(t;\omega)] = g(t;\omega), t \in [t_0, T] \tag{1}$$

where L is a linear random ordinary differential operator given by $\frac{d^n}{dt^n}$, n is a natural number, N is a nonlinear operator and $g(t;\omega)$ is a known analytical function, which contains random variables.

According to the VIM, the following correction functional related to equation (1) may be constructed:

$$x_{m+1}(t;\omega) = x_m(t;\omega) + \int_{t_0}^t \lambda(\xi,t) \left\{L(x_m(\xi;\omega)) + N(\tilde{x}_m(\xi;\omega)) - g(\xi;\omega)\right\} d\xi, \tag{2}$$

$m = 0, 1, \ldots$, where λ is the general Lagrange multiplier which can be identified optimally via the variational theory, the subscript m denotes the m^{th} approximation and $N(\tilde{x}_m(\xi;\omega))$ is considered as a restricted variation, that is, $\delta\tilde{x}_m(\xi;\omega) = 0$.

To start the approach, take the Laplace transform of equation (1), yields to:

$$\mathscr{L}\{L[x(t;\omega)]\} + \mathscr{L}\{N[x(t;\omega)]\} = \mathscr{L}\{g(t;\omega)\} \tag{3}$$

By using the Laplace transform of the differential operator of the correction functional (2), one may get:

$$\begin{aligned}\mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} \\ &+ \mathscr{L}\left\{\int_{t_0}^t \lambda(\xi,t) \left\{L(x_m(\xi;\omega)) + N(\tilde{x}_m(\xi;\omega)) - g(\xi;\omega)\right\} d\xi\right\},\end{aligned} \tag{4}$$

$m = 0, 1, \ldots$ Therefore, upon using the convolution theorem with respect to t, we have:

$$\begin{aligned}\mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} + \mathscr{L}\{\bar{\lambda}(t-s) * [L(x_m(t;\omega)) \\ &\quad + N(\tilde{x}_m(t;\omega)) - g(t;\omega)]\} \\ &= \mathscr{L}\{x_m(t;\omega)\} + \mathscr{L}\{\bar{\lambda}(s;t)\}\mathscr{L}\{L(x_m(t;\omega)) \\ &\quad + N(\tilde{x}_m(t;\omega)) - g(t;\omega)\}\end{aligned} \tag{5}$$

then equation (5) becomes:

$$\begin{aligned}\mathscr{L}\{x_{m+1}(t;\omega)\} &= \mathscr{L}\{x_m(t;\omega)\} + \Lambda(s)\{s^n \mathscr{L}[x_m(t;\omega)] \\ &- s^{n-1}x_m(0;\omega) - \ldots - x_m^{(n-1)}(0;\omega)\} \\ &+ \mathscr{L}\left\{N(\tilde{x}_m(t;\omega)) - g(t;\omega)\right\},\end{aligned} \tag{6}$$

where $\Lambda(s) = \mathscr{L}\left\{\bar{\lambda}(s;t)\right\}$. The iteration formula of equation (6) may be used to suggest the main scheme involving the Lagrange multiplier.

Now, consider the term $\mathscr{L}\left\{N(\tilde{x}_m(t;\omega)) - g(t;\omega)\right\}$ in equation (6) as restricted variation, this make equation (6) stationary with respect to x_m, and hence:

$$\mathscr{L}\{\delta x_{m+1}(t;\omega)\} = \mathscr{L}\{\delta x_m(t;\omega)\} + \Lambda(s)s^n \mathscr{L}[\delta x_m(t;\omega)], \tag{7}$$

where δ is the classical first variation, the optimality condition for the extremum is $\mathscr{L}\{\delta x_{m+1}(t;\omega)\} = 0$ and hence equation (7) leads to:

$$\Lambda(s) = \frac{-1}{s^n},\ s > 0.$$

The sequential approximations are obtained by taking the inverse Laplace transform to equation (6) after substituting $\lambda(s)$ yields:

$$\begin{aligned}x_{m+1}(t;\omega) = x_m(t;\omega) - \mathscr{L}^{-1}\Big\{\frac{1}{s^n}[X_m(s;\omega) - s^{n-1}x(0;\omega) - \ldots - x^{(n-1)}(0;\omega) \\ +\mathscr{L}\left(Nx_m(t;\omega) - g(t;\omega)\right)]\Big\},\end{aligned} \tag{8}$$

where $X_m(s;\omega) = \mathscr{L}\left\{x_m(t;\omega)\right\}$. Rearrange equation (8) will implies to:

$$\begin{aligned}x_{m+1}(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} + \ldots + \frac{x^{(n-1)}(0;\omega)}{s^n}\right\} \\ &- \mathscr{L}^{-1}\left\{\frac{1}{s^n}\left(\mathscr{L}\left[Nx_m(t;\omega) - g(t;\omega)\right]\right)\right\}\end{aligned} \tag{9}$$

with initial approximation to equation (9), the following equation can be considered:

$$x_0(t;\omega) = \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} + \frac{x'(0;\omega)}{s^2} + \ldots + \frac{x^{(n-1)}(0;\omega)}{s^n}\right\}. \tag{10}$$

After applying the inverse Laplace transform to equation (10), we get:

$$x_0(t;\omega) = x(0;\omega) + x'(0;\omega)t + \ldots + \frac{x^{(n-1)}(0;\omega)}{(n-1)!}t^{n-1}. \tag{11}$$

Finally, the values of $x_1(t;\omega), x_2(t;\omega), \ldots, x_m(t;\omega)$ are obtained. Therefore, the solution of equation (1) will be $x(t;\omega) = \lim_{m\to\infty} x_m(t;\omega)$.

4. Illustrative examples

In this section, for illustration purpose, the Laplace VIM introduced will be used to solve linear and nonlinear RODE's, but before indulging with these examples, it is necessary to note that the generation of different discretized Brownian motion over the unit interval [0,1] will be considered. These generations are given with total number of generation $N = 100$, 500 and 1000.

Example 4.1. Consider the linear RODE:

$$\frac{dx(t;\omega)}{dt} = -x(t;\omega) + sinW_t(\omega), t \in [0,1],$$

$x_0(t;\omega) = x_0(0,\omega) = 1$ which has the exact solution, for comparison purpose, given by [5]:

$$x(t;\omega) = x_0 e^{-(t-t_0)} + e^{-t}\int_0^t e^s \sin W_s(\omega)\, ds.$$

We can obtain the successive approximate solutions as:

$$\begin{aligned}
x_0(t;\omega) &= 1,\\
x_1(t;\omega) &= x_0(0;\omega) - tx_0(t;\omega) + t\sin W_t(\omega),\\
x_2(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} - \frac{1}{s}\mathscr{L}\left[1 - t + t\sin W_t(\omega) - \sin W_t(\omega)\right]\right\}\\
&= 1 - t + \frac{t^2}{2} - \frac{t^2}{2}\sin W_t(\omega) + t\sin W_t(\omega)\\
&= 1 - t(1-\sin W_t(\omega)) + \frac{t^2}{2}(1-\sin W_t(\omega))\\
x_3(t;\omega) &= \mathscr{L}^{-1}\left\{\frac{x(0;\omega)}{s} - \frac{1}{s}\mathscr{L}\left[1 - t + \frac{t^2}{2} - \frac{t^2}{2}\sin W_t(\omega)\right.\right.\\
&\quad \left.\left. + t\sin W_t(\omega) - \sin W_t(\omega)\right]\right\}\\
&= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^3}{6}\sin W_t(\omega) - \frac{t^2}{2}\sin W_t(\omega) + t\sin W_t(\omega)\\
&= 1 - t(1-\sin W_t(\omega)) + \frac{t^2}{2}(1-\sin W_t(\omega)) - \frac{t^3}{6}(1-\sin W_t(\omega)),\\
x_4(t;\omega) &= 1 - (1-\sin W_t(\omega)) + \frac{t^2}{2}(1-\sin W_t(\omega)) - \frac{t^3}{6}(1-\sin W_t(\omega))\\
&\quad + \frac{t^4}{24}(1-\sin W_t(\omega)),\\
x_5(t;\omega) &= 1 - (1-\sin W_t(\omega)) + \frac{t^2}{2}(1-\sin W_t(\omega)) - \frac{t^3}{6}(1-\sin W_t(\omega))\\
&\quad + \frac{t^4}{24}(1-\sin W_t(\omega)) - \frac{t^5}{120}(1-\sin W_t(\omega))
\end{aligned}$$

and so on by induction:

$$x_m(t;\omega) = 1 + \sum_{i=1}^{m} \frac{(-t)^i}{i!}(1-\sin W_t(\omega)), \quad m \in N. \tag{12}$$

The signal simulation of the generated discretized Brownian motion with total number of generations $N = 100$, 500 and 1000 over the unit interval [0,1] and the corresponding approximate solutions (12) for each generation are given in Figures (1)-(6).

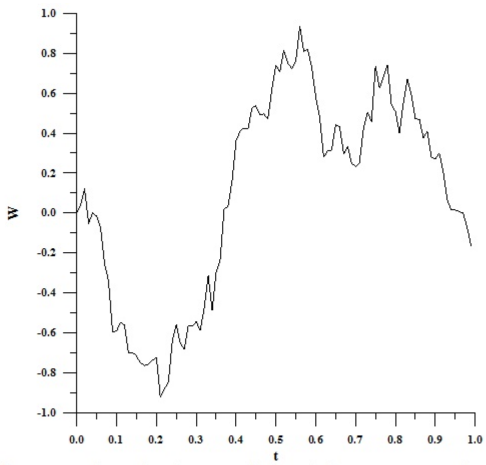

Figure 1: Discretized Brownian path for 100 generation.

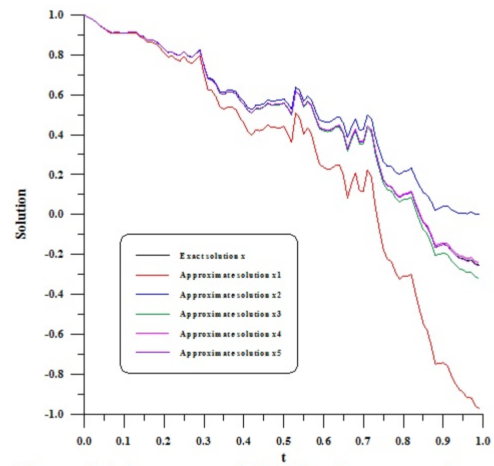

Figure 2: The exact and the first five approximate solutions of example 4.1 with 100 generation of Brownian motion.

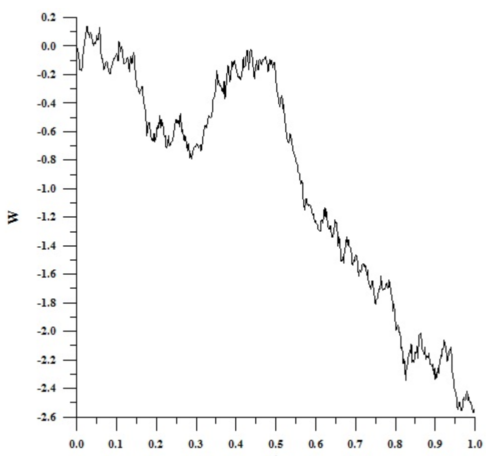

Figure 3: Discretized Brownian path for 500 generation.

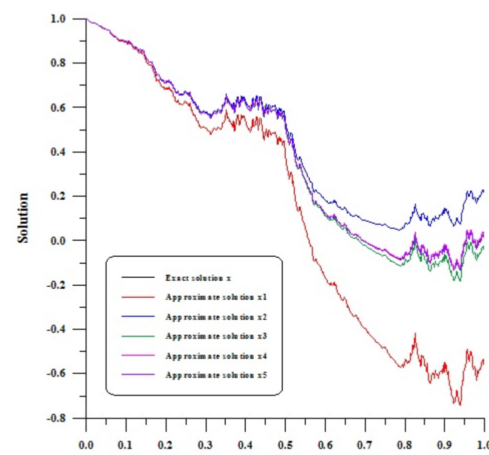

Figure 4: The exact and the first five approximate solutions of example 4.1 with 500 generation of Brownian motion.

Example 4.2. Consider the nonlinear RODE:

$$\frac{d^2x(t;\omega)}{dt^2} + W_t^2(\omega)x(t;\omega) = 0, t \in [0,1], \quad x(0,\omega) = 1, \frac{dx(0;\omega)}{dt} = 0.$$

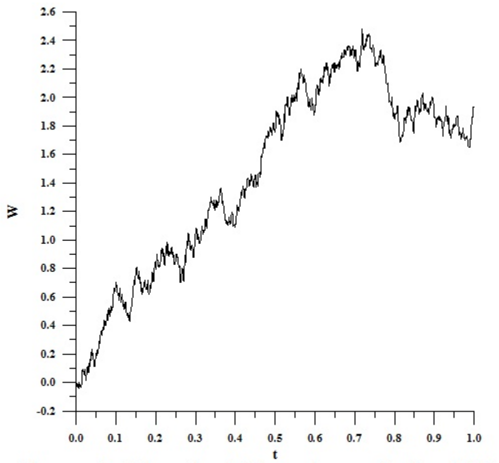

Figure 5: Discretized Brownian path for 1000 generation.

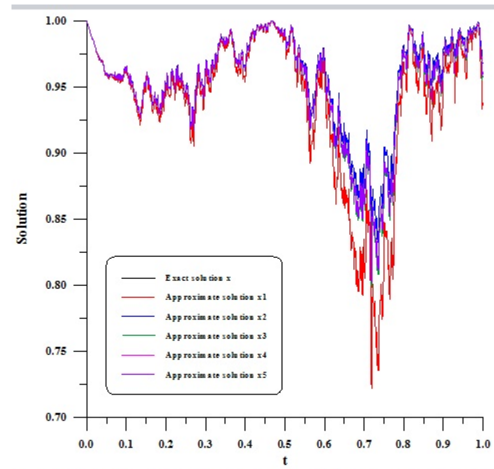

Figure 6: The exact and the first five approximate solutions of example 4.1 with 1000 generation of Brownian motion.

Starting with the initial guess solution $x_0(t;\omega) = 1$, we can find the successive approximate solutions using the proposed method in this paper, to get:

$$
\begin{aligned}
x_1(t;\omega) &= 1 - \frac{t^2}{2} W_t^2(\omega), \\
x_2(t;\omega) &= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)\left(1 - \frac{t^2}{2}W_t^2(\omega)\right)\right]\right\} \\
&= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega), \\
x_3(t;\omega) &= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}x'(0;\omega) + \frac{1}{s^2}x(0;\omega) + \frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)x_2(t;\omega)\right]\right\} \\
&= 1 - \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[W_t^2(\omega)\left(1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega)\right)\right]\right\} \\
&= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega) - \frac{t^6}{6!}W_t^6(\omega) \\
x_4(t;\omega) &= 1 - \frac{t^2}{2}W_t^2(\omega) + \frac{t^4}{24}W_t^4(\omega) - \frac{t^6}{6!}W_t^6(\omega) + \frac{t^8}{8!}W_t^8(\omega)
\end{aligned}
$$

and so on upon using mathematical induction, the general solution may be given as:

$$
x_m(t;\omega) = 1 + \sum_{i=1}^{m} \frac{(-1)^i t^{2i}}{(2i)!} W_t^{2i}(\omega). \tag{13}
$$

Similarly, as in example 1, the signal simulation of the generated discretized Brownian motion with total number of generations $N = 100$, 500 and 1000 over the unit interval [0,1] and the corresponding four approximate solutions given by (13) for each generation are given in Figures (7)-(12).

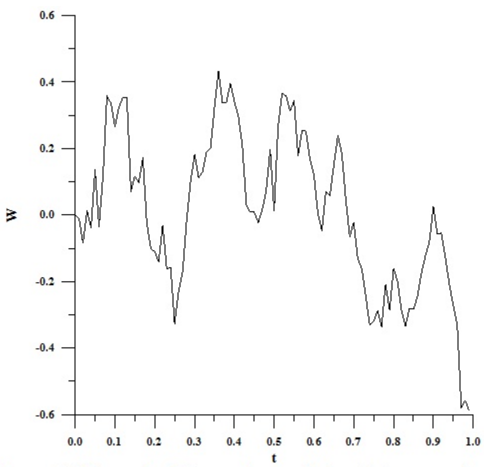

Figure 7: Discretized Brownian path for 100 generation.

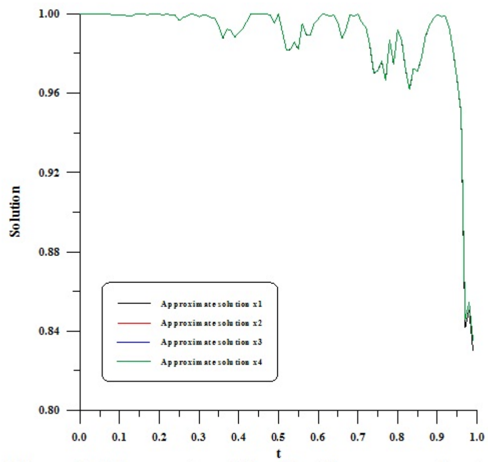

Figure 8: The exact and the first four approximate solutions of example 2 with 100 generation of Brownian motion.

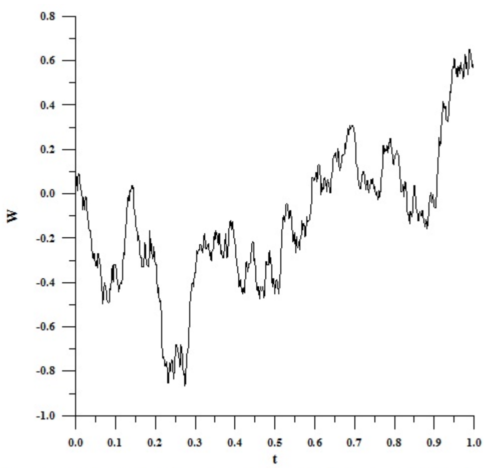

Figure 9: Discretized Brownian path for 500 generation.

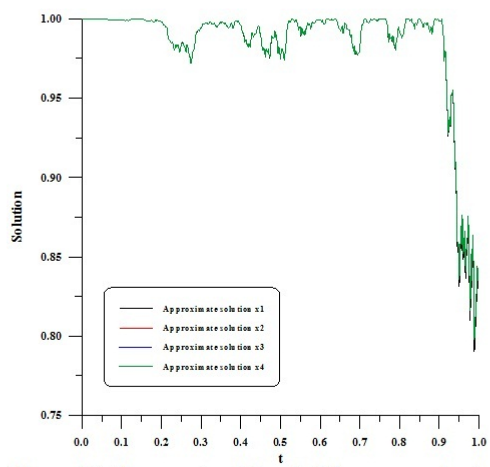

Figure 10: The exact and the first four approximate solutions of example 4.2 with 500 generation of Brownian motion.

5. Conclusions

The followed method proposed in this paper consists of a combination between the Laplace transformation method and VIM to solve linear and nonlinear RODEs, which give a closed form approximate solutions, while the VIM that gives only an approximate solutions that may be complicated in evaluation. This implies, more accurate and reliable results may be obtained for any number of iterations. Also, it is notable that the solution of the RODE's varies depending on varient number of generations of the Brownian motion and the signal of the Brownian path used in the simulation of the closed form solution.

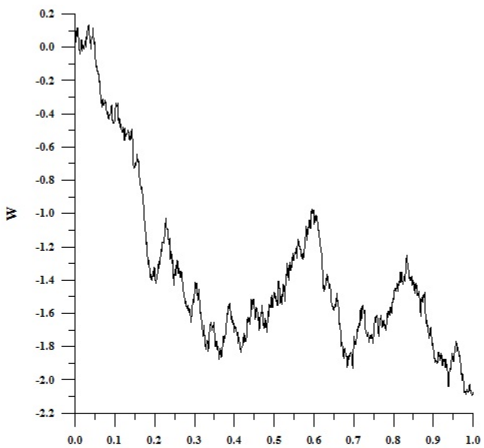

Figure 11: Discretized Brownian path for 1000 generation.

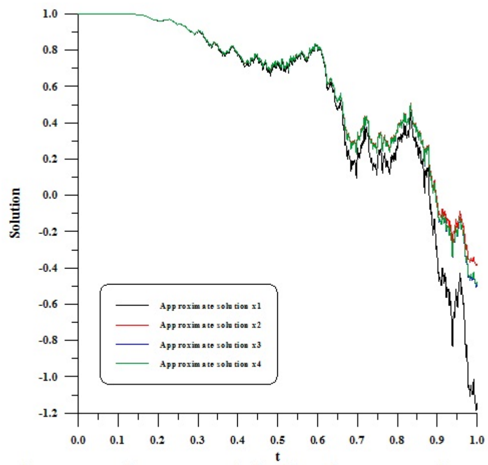

Figure 12: The exact and the first four approximate solutions of example 4.2 with 1000 generation of Brownian motion.

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