On condition \((P'_E)\)

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**Abstract.** In (Semigroup and Applications: Proceedings of Conference, world scientific, (1998) 72-77) Golchin and Renshaw introduced Condition \((P_E)\) and showed that this condition implies weak flatness, but not the converse. In this paper, we introduce a generalization of Conditions \((P_E)\), called Condition \((P'_E)\), and will show that this condition implies principal weak flatness. Also we give a characterization of monoids by this condition over their right acts.

**Keywords:** condition \((P'_E)\), condition \((W(WF)'\)), \(P'_E\)-left annihilating, weakly right reversible.

1. **Introduction**

Laan [12] introduced Condition \((E')\), a generalization of Condition \((E)\). Golchin and Renshaw [3] introduced a generalization of Condition \((P)\) called Condition \((P_E)\). After that Golchin and Mohammadzadeh [7] provided a characterization of monoids by this Condition of their right acts. They also introduced another generalization of Condition \((P)\) called Condition \((P')\), and gave a characterization of monoids by this property of their right acts [9]. In this paper, we introduce a generalization of Conditions \((P_E)\) and \((P')\), called Condition \((P'_E)\) and will give a characterization of monoids by this condition of their right acts.

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Throughout this paper $S$ will denote a monoid. We refer the reader to [11] for basic definitions and terminologies relating to semigroups and acts over monoids, one can see and [2] for definitions and results on flatness properties which are used here.

A nonempty set $A$ is called a right $S$-act, usually denoted $A_S$, if $S$ acts on $A$ unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$, and all $s, t \in S$. A right $S$-act $A_S$ satisfies Condition (E) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = vt$. $A_S$ satisfies Condition (E') if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = vt$. A right $S$-act $A_S$ satisfies Condition (PWP) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

A right $S$-act $A_S$ satisfies Condition (P) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. $A_S$ satisfies Condition (P') if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right $S$-act $A_S$ satisfies Condition (PE) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$, implies that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vs$. It is clear that Condition (P) implies Condition (PE).

2. General properties

Definition 2.1. A right $S$-act $A_S$ satisfies Condition (PE') if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$, imply that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vs$.

It is obvious that Conditions (PE) and (P') imply Condition (PE'). Examples 2.3 and 3.9 show that these implications are strict.

Definition 2.2 ([11]). A right $S$-act $A_S$ satisfies Condition (WFV), if $as = a't$ and $sz = tz$, for $a, a' \in A_S, s, t, z \in S$, imply that there exist $a'' \in A_S, w \in Ss \cap St$, such that $as = a't = a''w$.

Theorem 2.1. For any right $S$-act $A_S$ we have the following strict implications:

(1) Condition (PE') $\Rightarrow$ principal weak flatness;

(2) Condition (PE') $\Rightarrow$ Condition (WFV).

Proof. (1) Suppose that $A_S$ satisfies Condition (PE') and let $as = a's$, for $a, a' \in A_S, s \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vs$. Then we have

$$a \otimes s = a \otimes es = ae \otimes s = a''ue \otimes s = a'' \otimes ues = a'' \otimes us$$
$$= a' \otimes vs = a'' \otimes vfs = a''vf \otimes s = a'f \otimes s = a' \otimes fs = a' \otimes s$$
in $A_S \otimes SS$. That is, $A_S$ is principally weakly flat as required.

(2) Suppose that $A_S$ satisfies Condition $(P'_E)$, and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s, ft = t$ and $us = vt$. If $w = us = vt$ then $w \in Ss \cap St$, and $as = aes = a''ues = a''us = a''w$. Similarly, $a't = a''w$ and so $A_S$ satisfies Condition $(W_{(WF)'})$ as required. □

Example 3.1 shows that these implications are strict.

**Lemma 2.1.** Let $S$ be a monoid. Then:

1. if $\{ A_i \mid i \in I \}$ is a chain of subacts of an act $A_S$ and every $A_i, i \in I$ satisfies Condition $(P'_E)$, then $\bigcup_{i \in I} A_i$ satisfies Condition $(P'_E)$;
2. $A_S = \prod_{i \in I} A_i$ satisfies Condition $(P'_E)$ if and only if every $A_i, i \in I$ satisfies Condition $(P'_E)$;
3. the right $S$-act $S_S$ satisfies Condition $(P'_E)$.

**Proof.** It is clear from definition. □

**Theorem 2.2.** Any retract of a right $S$-act satisfying Condition $(P'_E)$ satisfies Condition $(P'_E)$.

**Proof.** Let $A_S$ be a retract of $B_S$ and suppose $B_S$ satisfies Condition $(P'_E)$. Let $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since $A_S$ is a retract of $B_S$, there are homomorphisms $\varphi : A_S \rightarrow B_S$ and $\varphi : B_S \rightarrow A_S$, such that $\varphi \varphi = 1_{A_S}$. Then we have $\varphi(as) = \varphi(a't)$ or $\varphi(a)s = \varphi(a)t$. Since $\varphi(a), \varphi(a') \in B_S$, by assumption there exist $b \in B_S, u, v \in S, e, f \in E(S)$ such that $\varphi(a)e = bue$, $\varphi(a')f = bf$, $es = s, ft = t$ and $us = vt$. Then $\varphi'((\varphi(a)e) = \varphi'(bue) = \varphi(b)ue$, and $\varphi'(\varphi(a')f) = \varphi'(bf) = \varphi(b)vf$. If $\varphi'(b) = a''$, then $ae = a''ue, a'f = a''vf$. It is obvious that $a'' \in A_S$ and so $A_S$ satisfies Condition $(P'_E)$ as required. □

**Definition 2.3** ([9]). Let $S$ be a monoid and let $P \subseteq S$ be a submonoid of $S$. $P$ is called weakly right reversible if

$$\forall s, t \in P)(\forall z \in S)(sz = tz \Rightarrow (\exists u, v \in P)(us = vt)).$$

**Proposition 2.1.** For any monoid $S$ the following statements are equivalent:

1. if $A_S = \prod_{i \in I} A_i$ satisfies Condition $(P'_E)$, then every $A_i, i \in I$ satisfies Condition $(P'_E)$;
2. the one-element right $S$-act $\Theta_S$ satisfies Condition $(P'_E)$;
3. the one-element right $S$-act $\Theta_S$ satisfies Condition $(W_{(WF)'})$;
4. $S$ is weakly right reversible;
(5) there exists a right \( S \)-act which contains a zero and satisfies Condition \((P'_{E})\).

**Proof.** Since \( \Theta_S \) is a retract of any right \( S \)-act containing a zero, (2) and (5) are equivalent by Theorem 2.2.

(1) \( \Rightarrow \) (2). Since \( S_S \cong S_S \times \Theta_S \), it follows from Lemma 2.1(3).

(2) \( \Rightarrow \) (3). It follows from Theorem 2.1(2).

(3) \( \Rightarrow \) (4). Let \( sz = tz \) for \( s, t, z \in S \). Since \( \theta s = \theta t \), and the right \( S \)-act \( \Theta_S \) satisfies Condition \((W'_F)\), there exist \( u, v \in S \) such that \( us = vt \), that is, \( S \) is weakly right reversible.

(4) \( \Rightarrow \) (1). Suppose that \( A_S = \prod_{i \in I} A_i \) satisfies Condition \((P'_{E})\), and let \( a_is = a'_it, sz = tz \), for \( a_i, a'_i \in A_i, s, t, z \in S \). Since \( S \) is weakly right reversible, there exist \( u_1, v_1 \in S \) such that \( u_is = v_it \). For every \( j \in I \setminus \{i\} \), let \( a_j \) be a fix element of \( A_j \). Define

\[
    c_j = \begin{cases} 
        a_ju_1, & j \neq i \\
        a_i, & j = i 
    \end{cases}, \quad d_j = \begin{cases} 
        a_jv_1, & j \neq i \\
        a'_i, & j = i 
    \end{cases}.
\]

Thus \( (c_j)_is = (d_j)_it \), and so by assumption there exist \( (a''_{ji}) \in \prod_{i \in I} A_i \), \( u, v \in S, e, f \in E(S) \) such that \( (c_j)_ie = (a''_{ji})ue, (d_j)_if = (a''_{ji})vf, es = s, ft = t \), and \( us = vt \). Hence \( a_ise = a''_{ji}ue, a'_if = a''_{ji}vf \), and so \( A_i \) satisfies Condition \((P'_{E})\). □

**Corollary 2.1.** Let \( S \) be a commutative monoid and \( \{A_i \mid i \in I\} \) be a family of right \( S \)-acts. If \( A_S = \prod_{i \in I} A_i \) satisfies Condition \((P'_{E})\), then every \( A_i \), \( i \in I \) satisfies Condition \((P'_{E})\).

**Example 2.3.** Let \( S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \right\} \mid a, b \in \mathbb{Z}, a \neq 0 \right\} \) with matrix product as operation, then \( S \) is a right cancellative monoid, and so it is weakly right reversible, but \( S \) is not right reversible, since for every \( a, b, c, d \in \mathbb{Z} \) with \( a, c \neq 0 \),

\[
    \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Now the one-element right \( S \)-act \( \Theta_S \) satisfies Condition \((P'_{E})\) by Theorem 2.1, but it does not satisfy Condition \((P_{E})\), since otherwise it is weakly flat which is not the case.

### 3. Homological classifications

In this section we will classify monoids by Condition \((P'_{E})\) of their (cyclic, Rees factor) acts.

Let \( K \) be a proper right ideal of a monoid \( S \). If \( x, y \) and \( z \) denote elements not belonging to \( S \), define

\[
    A(K) = \left( \{x, y \times (S \setminus K) \} \cup \{z \times K \} \right)
\]
and define a right $S$-action on $A(K)$ by

$$(x, u)s = \begin{cases} (x, us), & us \notin K \\ (z, us), & us \in K \end{cases}$$

$$(y, u)s = \begin{cases} (y, us), & us \notin K \\ (z, us), & us \in K \end{cases}$$

$$(z, u)s = (z, us).$$

$A(K)$ is a right $S$-act, which called the amalgamated coproduct of $S_S$ and $S_S$ by the core $K_S$, and usually denoted by $S_S \coprod^{K_S} S_S$.

Recall from [11] that a right ideal $K_S$ of a monoid $S$ is called left stabilizing if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$. The following example shows that implications in Theorem 2.1 are strict.

**Example 3.1.** Let $S = \{0, 1, e, a\}$ be a monoid with the identity element 1 and zero element 0, and $e^2 = e, ae = a, a^2 = ea = 0$. Let $K = \{0, e\}$. Now the right $S$-act $A(K) = S_S \coprod^{K_S} S_S$ does not satisfy Condition $(P_E')$, since the equality $(x, a)a = (y, a)a$ implies $x = y$ which is a contradiction. It is a routine matter to verify that $A_S$ satisfies Condition $(W_{(W_E'})$. Since $K_S$ is a left stabilizing right ideal, $A_S$ is principally weakly flat by ([11], III, 12.19).

**Lemma 3.1.** Let $K_S$ be a proper right ideal of a monoid $S$. If the right $S$-act $S_S \coprod^{K_S} S_S$ satisfies Condition $(P_E')$, then $K_S$ is left stabilizing.

**Proof.** Suppose that the right $S$-act $S_S \coprod^{K_S} S_S$ satisfies Condition $(P_E')$ and let $k \in K_S$. Since $(x, 1)k = (y, 1)k$, there exist $w \in \{x, y, z\}, r, u, v \in S$, and $e, f \in E(S)$ such that $(x, 1)e = (w, r)ue$, $(y, 1)f = (w, r)vf, ek = fk = k$ and $uk = vk$. If $e, f \notin K_S$, then $(x, e) = (w, rue)$ and $(y, f) = (w, ruf)$, which implies that $w = x = y$, a contradiction. Thus at least one of $e$ or $f$ belongs to $K_S$, and so the equality $ek = fk = k$ shows that $K_S$ is left stabilizing. \[\square\]

We recall from [11] that a monoid $S$ is called left $PP$ if every principal left ideal of $S$ (as a left $S$-act) is projective, it is equivalent to say that for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$.

**Theorem 3.2 ([3]).** Let $S$ be a left $PP$ monoid and $A_S$ be a right $S$-act. Then $A_S$ is weakly flat if and only if it satisfies Condition $(P_E)$.

**Theorem 3.3.** For any monoid $S$ the following statements are equivalent:

1. all right $S$-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition $(P_E)$;
2. all right $S$-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition $(P_E)$;
3. $S$ is a regular monoid.
Proof. The implication (1) ⇒ (2) is obvious.

(2) ⇒ (3). Let \( s \in S \). If \( sS = S \) then \( s \) is obviously regular. Assume that \( sS \neq S \), then by assumption the right \( S \)-act \( sS \{ S \} \) satisfies Condition \((P'_k)\) and so by Lemma 3.1, \( sS \) is left stabilizing. Thus there exists \( l \in sS \) such that \( ls = s \), that is \( s \) is regular as required.

(3) ⇒ (1). Let \( K_S \) be a proper right ideal of \( S \) and \( k \in K_S \). By assumption there exist \( k' \in S \) such that \( k = kk'k \) and so \( K_S \) is left stabilizing. Thus \( S \{ K_S \} \{ S \} \) is weakly flat by ([11], III, 12.19). Since \( S \) is regular, it is left PP and so by Theorem 3.2, \( S \{ K_S \} \{ S \} \) satisfies Condition \((P_E)\).

Theorem 3.4. Let \( S \) be a monoid and \( \rho \) be a right congruence on \( S \). Then the right \( S \)-act \( S/\rho \) satisfies Condition \((P'_E)\) if and only if, for all \( x, y, z, s, t \in S \) with \((xs)\rho(zt)\) and \( sz = tz \), there exist \( u, v, e, f \in E(S) \), such that \((xe)\rho(ue)\), \((yf)\rho(vf)\) and \( uS = vt \).

Proof. Necessity. Let \((xs)\rho(zt)\) and \( sz = tz \), for \( x, y, z, s, t \in S \). Then \([x]_\rho s = [y]_\rho t\), and so Condition \((P'_E)\) of \( S/\rho \) implies that there exist \( u', v', z' \in S \), \( e, f \in E(S) \), such that \([x]_\rho e = [z']_\rho u'e = [z'u']_\rho e\), \([y]_\rho f = [z']_\rho v'f = [z'v']_\rho f\), \( es = s, ft = t \) and \( u's = v't \). If \( u = z'u' \) and \( v = z'v' \), then \((xe)\rho(ue), (yf)\rho(vf)\) and \( uS = vt \), as required.

Sufficiency. Let \([x]_\rho s = [y]_\rho t\) and \( sz = tz \), for \( x, y, z, s, t \in S \). Then \((xs)\rho(zt)\), and so by assumption there exist \( u, v \in S \), \( e, f \in E(S) \), such that \((xe)\rho(ue)\), \((yf)\rho(vf)\), \( es = s, ft = t \) and \( uS = vt \). Then \([x]_\rho e = [1]_\rho ue\) and \([y]_\rho f = [1]_\rho vf\), and so \( S/\rho \) satisfies Condition \((P'_E)\) as required.

Corollary 3.1. Let \( S \) be a monoid and \( u \in S \). Then the principal right ideal \( uS \) satisfies Condition \((P'_E)\) if and only if, for all \( x, y, z, s, t \in S \), \( uxs = uyt \) and \( sz = tz \) imply the existence of \( v, r \in S \), \( e, f \in E(S) \), such that \( uxe = uve, uf = urf, es = s, ft = t, \) and \( vs = rt \).

Proof. Since \( uS \cong S/\ker \lambda_u \), it suffices to take \( \rho = \ker \lambda_u \).

Theorem 3.5 ([9]). Let \( S \) be a monoid and \( \rho \) be a right congruence on \( S \). Then the right \( S \)-act \( S/\rho \) satisfies Condition \((P')\) if and only if, for all \( x, y, z, s, t \in S \) with \((xs)\rho(zt)\) and \( sz = tz \), there exist \( u, v \in S \), such that \( xpu, ypv \) and \( us = vt \).

Proof. It is clear that Condition \((P')\) implies Condition \((P'_E)\). Now suppose that \( S/\rho \) satisfies Condition \((P'_E)\) and let \((xs)\rho(zt)\), \( sz = tz \), for \( x, y, z, s, t \in S \). By assumption there exist \( u, v \in S \), \( e, f \in E(S) \) such that \((xe)\rho(ue)\), \((yf)\rho(vf)\), \( es = s, ft = t \) and \( us = vt \). If \( e = f = 1 \) then the result follows from Theorem 3.5. If \( e = 1, f \neq 1 \), then \( f \) and so \( t \) are right zeros, thus \( xpu, ypv \) and \( us = vt \).
If $e \neq 1, f = 1$, a similar argument as the previous case shows the result. If $e \neq 1, f \neq 1$ then $e$ and $f$ are both right zero and so are $s$ and $t$. Then $x \rho x, y \rho y$ and $xs = yt$, thus $S/\rho$ satisfies Condition $(P')$ by Theorem 3.5.

**Definition 3.1.** Let $S$ be a monoid. A right ideal $K_S$ of $S$ is called $P'_E$-left annihilating if for all $x, y, z, s, t \in S$,

\[
[(xs \neq yt) \land (sz = tz)] \Rightarrow [(xs \notin K_S) \lor (yt \notin K_S)] \lor,
\]

\[
(\exists u, v \in S, e, f \in E(S) : es = s, ft = t, us = vt,
(xe \neq ue \Rightarrow xe, ue \in K_S), (yf \neq vf \Rightarrow yf, vf \in K_S)]
\]

**Lemma 3.2.** Let $S$ be a monoid and $K_S$ be a right ideal of $S$. If $K_S$ is $P'_E$-left annihilating, then $K_S$ is left stabilizing.

**Proof.** If $K_S = S$, then obviously $K_S$ is left stabilizing. Thus we assume that $K_S$ is a proper $P'_E$-left annihilating right ideal and let $k \in K_S$. For every $l \in K_S$ either $lk = k$, and so we are done, or $lk \neq k$. Since $k, lk \in K_S$ by assumption there exist $u, v \in S, e, f \in E(S)$ such that $ek =fk = k, uk = vk$, and $le \neq ue$ implies that $le, ue \in K_S$, and $f \neq vf$ implies that $f, vf \in K_S$. If $f \neq vf$ then $f \in K_S$ and $fk = k$ implies that $K_S$ is left stabilizing. If $f = vf$, then $le \neq ue$, otherwise

\[
lk = lek = uek = uk = vk = vf k = fk = k,
\]

a contradiction. So $ue \in K_S$ and

\[
k =fk =vf k = vk = uk = uek,
\]

thus $K_S$ is left stabilizing as required. \[\square\]

**Theorem 3.7.** Let $S$ be a monoid and $K_S$ be a right ideal of $S$. Then $S/K_S$ satisfies Condition $(P'_E)$, if and only if $S$ is weakly right reversible and $K_S$ is $P'_E$-left annihilating.

**Proof.** Necessity. Suppose that $S/K_S$ satisfies Condition $(P'_E)$, for the right ideal $K_S$ of $S$, and let $sz = tz$, for $s, t, z \in S$. Let $k \in K_S$, since $[k]_{\rho K} = [k]_{\rho K}, t$, Condition $\langle W_{(W_F)} \rangle$, implies that there exist $u, v \in S$ such that $us = vt$, thus $S$ is weakly right reversible. To show that $K_S$ is $P'_E$-left annihilating, suppose that $xs \neq yt, sz = tz$, for $x, y, z, s, t \in S$. If $xs \notin K_S$ or $yt \notin K_S$ then the result follows, otherwise $xs, yt \in K_S$ and so $(xs)_{\rho K} (yt)$. Thus by Theorem 3.4 there exist $u, v \in S, e, f \in E(S)$ such that $(xe)_{\rho K} (ue), (yf)_{\rho K} (vf), es = s, ft = t$ and $us = vt$. Since $(xe)_{\rho K} (ue)$, it follows that $xe = ue$ or $xe, ue \in K_S$. A similar argument shows that $yf = vf$ or $yf, vf \in K_S$.

Sufficiency. Suppose that $S$ is a weakly right reversible monoid and $K_S$ is a $P'_E$-left annihilating right ideal of $S$. Then there are two cases as follow:

**Case 1.** $K_S = S$. Since $S$ is weakly right reversible, $S/K_S \cong \Theta_S$ satisfies Condition $(P'_E)$ by Theorem 2.1.
Case 2. \( K_S \neq S \). Suppose that \((xs)\rho_{K_S}(yt)\) and let \( sz = tz \), for \( x, y, z, s, t \in S \). Then there are two possibilities that can arise:

1. \( xs = yt \). If \( u = x, v = y \) and \( e = f = 1 \), then by Theorem 3.4, \( S/K_S \) satisfies Condition \((P'_E)\).

2. \( xs \neq yt \). Then \( xs, yt \in K_S \). Since \( K_S \) is \( P'_E \)-left annihilating, there exist \( u, v \in S, e, f \in E(S) \) such that \( es = s, ft = t, us = vt \), if \( xe \neq ue \) then \( xe, ue \in K_S \), and if \( yf \neq vf \) then \( yf, vf \in K_S \). If \( xe \neq ue \), then by assumption \((xe)\rho_{K_S}(ue)\), otherwise \( xe = ue \), and again \((xe)\rho_{K_S}(ue)\). A similar argument shows that \((yf)\rho_{K_S}(vf)\), and so by Theorem 3.4, \( S/K_S \) satisfies Condition \((P'_E)\).

\[
[(xs \neq yt) \land (sz = tz)] \Rightarrow [(xs \notin K_S) \lor (yt \notin K_S) \lor (x \in K_S) \lor (y \in K_S)]
\]

**Theorem 3.8** ([9]). Let \( S \) be a monoid and \( K_S \) be a right ideal of \( S \). \( K_S \) is called completely left annihilating if for all \( x, y, z, s, t \in S \),

\[
[(xs \neq yt) \land (sz = tz)] \Rightarrow [(xs \notin K_S) \lor (yt \notin K_S) \lor (x \in K_S) \lor (y \in K_S)]
\]

The following example shows that Condition \((P'_E)\) does not imply Condition \((P')\).

**Example 3.9.** Let \( S \) and \( K_S \) be as in Example 3.1. Then \( 0 = aa \neq 1e = e \) and \( 0, e \in K_S, a, 1 \notin K_S \), and so \( K_S \) is not completely left annihilating, thus \( S/K_S \) does not satisfy Condition \((P')\) by Theorem 3.8. It can be seen that \( K_S \) is \( P'_E \)-left annihilating and \( S \) is weakly right reversible, and so \( S/K_S \) satisfies Condition \((P'_E)\).

**Definition 3.3** ([1]). A monoid \( S \) satisfies Condition \((R_{(WF')})\), if for all \( x, y, z, s, t \in S \), \( sz = tz \) implies the existence of \( w \in Ss \cap St \) such that \( wp(xs, yt)xs \).

**Theorem 3.10.** For any monoid \( S \) the following statements are equivalent:

1. all right \( S \)-acts satisfy Condition \((P'_E)\);
2. all finitely generated right \( S \)-acts satisfy Condition \((P'_E)\);
3. all cyclic right \( S \)-acts satisfy Condition \((P'_E)\);
4. all monocyclic right \( S \)-acts satisfy Condition \((P'_E)\);
5. \( S \) is regular and satisfies Condition \((R_{(WF')})\).
Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). Since by assumption all monocyclic right $S$-acts are principally weakly flat, $S$ is regular by ([11], IV, 6.6). Now let $sz = tz$, and suppose that $\rho = \rho(xs, yt), x, y, z, s, t \in S$. By assumption $S/\rho$ satisfies Condition $(P_E)$. Since $(xs)\rho(yt)$, by Theorem 3.4, there exist $u, v \in S(e, f \in E(S)$, such that $(xe)\rho(ue), (yf)\rho(vf), es = s, ft = t$ and $us = vt$. Then $xs = (xes)\rho(ues) = us$. Now $w = us = vt \in Ss \cap St$, and $w\rho(xs)$ as required.

(5) ⇒ (3). Let $\rho$ be a right congruence on $S$, and suppose that $(xs)\rho(yt), sz = tz, x, y, z, s, t \in S$. Since $S$ satisfies Condition $(R_{\langle WF \rangle})$, there exist $u, v \in S$ such that $w = us = vt, xsp(xs, yt)us$ and $ytp(xs, yt)vt$. Since $S$ is regular there exist $s', t' \in S$ such that $s = ss's, t = tt't$ and so $(xs's)\rho(xs, yt)(uss'), (ytt')\rho(xs, yt)(vtt').$ If $e = ss', f = tt'$, then $xe \rho(xs, yt)ue, yf \rho(xs, yt)vf, es = s$ and $ft = t$. But $\rho(xs, yt) \subseteq \rho$, and so $S/\rho$ satisfies Condition $(P_E)$ by Theorem 3.4.

(3) ⇒ (1). Let $A_S$ be a right $S$-act and $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since (3) ⇔ (5), $S$ is regular and so there exist $s', t'$ such that $s = ss's$ and $t = tt't$. Then $a(ss's) = a'(tt't)$, and so $(bs')s = (bt')t$, where $b = as = a't$. By assumption $bs$ satisfies Condition $(P_E)$, and so there exist $b'' \in bS \subseteq A_S, u, v \in S, e', f' \in E(S)$ such that

$$bs'e' = b''ue', bt'f' = b''vf', e's = s, f't = t, us = vt.$$  

If $e = ss', f = tt'$, then

$$ae = ass' = a(ss's)s' = (ass'e')ss' = (bs'e')ss' = (b''ue')ss' = b''uss' = b''ue.$$  

Similarly, $a'f = b''vf$. It is clear that $es = s$ and $ft = t$ and so $A_S$ satisfies Condition $(P_E)$ as required.  

Definition 3.4 ([11]). Let $L$ denote the set of all left ideals of a monoid $S$. $S$ is called right $L$-reductive if for all $S\mathbf{K} \subseteq L$ and for all $a, b \in S\mathbf{K}, a \neq b$, there exists $x \in S\mathbf{K}$ such that $ax \neq bx$.

Theorem 3.11. Let $S$ be a monoid. If all right $S$-acts satisfy Condition $(P_E)$, then $S$ is right $L$-reductive.

Proof. Suppose that $S$ is not $L$-reductive, thus there exist a left ideal $S\mathbf{K}$ and $a, b \in S\mathbf{K}$ such that $a \neq b$ and for all $x \in S\mathbf{K}, ax = bx$. By Theorem 3.10, $S$ satisfies Condition $(R_{\langle WF \rangle})$, so there exists $w \in Sa \cap Sb$ such that $w\rho(a, b)a$. Thus by ([11], I, 4.37) either $w = a$ or there exist $u_1, ..., u_n, a_1, b_1, ..., a_n, b_n \in S$, such that $\{a_i, b_i\} = \{a, b\}$ for $i = 1, ..., n$, and

$$a = a_1u_1, b_2u_2 = a_3u_3, ..., b_nu_n = w.$$  

$$b_1u_1 = a_2u_2, b_3u_3 = a_4u_4, ...$$
Since $S$ is regular by Theorem 3.10, there exist $a', b' \in S$ such that $a = aa'a, b = bb'b$. Now $a \in sK$, implies that $u_1a' \in sK$. Then

$$a = aa'a = a_1u_1a' = b_1u_1a' = a_2u_2a' = \ldots = b_nu_n\lambda = wa' = w.$$ 

Thus in both cases $a = w$, also $b \in sK$ implies that $b'b \in sK$, and so

$$b = bb'b = ab'b = wb'b = w.$$ 

Hence $a = b$, which is a contradiction. \qed

We recall from [11] that a band is an idempotent semigroup. Let $A$ and $B$ be nonempty sets and $S = A \times B$, define a multiplication on $S$ by $(a, b)(c, d) = (a, d)$ for $a, c \in A, b, d \in B$. This semigroup is called a rectangular band. An idempotent monoid $S$ is called left regular if $st = sts$, for every $s, t \in S$.

**Theorem 3.12** ([11]). A band is a semilattice of rectangular bands.

**Theorem 3.13.** Let $S$ be an idempotent monoid. Then all right $S$-acts satisfy Condition $(P'_E)$, if and only if $S$ is left regular.

**Proof.** Necessity. Since $S$ is an idempotent monoid, it is a semilattice of rectangular bands by Theorem 3.12. Let $S = \bigcup_{\Gamma} S_{\gamma}$ be a semilattice such that each $S_{\gamma}$ is a rectangular band. By dual of ([11], I, 3.46) we need to show that each $S_{\gamma}$ is a left zero band, so let $x, y \in S_{\gamma}$, $(\gamma \in \Gamma)$. Let $z \in S_{\gamma}$ be an arbitrary element, and let $\lambda \in \Gamma$ be such that $1 \in S_{\lambda}$, thus $1xz, 1yz \in S_{\lambda\gamma}$, and since $S_{\lambda\gamma}$ is a rectangular band we have $1xz = 1yz = 1z = z$. By assumption all right $S$-acts satisfy Condition $(P'_E)$, and so by Theorem 3.10, $S$ satisfies Condition $(R_{(W,F')})$. Thus there exists $w = ux = vy \in Sx \cap Sy$ such that $wp(x, y)x$, and so by ([11], I, 4.37) either $x = w = ux$ or there exist $u_1, \ldots, u_n, x_1, y_1, \ldots, x_n, y_n \in S$, such that $\{x_i, y_i\} = \{x, y\}$, for $i = 1, \ldots, n$ and

$$x = x_1u_1 \quad y_2u_2 = x_3u_3 \quad \ldots \quad y_nu_n = vy.$$ 

$$y_1u_1 = x_2u_2 \quad y_3u_3 = x_4u_4 \quad \ldots$$

In first case $xy = xvy = xux = xx = x$. In the second case it can be seen that $w = vy$ belongs to $S_{\gamma}$. Thus $xy = x(vy)y = xvy = xux = xx = x$ as required.

Sufficiency. Since $S$ is an idempotent monoid, obviously it is regular. Now let $x, y, z, s, t \in S$, and $sz = tz$, then $xsp(xs, yt)yt$ implies that $xstp(xs, yt)yt$. If $w = xsts = xst$ then $w \in Ss \cap St$ and $wp(xs, yt)xs$. Thus $S$ satisfies Condition $(R_{(W,F')})$, and so every right $S$-act satisfies Condition $(P'_E)$ by Theorem 3.10. \qed

**Lemma 3.3.** Let $S$ be a weakly right reversible regular monoid. Then every right Rees factor $S$-act satisfies Condition $(W_{(W,F')})$.

**Proof.** Suppose that $S$ is a weakly right reversible regular monoid and $K_S$ is a right ideal of $S$. Then there are two cases as follow:
Case 1. \( K_S = S \). Since \( S \) is weakly right reversible, \( S/K_S \cong \Theta_S \) satisfies Condition \((W_{WF'})\) by Theorem 2.1.

Case 2. \( K_S \neq S \). Since \( S \) is regular, \( S/K_S \) is principally weakly flat by ([11], IV, 6.6) and so \( K_S \) is left stabilizing. Let \([x]_{\rho_K} = [y]_{\rho_K} \) and \( sz = tz \), for \( x, y, z, s, t \in S \). Since \((xs)_{\rho_K}(yt)\) then \( xs = yt \) or \( xs, yt \in K_S \). If \( xs = yt \), set \( u = x \) and \( v = y \), then \( x(\rho_K \cup \ker\rho_s)u \) and \( y(\rho_K \cup \ker\rho_t)v \). If \( xs, yt \in K_S \), then there exist \( l_1, l_2 \in K_S \) such that \( l_1xs = xs \) and \( l_2yt = yt \). Thus, \((l_1x)\ker\rho_s(x)\) and \((l_2y)\ker\rho_t(y)\). Since \( S \) is weakly right reversible there exist \( u', v' \in S \) such that \( u's = v't \). If \( u = l_1u', v = l_1v' \), then \((x)\ker\rho_s(l_1x)\rho_K(u)\) and so \( x(\rho_K \cup \ker\rho_u)u \). Similarly, \( y(\rho_K \cup \ker\rho_v)v \), and \( us = l_1u's = l_1v't = vt \). So by ([11], III, 10.6) in both cases \([x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s \), and \([y]_{\rho_K} \otimes t = [v]_{\rho_K} \otimes t \) in \( S/\rho_K \otimes SSs \), and \( S/\rho_K \otimes sSt \), respectively. Hence

\[
[x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s = [1]_{\rho_K} \otimes us = [1]_{\rho_K} \otimes vt = [v]_{\rho_K} \otimes t = [y]_{\rho_K} \otimes t,
\]

in \( S/\rho_K \otimes SSs(Ss \cup St) \). Thus, there exist \( s_1, ..., s_k, t_1, ..., t_k, b_1, ..., b_{k-1} \in S, u_1, ..., u_k \in Ss \cup St \), such that

\[
\begin{align*}
[x]_{\rho_K} s_1 &= [b_1]_{\rho_K} t_1, & s_1u_1 &= s \\
[b_1]_{\rho_K} s_2 &= [b_2]_{\rho_K} t_2, & s_2u_2 &= t_1u_1 \\
... \\
[b_{k-1}]_{\rho_K} s_k &= [y]_{\rho_K} t_k, & t &= t_ku_k.
\end{align*}
\]

Let \( j \) be the first index such that \( u_j \in St \). If \( j = 1 \), then \( s = s_1u_1 \in St \), and so \( s = v_1t \), for some \( v_1 \in S \), thus Condition \((W_{WF'})\) holds for \( w = s \) and \( a'' = x \). Suppose now that \( j > 1 \). Then \( u_{j-1} \in Ss \), and since \( s_ju_j = t_{j-1}u_{j-1} \), we have \( w = s_ju_j \in Ss \cap St \) and so

\[
[x]_{\rho_K} s = [x]_{\rho_K} s_1u_1 = [b_1]_{\rho_K} t_1u_1 = \ldots = [b_{j-1}]_{\rho_K} s_ju_j = [b_{j-1}]_{\rho_K} w.
\]

Thus Condition \((W_{WF'})\) holds for \( a'' = b_{j-1} \).

\[\square\]

Theorem 3.14 ([11]). Let \( S \) be a left \( PP \) monoid. A right \( S \)-act \( A_S \) is principally weakly flat if and only if for every \( a, a' \in A_S, s \in S, as = a's \) implies that there exists \( e \in E(S) \) such that \( es = s \) and \( ae = a'e \).

Theorem 3.15. For any monoid \( S \) the following statements are equivalent:

(1) all right Rees factor \( S \)-acts satisfy Condition \((P_E')\);

(2) \( S \) is a weakly right reversible regular monoid.

Proof. (1) \(\Rightarrow\) (2). Suppose that all right Rees factor \( S \)-acts satisfy Condition \((P_E')\), then by Theorem 2.1(1) all right Rees factor \( S \)-acts are principally weakly flat and so \( S \) is regular by ([11], IV, 6.6). Also by assumption the one-element
right $S$-act $\Theta_S$ satisfies Condition $(P_E')$, and so $S$ is weakly right reversible by Theorem 2.1.

(2) $\Rightarrow$ (1). Suppose that $S$ is a weakly right reversible regular monoid and let $K_S$ be a right ideal of $S$. Then there are two cases that can arise:

**Case 1.** $K_S = S$. Then $S/K_S \cong \Theta_S$ satisfies Condition $(P_E')$ by Theorem 2.1.

**Case 2.** $K_S \neq S$. Then $S/K_S$ is principally weakly flat by ([11], IV, 6.6). Let $[x]_{\rho_K} s = [y]_{\rho_K} t$, and $sz = tz$, for $x, y, z, s, t \in S$. By Lemma 3.3, $S/K_S$ satisfies Condition $(W_{(WF)})$, so there exist $w, u, v \in S$ such that $[x]_{\rho_K} s = [w]_{\rho_K} us, [y]_{\rho_K} t = [w]_{\rho_K} vt$ and $us = vt$. Since $S$ is regular it is left $PP$ and so there exist $e, f \in E(S)$ such that $es = s, ft = t, [x]_{\rho_K} e = [w]_{\rho_K} ue$, and $[y]_{\rho_K} f = [w]_{\rho_K} vf$ by Theorem 3.14. Thus $S/K_S$ satisfies Condition $(P_E)$ as required. 

In what follows we give a characterization of monoids over which all right acts satisfying Condition $(P_E')$ have some other flatness properties and vice versa.

**Theorem 3.16.** For any monoid $S$ the following statements are equivalent:

1. all right $S$-acts satisfying Condition $(P_E')$ are free;
2. all finitely generated right $S$-acts satisfying Condition $(P_E')$ are free;
3. all (mono)cyclic right $S$-acts satisfying Condition $(P_E')$ are free;
4. all right $S$-acts satisfying Condition $(P_E)$ are projective generators;
5. all finitely generated right $S$-acts satisfying Condition $(P_E)$ are projective generators;
6. all (mono)cyclic right $S$-acts satisfying Condition $(P_E)$ are projective generators;
7. $S = \{1\}$.

**Proof.** Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), are obvious.

(6) $\Rightarrow$ (7). Suppose that all (mono)cyclic right $S$-acts satisfying Condition $(P_E')$ are projective generators. Then all (mono)cyclic right $S$-acts satisfying Condition $(P)$ are projective generators and so $S = \{1\}$ by ([11], IV, 12.8).

(7) $\Rightarrow$ (1). It is obvious.

**Theorem 3.17 ([3]).** Let $S$ be a left $PP$ monoid and $A_S$ be a right $S$-act. Then $A_S$ is weakly flat if and only if it satisfies Condition $(P_E)$. 

From Theorems 3.21, 3.17 and ([11], IV, 8.1) we have the following result:

**Proposition 3.1.** Let $S$ be a left $PP$ right collapsible monoid. If $S$ satisfies Condition

\[(L'): \forall e, f \in E(S), \exists z \in eS \cap fS : z\lambda(e, f)e\]

then every right $S$-act satisfying Condition $(P_E')$ is flat.

We recall from [6] that a monoid $S$ satisfies Condition $(FP_2)$ if every left collapsible submonoid of $< E(S) >$ contains a left zero.

**Theorem 3.18** ([6]). Let $S$ be a monoid with $E(S) \subseteq C(S)$ and let $U$ be a property of $S$-acts implied by Condition $(P)$. If all cyclic right $S$-acts with property $U$ are regular then $S$ is an idempotent monoid which satisfies Condition $(FP_2)$.

**Theorem 3.19.** For any monoid $S$ with $E(S) \subseteq C(S)$, the following statements are equivalent:

1. all right $S$-acts satisfying Condition $(P_E')$ are regular;
2. all finitely generated right $S$-acts satisfying Condition $(P_E')$ are regular;
3. all cyclic right $S$-acts satisfying Condition $(P_E')$ are regular;
4. $S = \{0, 1\}$ or $S = \{1\}$.

**Proof.** Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Theorem 3.18, $S$ is an idempotent monoid, thus $S$ is commutative and left $PP$, and so by Theorem 3.2 Condition $(P_E)$ and weak flatness coincide. Thus by assumption all weakly flat cyclic right $S$-acts are regular, and so by [[6], Theorem 1.14] every element of $S \setminus \{1\}$ is a right zero. Since $S$ is commutative every element of $S$ different from 1 is a zero, and so $S = \{0, 1\}$ or $S = \{1\}$.

(4) $\Rightarrow$ (1). It follows from ([11], IV, 14.4).

**Theorem 3.20.** Let $S$ be a simple monoid, then the following statements are equivalent:

1. all right $S$-acts satisfying Condition $(P_E')$ are regular;
2. all finitely generated right $S$-acts satisfying Condition $(P_E')$ are regular;
3. all cyclic right $S$-acts satisfying Condition $(P_E')$ are regular;
4. $S = \{1\}$.

**Proof.** It follows from [[6], Theorem 4.1].
Theorem 3.21. Let $S$ be a monoid. If $S$ is right collapsible, then every right $S$-act satisfying Condition $(P'_E)$ satisfies Condition $(P_E)$.

Proof. Suppose that $A_S$ satisfies Condition $(P'_E)$, and let $as = a't$, for $a, a' \in A_S$, and $s, t \in S$. Since $S$ is right collapsible, there exists $z \in S$ such that $sz = t\z$. Thus by assumption there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s$, $ft = t$ and $us = vt$, that is $A_S$ satisfies Condition $(P_E)$ as required. 

We recall from [13] that an act $A_S$ is called strongly torsion free if for every $a, a' \in A_S$ and every $s \in S$, the equality $as = a's$ implies $a = a'$. By [[13], Proposition 2.1] $S_S$ is strongly torsion free if and only if $S$ is right cancellative, while for any monoid $S$, $S_S$ satisfies Condition $(P'_E)$, so Condition $(P'_E)$ does not imply strong torsion freeness in general.

Proposition 3.2. For any monoid $S$ the following statements are equivalent:

1. all right $S$-acts satisfying Condition $(P'_E)$ are strongly torsion free;
2. all finitely generated right $S$-acts satisfying Condition $(P'_E)$ are strongly torsion free;
3. all cyclic right $S$-acts satisfying Condition $(P'_E)$ are strongly torsion free;
4. $S$ is right cancellative.

Proof. It follows from [[13], Theorem 3.1].

Recall from [11] that a right $S$-act $A_S$ is divisible if $Ac = A$ for every left cancellable element $c \in S$. By Lemma 2.1, $S_S$ satisfies Condition $(P'_E)$, while it is not divisible in general. Now it is natural to ask for which monoids Condition $(P'_E)$ implies divisibility.

Proposition 3.3. For any monoid $S$ the following statements are equivalent:

1. all right $S$-acts satisfying Condition $(P'_E)$ are divisible;
2. all finitely generated right $S$-acts satisfying Condition $(P'_E)$ are divisible;
3. all cyclic right $S$-acts satisfying Condition $(P'_E)$ are divisible;
4. all left cancellable elements of $S$ are left invertible.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Lemma 2.1(3), $S_S$ satisfies Condition $(P'_E)$, and so by assumption it is divisible. Thus $Sc = S$, for every left cancellable element $c \in S$. That is, there exists $x \in S$ such that $xc = 1$.

(4) $\Rightarrow$ (1). It is clear from ([11], III, 2.2).
We recall from [5] that a right $S$-act $A_S$ satisfies Condition $(EP)$ if for all $a \in A_S, s, t \in S$, as $= at$ implies that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also we recall from [4] that a right $S$-act $A_S$ satisfies Condition $(E'P)$ if for all $a \in A_S, s, t, z \in S$, as $= at$ and $sz = tz$ imply that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

Notice that Condition $(E)$ does not imply Condition $(P_E')$, otherwise Condition $(E)$ implies principal weak flatness, which is not the case.

**Theorem 3.22.** For any monoid $S$, the following statements are equivalent:

1. $S$ is regular;
2. all right $S$-acts satisfying Condition $(E'P)$ satisfy Condition $(P_E')$;
3. all right $S$-acts satisfying Condition $(EP)$ satisfy Condition $(P_E')$;
4. all right $S$-acts satisfying Condition $(E')$ satisfy Condition $(P_E')$;
5. all right $S$-acts satisfying Condition $(E)$ satisfy Condition $(P_E')$.

**Proof.** Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2). Suppose that the right $S$-act $A_S$ satisfies Condition $(E'P)$ and let as $= at$, sz $= tz$, for $a, a' \in A_S$, $s, t, z \in S$. Since $S$ is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$, thus $a't = ass's = a'ts's$ and $ts' = ts'ss'$. Since $A_S$ satisfies Condition $(E'P)$, there exist $a'' \in A_S$, $u, v \in S$, such that $a' = a''u = a''v$ and $ut = vts's$. If $w = vts', e = ss', f = 1$, then $ac = a(ss')e = (a'ts')e = (a''vts')e = a''we$, $af = a''uf$, $es = s, ft = t$ and $ws = ut$. Thus $A_S$ satisfies Condition $(P_E')$ as required.

(5) $\Rightarrow$ (1). Let $s \in S$. If $sS = S$ then obviously $s$ is regular. Otherwise $sS$ is a proper right ideal of $S$ and the right $S$-act $A(sS) = S_S \prod^{\infty} S_S$ satisfies Condition $(E)$ by ([11], III, 14.3(3)). Thus by assumption it satisfies Condition $(P_E')$ and so $sS$ is left stabilizing by Lemma 3.1. That is, there exists $t \in sS$ such that $s = ts$ which implies that $s$ is regular.

**Remark 3.23.** By Theorem 2.1, the one-element right $S$-act $\Theta_S$ satisfies Condition $(P_E')$, if and only if $S$ is a weakly right reversible monoid, but as we know for any monoid $S, \Theta_S$ satisfies Condition $(PWP)$.

**Theorem 3.24.** Let $S$ be a right cancellative monoid. Then every right $S$-act satisfying Condition $(PWP)$ satisfies Condition $(P_E')$.

**Proof.** Suppose that the right $S$-act $A_S$ satisfies Condition $(PWP)$ and let as $= a't$, sz $= tz$, for $a, a' \in A_S, s, t, z \in S$. Since $S$ is right cancellative $s = t$, and so as $= a's$. Thus by assumption there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a'u, a' = a''v$ and $us = vs$. We take $e = f = 1$ and then $A_S$ satisfies Condition $(P_E')$ as required.
Recall from [8] that a right $S$-act $A_S$ satisfies Condition $(PWP_E)$ if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, and $u, v \in S, e, f \in E(S)$ such that $ae = a''ue, af = a''vf, es = s = fs$ and $us = vs$.

It is obvious that Condition $(P_E') \Rightarrow$ Condition $(PWP_E)$.

**Theorem 3.25** ([8]). Let $S$ be a monoid and $A_S$ be a right $S$-act. If $A_S$ satisfies Condition $(PWP_E)$, then it is principally weakly flat. If $S$ is a left PP monoid, then $A_S$ is principally weakly flat if and only if it satisfies Condition $(PWP_E)$.

**Theorem 3.26.** For any monoid $S$ the following statements are equivalent:

1. all torsion free right $S$-acts satisfy Condition $(P_E')$;
2. all torsion free right $S$-acts are principally weakly flat and satisfy Condition $(W_{(WF)'})$;
3. all torsion free right $S$-acts satisfy Condition $(W_{(WF)'}$) and $S$ is left almost regular;
4. all principally weakly flat right $S$-acts satisfy Condition $(P_E')$ and $S$ is left almost regular;
5. all principally weakly flat right $S$-acts satisfy Condition $(W_{(WF)'}$) and $S$ is left almost regular;
6. all right $S$-acts satisfying Condition $(PWP_E)$ satisfy Condition $(P_E')$ and $S$ is left almost regular;
7. all right $S$-acts satisfying Condition $(PWP_E)$ satisfy Condition $(W_{(WF)'}$) and $S$ is left almost regular.

**Proof.** Implications (4) $\Rightarrow$ (6), (5) $\Rightarrow$ (7) follow from Theorem 3.25. Implications (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) follow from Theorem 2.1.

(2) $\Rightarrow$ (3). By assumption all torsion free right $S$-acts are principally weakly flat, and so by ([11], IV, 6.5) $S$ is left almost regular.

(3) $\Rightarrow$ (4). Suppose that $A_S$ is a principally weakly flat right $S$-act and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. Since by assumption $A_S$ satisfies Condition $(W_{(WF)'}$), there exist $a'' \in A_S, u, v \in S$ such that $as = a''us, a't = a''vt$ and $us = vt$. Since also $S$ is left almost regular it is left $PP$ and so by Theorem 3.14 there exist $e, f \in E(S)$ such that $es = s, ft = t, ae = a''ue$, and $a'f = a''vf$. Thus $A_S$ satisfies Condition $(P_E')$ as required.

(7) $\Rightarrow$ (1). Since $S$ is left almost regular, all torsion free right $S$-acts are principally weakly flat by ([11], IV, 6.5). Also $S$ is left $PP$ and so by Theorem 3.25 all principally weakly flat right $S$-acts satisfy Condition $(PWP_E)$, thus by assumption all torsion free right $S$-acts satisfy Condition $(W_{(WF)'}$). The result follows similar to the proof of (3) $\Rightarrow$ (4). 

□
By [[13], Proposition 2.1] \( \Theta_S \) is strongly torsion free for any monoid \( S \), while it satisfies Condition \( (P'_E) \) if and only if \( S \) is weakly right reversible. So strong torsion freeness does not imply Condition \( (P'_E) \) in general.

**Theorem 3.27.** Let \( S \) be an idempotent monoid. Then the following statements are equivalent:

1. all strongly torsion free right \( S \)-acts satisfy Condition \( (P'_E) \);
2. all strongly torsion free right \( S \)-acts satisfy Condition \( (W_{(WF)'}) \);
3. all finitely generated strongly torsion free right \( S \)-acts satisfy Condition \( (P'_E) \);
4. all finitely generated strongly torsion free right \( S \)-acts satisfy Condition \( (W_{(WF)'}) \);
5. all cyclic strongly torsion free right \( S \)-acts satisfy Condition \( (P'_E) \);
6. all cyclic strongly torsion free right \( S \)-acts satisfy Condition \( (W_{(WF)'}) \);
7. \( S \) is weakly right reversible.

**Proof.** Implications (1) \( \Rightarrow \) (3) \( \Rightarrow \) (5), (2) \( \Rightarrow \) (4) \( \Rightarrow \) (6) are obvious. Implications (1) \( \Rightarrow \) (2) and (5) \( \Rightarrow \) (6) follow from Theorem 2.1.

(6) \( \Rightarrow \) (7). The one-element right \( S \)-act \( \Theta_S \) is strongly torsion free by [[13], Proposition 2.1], and so by assumption it satisfies Condition \( (W_{(WF)'}) \). Thus \( S \) is weakly right reversible by Theorem 2.1.

(7) \( \Rightarrow \) (1). Suppose that the right \( S \)-act \( A_S \) is strongly torsion free. Then the equality \( ae = (ae)e \), for \( a \in A_S \) and \( e^2 = e \in S \), implies that \( ae = a \). Hence \( aS = \{a\} \), for every \( a \in A_S \). Now let \( as = a't \), \( sz = tz \), for \( a, a' \in A_S \), and \( s, t, z \in S \). Then \( a = a' \). Since \( S \) is weakly right reversible, there exist \( u, v \in S \) such that \( us = vt \). Let \( a'' = a \), then \( ae = a''ue \), \( a'f = a''vf \), \( es = s \) and \( ft = t \) for \( e = f = 1 \), and so \( A_S \) satisfies Condition \( (P'_E) \) as required.

**References**


Accepted: 28.05.2019