On condition (P'_E)

Mahdiyeh Abbasi*
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
abbasi.m@iauzah.ac.ir

Akbar Golchin
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
agdm@math.usb.ac.ir

Hossein Mohammadzadeh Saany
Department of Mathematics
University of Sistan and Baluchestan
Zahedan
Iran
saany582@gmail.com

Abstract. In (Semigroup and Applications: Proceedings of Conference, world scientific, (1998) 72-77) Golchin and Renshaw introduced Condition (P_E) and showed that this condition implies weak flatness, but not the converse. In this paper, we introduce a generalization of Conditions (P_E), called Condition (P'_E), and will show that this condition implies principal weak flatness. Also we give a characterization of monoids by this condition over their right acts.

Keywords: condition (P'_E), condition $(W_{(WF)'})$, P'_E-left annihilating, weakly right reversible.

1. Introduction

Laan [12] introduced Condition (E'), a generalization of Condition (E). Golchin and Renshaw [3] introduced a generalization of Condition (P) called Condition (P_E). After that Golchin and Mohammadzadeh [7] provided a characterization of monoids by this Condition of their right acts. They also introduced another generalization of Condition (P) called Condition (P'), and gave a characterization of monoids by this property of their right acts [9]. In this paper, we introduce a generalization of Conditions (P_E) and (P'), called Condition (P'_E) and will give a characterization of monoids by this condition of their right acts.

∗. Corresponding author

Throughout this paper S will denote a monoid. We refer the reader to [11] for basic definitions and terminologies relating to semigroups and acts over monoids, one can see and [2] for definitions and results on flatness properties which are used here.

A nonempty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$, and all $s, t \in S$. A right S-act A_S satisfies *Condition* (E) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A_S satisfies *Condition* (E') if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (PWP) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A_S satisfies *Condition* (P') if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (P_E) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$, implies that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$. It is clear that Condition (P) implies Condition (P_E).

2. General properties

Definition 2.1. *A right S-act A_S satisfies Condition (P'_E) if for all $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$, imply that there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$.*

It is obvious that Conditions (P_E) and (P') imply Condition (P'_E). Examples 2.3 and 3.9 show that these implications are strict.

Definition 2.2 ([1])**.** *A right S-act A_S satisfies Condition $(W_{(WF)'})$, if $as = a't$ and $sz = tz$, for $a, a' \in A_S, s, t, z \in S$, imply that there exist $a'' \in A_S, w \in Ss \cap St$, such that $as = a't = a''w$.*

Theorem 2.1. *For any right S-act A_S we have the following strict implications:*

(1) *Condition $(P'_E) \Rightarrow$ principal weak flatness;*

(2) *Condition $(P'_E) \Rightarrow$ Condition $(W_{(WF)'})$.*

Proof. (1) Suppose that A_S satisfies Condition (P'_E) and let $as = a's$, for $a, a' \in A_S, s \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = fs = s$ and $us = vs$. Then we have

$$\begin{aligned} a \otimes s &= a \otimes es = ae \otimes s = a''ue \otimes s = a'' \otimes ues = a'' \otimes us \\ &= a'' \otimes vs = a'' \otimes vfs = a''vf \otimes s = a'f \otimes s = a' \otimes fs = a' \otimes s \end{aligned}$$

in $A_S \otimes {}_S Ss$. That is, A_S is principally weakly flat as required.

(2) Suppose that A_S satisfies Condition (P'_E), and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. By assumption there exist $a'' \in A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s, ft = t$ and $us = vt$. If $w = us = vt$ then $w \in Ss \cap St$, and $as = aes = a''ues = a''us = a''w$. Similarly, $a't = a''w$ and so A_S satisfies Condition ($W_{(WF)'}$) as required. □

Example 3.1 shows that these implications are strict.

Lemma 2.1. *Let S be a monoid. Then:*

(1) *if $\{A_i \mid i \in I\}$ is a chain of subacts of an act A_S and every $A_i, i \in I$ satisfies Condition (P'_E), then $\bigcup_{i\in I} A_i$ satisfies Condition (P'_E);*

(2) *$A_S = \coprod_{i\in I} A_i$ satisfies Condition (P'_E) if and only if every $A_i, i \in I$ satisfies Condition (P'_E);*

(3) *the right S-act S_S satisfies Condition (P'_E).*

Proof. It is clear from definition. □

Theorem 2.2. *Any retract of a right S-act satisfying Condition (P'_E) satisfies Condition (P'_E).*

Proof. Let A_S be a retract of B_S and suppose B_S satisfies Condition (P'_E). Let $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since A_S is a retract of B_S, there are homomorphisms $\varphi : A_S \to B_S$ and $\varphi' : B_S \longrightarrow A_S$, such that $\varphi'\varphi = 1_A$. Then we have $\varphi(as) = \varphi(a't)$ or $\varphi(a)s = \varphi(a')t$. Since $\varphi(a), \varphi(a') \in B_S$, by assumption there exist $b \in B_S, u, v \in S$, $e, f \in E(S)$ such that $\varphi(a)e = bue$, $\varphi(a')f = bvf$, $es = s, ft = t$ and $us = vt$. Then $\varphi'(\varphi(a)e) = \varphi'(bue) = \varphi'(b)ue$, and $\varphi'(\varphi(a')f) = \varphi'(bvf) = \varphi'(b)vf$. If $\varphi'(b) = a''$, then $ae = a''ue$, $a'f = a''vf$. It is obvious that $a'' \in A_S$ and so A_S satisfies Condition (P'_E) as required. □

Definition 2.3 ([9]). *Let S be a monoid and let $P \subseteq S$ be a submonoid of S. P is called weakly right reversible if*

$$(\forall s, t \in P)(\forall z \in S)(sz = tz \Rightarrow (\exists u, v \in P)(us = vt)).$$

Proposition 2.1. *For any monoid S the following statements are equivalent:*

(1) *if $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), then every A_i, $i \in I$ satisfies Condition (P'_E);*

(2) *the one-element right S-act Θ_S satisfies Condition (P'_E);*

(3) *the one-element right S-act Θ_S satisfies Condition ($W_{(WF)'}$);*

(4) *S is weakly right reversible;*

(5) *there exists a right S-act which contains a zero and satisfies Condition* (P'_E).

Proof. Since Θ_S is a retract of any right S-act containing a zero, (2) and (5) are equivalent by Theorem 2.2.

(1) $\Rightarrow$ (2). Since $S_S \cong S_S \times \Theta_S$, it follows from Lemma 2.1(3).

(2) $\Rightarrow$ (3). It follows From Theorem 2.1(2).

(3) $\Rightarrow$ (4). Let $sz = tz$ for $s, t, z \in S$. Since $\theta s = \theta t$, and the right S-act Θ_S satisfies Condition $(W_{(WF)'})$, there exist $u, v \in S$ such that $us = vt$, that is, S is weakly right reversible.

(4) $\Rightarrow$ (1). Suppose that $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), and let $a_i s = a'_i t, sz = tz$, for $a_i, a'_i \in A_i, s, t, z \in S$. Since S is weakly right reversible, there exist $u_1, v_1 \in S$ such that $u_1 s = v_1 t$. For every $j \in I \setminus \{i\}$, let a_j be a fix element of A_j. Define

$$c_j = \begin{cases} a_j u_1, & j \neq i \\ a_i, & j = i \end{cases} \quad , \qquad d_j = \begin{cases} a_j v_1, & j \neq i \\ a'_i, & j = i \end{cases}.$$

Thus $(c_j)_I s = (d_j)_I t$, and so by assumption there exist $(a''_j)_I \in \prod_{i\in I} A_i$, $u, v \in S, e, f \in E(S)$ such that $(c_j)_I e = (a''_j)_I ue, (d_j)_I f = (a''_j)_I vf, es = s, ft = t$, and $us = vt$. Hence $a_i e = a''_i ue, a'_i f = a''_i vf$, and so A_i satisfies Condition (P'_E). $\square$

Corollary 2.1. *Let S be a commutative monoid and $\{A_i |\ i \in I\}$ be a family of right S-acts. If $A_S = \prod_{i\in I} A_i$ satisfies Condition (P'_E), then every A_i, $i \in I$ satisfies Condition (P'_E).*

Example 2.3. Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}, a \neq 0 \right\}$ with matrix product as operation, then S is a right cancellative monoid, and so it is weakly right reversible, but S is not right reversible, since for every $a, b, c, d \in Z$ with $a, c \neq 0$,

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix} \neq \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}.$$

Now the one-element right S-act Θ_S satisfies Condition (P'_E) by Theorem 2.1, but it does not satisfy Condition (P_E), since otherwise it is weakly flat which is not the case.

3. Homological classifications

In this section we will classify monoids by Condition (P'_E) of their (cyclic, Rees factor) acts.

Let K be a proper right ideal of a monoid S. If x, y and z denote elements not belonging to S, define

$$A(K) = (\{x, y\} \times (S \setminus K)) \cup (\{z\} \times K)$$

and define a right S-action on $A(K)$ by

$$(x,u)s = \begin{cases} (x,us), & us \notin K \\ (z,us), & us \in K \end{cases}$$
$$(y,u)s = \begin{cases} (y,us), & us \notin K \\ (z,us), & us \in K \end{cases}$$
$$(z,u)s = (z,us).$$

$A(K)$ is a right S-act, which called the amalgamated coproduct of S_S and S_S by the core K_S, and usually denoted by $S_S \coprod^{K_S} S_S$.

Recall from [11] that a right ideal K_S of a monoid S is called *left stabilizing* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$. The following example shows that implications in Theorem 2.1 are strict.

Example 3.1. Let $S = \{0, 1, e, a\}$ be a monoid with the identity element 1 and zero element 0, and $e^2 = e, ae = a, a^2 = ea = 0$. Let $K = \{0, e\}$. Now the right S-act $A(K) = S_S \coprod^{K_S} S_S$ does not satisfy Condition (P'_E), since the equality $(x,a)a = (y,a)a$ implies $x = y$ which is a contradiction. It is a routine matter to verify that A_S satisfies Condition $(W_{(WF)'})$. Since K_S is a left stabilizing right ideal, A_S is principally weakly flat by ([11], III, 12.19).

Lemma 3.1. *Let K_S be a proper right ideal of a monoid S. If the right S-act $S_S \coprod^{K_S} S_S$ satisfies Condition (P'_E), then K_S is left stabilizing.*

Proof. Suppose that the right S-act $S_S \coprod^{K_S} S_S$ satisfies Condition (P'_E) and let $k \in K_S$. Since $(x,1)k = (y,1)k$, there exist $w \in \{x,y,z\}, r, u, v \in S$, and $e, f \in E(S)$ such that $(x,1)e = (w,r)ue$, $(y,1)f = (w,r)vf, ek = fk = k$ and $uk = vk$. If $e, f \notin K_S$, then $(x,e) = (w,rue)$ and $(y,f) = (w,rvf)$, which implies that $w = x = y$, a contradiction. Thus at least one of e or f belongs to K_S, and so the equality $ek = fk = k$ shows that K_S is left stabilizing. □

We recall from [11] that a monoid S is called *left PP* if every principal left ideal of S (as a left S-act) is projective, it is equivalent to say that for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$.

Theorem 3.2 ([3])**.** *Let S be a left PP monoid and A_S be a right S-act. Then A_S is weakly flat if and only if it satisfies Condition (P_E).*

Theorem 3.3. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition (P_E);*

(2) *all right S-acts of the form $S_S \coprod^{K_S} S_S$ satisfy Condition (P'_E);*

(3) *S is a regular monoid.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Let $s \in S$. If $sS = S$ then s is obviously regular. Assume that $sS \neq S$, then by assumption the right S-act $S_S \coprod^{sS} S_S$ satisfies Condition (P'_E) and so by Lemma 3.1, sS is left stabilizing. Thus there exists $l \in sS$ such that $ls = s$, that is s is regular as required.

(3) $\Rightarrow$ (1). Let K_S be a proper right ideal of S and $k \in K_S$. By assumption there exist $k' \in S$ such that $k = kk'k$ and so K_S is left stabilizing. Thus $S_S \coprod^{K_S} S_S$ is weakly flat by ([11], III, 12.19). Since S is regular, it is left PP and so by Theorem 3.2, $S_S \coprod^{K_S} S_S$ satisfies Condition (P_E). □

Theorem 3.4. *Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition (P'_E) if and only if, for all $x, y, z, s, t \in S$ with $(xs)\rho(yt)$ and $sz = tz$, there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue), (yf)\rho(vf), es = s, ft = t$ and $us = vt$.*

Proof. *Necessity.* Let $(xs)\rho(yt)$ and $sz = tz$, for $x, y, z, s, t \in S$. Then $[x]_\rho s = [y]_\rho t$, and so Condition (P'_E) of S/ρ implies that there exist $u', v', z' \in S, e, f \in E(S)$, such that $[x]_\rho e = [z']_\rho u'e = [z'u']_\rho e$, $[y]_\rho f = [z']_\rho v'f = [z'v']_\rho f, es = s, ft = t$ and $u's = v't$. If $u = z'u'$ and $v = z'v'$, then $(xe)\rho(ue)$, $(yf)\rho(vf)$ and $us = vt$, as required.

Sufficiency. Let $[x]_\rho s = [y]_\rho t$ and $sz = tz$, for $x, y, z, s, t \in S$. Then $(xs)\rho(yt)$, and so by assumption there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue)$, $(yf)\rho(vf), es = s, ft = t$ and $us = vt$. $[x]_\rho e = [1]_\rho ue$ and $[y]_\rho f = [1]_\rho vf$, and so S/ρ satisfies Condition (P'_E) as required. □

Corollary 3.1. *Let S be a monoid and $u \in S$. Then the principal right ideal uS satisfies Condition (P'_E) if and only if, for all $x, y, z, s, t \in S$, $uxs = uyt$ and $sz = tz$ imply the existence of $v, r \in S, e, f \in E(S)$, such that $uxe = uve, uyf = urf, es = s, ft = t$, and $vs = rt$.*

Proof. Since $uS \cong S/ker\lambda_u$, it suffices to take $\rho = ker\lambda_u$. □

Theorem 3.5 ([9])**.** *Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition (P') if and only if, for all $x, y, z, s, t \in S$ with $(xs)\rho(yt)$ and $sz = tz$, there exist $u, v \in S$, such that $x\rho u, y\rho v$ and $us = vt$.*

Theorem 3.6. *Let S be a monoid such that every element of $E(S) \setminus \{1\}$ is a right zero and let ρ be a right congruence on S. Then S/ρ satisfies Condition (P') if and only if it satisfies Condition (P'_E).*

Proof. It is clear that Condition (P') implies Condition (P'_E). Now suppose that S/ρ satisfies Condition (P'_E) and let $(xs)\rho(yt), sz = tz$, for $x, y, z, s, t \in S$. By assumption there exist $u, v \in S, e, f \in E(S)$ such that $(xe)\rho(ue), (yf)\rho(vf), es = s, ft = t$ and $us = vt$. If $e = f = 1$ then the result follows from Theorem 3.5. If $e = 1, f \neq 1$, then f and so t are right zeros, thus $x\rho u, y\rho y$ and $us = yt$.

If $e \neq 1, f = 1$, a similar argument as the previous case shows the result. If $e \neq 1, f \neq 1$ then e and f are both right zero and so are s and t. Then $x\rho x, y\rho y$ and $xs = yt$, thus S/ρ satisfies Condition (P') by Theorem 3.5. □

Definition 3.1. *Let S be a monoid. A right ideal K_S of S is called P'_E-left annihilating if for all $x, y, z, s, t \in S$,*

$$\begin{aligned}
&[(xs \neq yt) \wedge (sz = tz)] \Rightarrow [(xs \notin K_S) \vee (yt \notin K_S) \vee, \\
&(\exists u, v \in S, e, f \in E(S) : es = s, ft = t, us = vt, \\
&(xe \neq ue \Rightarrow xe, ue \in K_S), (yf \neq vf \Rightarrow yf, vf \in K_S))]
\end{aligned}$$

Lemma 3.2. *Let S be a monoid and K_S be a right ideal of S. If K_S is P'_E-left annihilating, then K_S is left stabilizing.*

Proof. If $K_S = S$, then obviously K_S is left stabilizing. Thus we assume that K_S is a proper P'_E-left annihilating right ideal and let $k \in K_S$. For every $l \in K_S$ either $lk = k$, and so we are done, or $lk \neq k$. Since $k, lk \in K_S$ by assumption there exist $u, v \in S, e, f \in E(S)$ such that $ek = fk = k, uk = vk$, and $le \neq ue$ implies that $le, ue \in K_S$, and $f \neq vf$ implies that $f, vf \in K_S$. If $f \neq vf$ then $f \in K_S$ and $fk = k$ implies that K_S is left stabilizing. If $f = vf$, then $le \neq ue$, otherwise

$$lk = lek = uek = uk = vk = vfk = fk = k,$$

a contradiction. So $ue \in K_S$ and

$$k = fk = vfk = vk = uk = uek,$$

thus K_S is left stabilizing as required. □

Theorem 3.7. *Let S be a monoid and K_S be a right ideal of S. Then S/K_S satisfies Condition (P'_E), if and only if S is weakly right reversible and K_S is P'_E-left annihilating.*

Proof. *Necessity.* Suppose that S/K_S satisfies Condition (P'_E), for the right ideal K_S of S, and let $sz = tz$, for $s, t, z \in S$. Let $k \in K_S$, since $[k]_{\rho_K} s = [k]_{\rho_K} t$, Condition ($W_{(WF)'}$), implies that there exist $u, v \in S$ such that $us = vt$, thus S is weakly right reversible. To show that K_S is P'_E-left annihilating, suppose that $xs \neq yt, sz = tz$, for $x, y, z, s, t \in S$. If $xs \notin K_S$ or $yt \notin K_S$ then the result follows, otherwise $xs, yt \in K_S$ and so $(xs)\rho_K(yt)$. Thus by Theorem 3.4 there exist $u, v \in S, e, f \in E(S)$ such that $(xe)\rho_K(ue), (yf)\rho_K(vf), es = s, ft = t$ and $us = vt$. Since $(xe)\rho_K(ue)$, it follows that $xe = ue$ or $xe, ue \in K_S$. A similar argument shows that $yf = vf$ or $yf, vf \in K_S$.

Sufficiency. Suppose that S is a weakly right reversible monoid and K_S is a P'_E-left annihilating right ideal of S. Then there are two cases as follow:

Case 1. $K_S = S$. Since S is weakly right reversible, $S/K_S \cong \Theta_S$ satisfies Condition (P'_E) by Theorem 2.1.

Case 2. $K_S \neq S$. Suppose that $(xs)\rho_K(yt)$ and let $sz = tz$, for $x, y, z, s, t \in S$. Then there are two possibilities that can arise:

(1) $xs = yt$. If $u = x, v = y$ and $e = f = 1$, then by Theorem 3.4, S/K_S satisfies Condition (P'_E).

(2) $xs \neq yt$. Then $xs, yt \in K_S$. Since K_S is P'_E-left annihilating, there exist $u, v \in S, e, f \in E(S)$ such that $es = s, ft = t, us = vt$, if $xe \neq ue$ then $xe, ue \in K_S$, and if $yf \neq vf$ then $yf, vf \in K_S$. If $xe \neq ue$, then by assumption $(xe)\rho_K(ue)$, otherwise $xe = ue$, and again $(xe)\rho_K(ue)$. A similar argument shows that $(yf)\rho_K(vf)$, and so by Theorem 3.4, S/K_S satisfies Condition (P'_E). □

Definition 3.2 ([9])**.** *Let S be a monoid and K_S be a right ideal of S. K_S is called completely left annihilating if for all $x, y, z, s, t \in S$,*

$$[(xs \neq yt) \wedge (sz = tz)] \Rightarrow [(xs \notin K_S) \vee (yt \notin K_S) \vee (x \in K_S) \vee (y \in K_S)]$$

Theorem 3.8 ([9])**.** *Let S be a monoid and K_S be a right ideal of S. Then S/K_S satisfies Condition (P'), if and only if $K_S = S$ and S is weakly right reversible or K_S is a completely left annihilating and left stabilizing proper right ideal.*

The following example shows that Condition (P'_E) does not imply Condition (P').

Example 3.9. Let S and K_S be as in Example 3.1. Then $0 = aa \neq 1e = e$ and $0, e \in K_S, a, 1 \notin K_S$, and so K_S is not completely left annihilating, thus S/K_S does not satisfy Condition (P') by Theorem 3.8. It can be seen that K_S is P'_E-left annihilating and S is weakly right reversible, and so S/K_S satisfies Condition (P'_E).

Definition 3.3 ([1])**.** *A monoid S satisfies Condition $(R_{(WF)'})$, if for all $x, y, z, s, t \in S$, $sz = tz$ implies the existence of $w \in Ss \cap St$ such that $w\rho(xs, yt)xs$.*

Theorem 3.10. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfy Condition (P'_E);*

(2) *all finitely generated right S-acts satisfy Condition (P'_E);*

(3) *all cyclic right S-acts satisfy Condition (P'_E);*

(4) *all monocyclic right S-acts satisfy Condition (P'_E);*

(5) *S is regular and satisfies Condition $(R_{(WF)'})$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Since by assumption all monocyclic right S-acts are principally weakly flat, S is regular by ([11], IV, 6.6). Now let $sz = tz$, and suppose that $\rho = \rho(xs, yt), x, y, z, s, t \in S$. By assumption S/ρ satisfies Condition (P'_E). Since $(xs)\rho(yt)$, by Theorem 3.4, there exist $u, v \in S, e, f \in E(S)$, such that $(xe)\rho(ue), (yf)\ \rho(vf), es = s, ft = t$ and $us = vt$. Then $xs = (xes)\rho(ues) = us$. Now $w = us = vt \in Ss \cap St$, and $w\rho(xs)$ as required.

(5) $\Rightarrow$ (3). Let ρ be a right congruence on S, and suppose that $(xs)\rho(yt), sz = tz, x, y, z, s, t \in S$. Since S satisfies Condition ($R_{(WF)'}$), there exist $u, v \in S$ such that $w = us = vt$, $xs\rho(xs, yt)us$ and $yt\rho(xs, yt)\ vt$. Since S is regular there exist $s', t' \in S$ such that $s = ss's, t = tt't$ and so $(xss')\rho(xs, yt)(uss')$, $(ytt')\rho(xs, yt)(vtt')$. If $e = ss', f = tt'$, then $xe\rho(xs, yt)ue$, $yf\rho(xs, yt)vf$, $es = s$ and $ft = t$. But $\rho(xs, yt) \subseteq \rho$, and so S/ρ satisfies Condition (P'_E) by Theorem 3.4.

(3) $\Rightarrow$ (1). Let A_S be a right S-act and $as = a't, sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since (3) $\Leftrightarrow$ (5), S is regular and so there exist s', t' such that $s = ss's$ and $t = tt't$. Then $a(ss's) = a'(tt't)$, and so $(bs')s = (bt')t$, where $b = as = a't$. By assumption bS satisfies Condition (P'_E), and so there exist $b'' \in bS \subseteq A_S, u, v \in S, e', f' \in E(S)$ such that

$$bs'e' = b''ue', bt'f' = b''vf', e's = s, f't = t, us = vt.$$

If $e = ss', f = tt'$, then

$$ae = ass' = a(ss's)s' = (ass'e')ss' = (bs'e')ss' = (b''ue')ss' = b''uss' = b''ue.$$

Similarly, $a'f = b''vf$. It is clear that $es = s$ and $ft = t$ and so A_S satisfies Condition (P'_E) as required. □

Definition 3.4 ([11]). *Let L denotes the set of all left ideals of a monoid S. S is called right L-reductive if for all ${}_SK \in L$ and for all $a, b \in {}_SK, a \neq b$, there exists $x \in {}_SK$ such that $ax \neq bx$.*

Theorem 3.11. *Let S be a monoid. If all right S-acts satisfy Condition (P'_E), then S is right L-reductive.*

Proof. Suppose that S is not L-reductive, thus there exist a left ideal ${}_SK$, and $a, b \in {}_SK$ such that $a \neq b$ and for all $x \in {}_SK$, $ax = bx$. By Theorem 3.10, S satisfies Condition ($R_{(WF)'}$), so there exists $w \in Sa \cap Sb$ such that $w\rho(a, b)a$. Thus by ([11], I, 4.37) either $w = a$ or there exist $u_1, ..., u_n, a_1, b_1, ..., a_n, b_n \in S$, such that $\{a_i, b_i\} = \{a, b\}$ for $i = 1, ..., n$, and

$$\begin{array}{llll} a = a_1u_1 & b_2u_2 = a_3u_3 & \ldots. & b_nu_n = w. \\ & b_1u_1 = a_2u_2 & b_3u_3 = a_4u_4 & \ldots \end{array}$$

Since S is regular by Theorem 3.10, there exist $a', b' \in S$ such that $a = aa'a, b = bb'b$. Now $a \in {}_SK$, implies that $u_1a'a \in {}_SK$. Then

$$a = aa'a = a_1u_1a'a = b_1u_1a'a = a_2u_2a'a = ... = b_nu_na'a = wa'a = w.$$

Thus in both cases $a = w$, also $b \in {}_SK$ implies that $b'b \in {}_SK$, and so

$$b = bb'b = ab'b = wb'b = w.$$

Hence $a = b$, which is a contradiction. □

We recall from [11] that a *band* is an idempotent semigroup. Let A and B be nonempty sets and $S = A \times B$, define a multiplication on S by $(a, b)(c, d) = (a, d)$ for $a, c \in A, b, d \in B$. This semigroup is called a *rectangular band*. An idempotent monoid S is called *left regular* if $st = sts$, for every $s, t \in S$.

Theorem 3.12 ([11]). *A band is a semilattice of rectangular bands.*

Theorem 3.13. *Let S be an idempotent monoid. Then all right S-acts satisfy Condition (P'_E), if and only if S is left regular.*

Proof. *Necessity.* Since S is an idempotent monoid, it is a semilattice of rectangular bands by Theorem 3.12. Let $S = \dot{\bigcup}_{\gamma\in\Gamma} S_\gamma$ be a semilattice such that each S_γ is a rectangular band. By dual of ([11], I, 3.46) we need to show that each S_γ is a left zero band, so let $x, y \in S_\gamma$, $(\gamma \in \Gamma)$. Let $z \in S_\gamma$ be an arbitrary element, and let $\lambda \in \Gamma$ be such that $1 \in S_\lambda$, thus $1xz, 1yz \in S_{\lambda\gamma}$, and since $S_{\lambda\gamma}$ is a rectangular band we have $1xz = 1yz = 1z = z$. By assumption all right S-acts satisfy Condition (P'_E), and so by Theorem 3.10, S satisfies Condition $(R_{(WF)'})$. Thus there exists $w = ux = vy \in Sx \cap Sy$ such that $w\rho(x, y)x$, and so by ([11], I, 4.37) either $x = w = ux$ or there exist $u_1, ..., u_n, x_1, y_1, ..., x_n, y_n \in S$, such that $\{x_i, y_i\} = \{x, y\}$, for $i = 1, ..., n$ and

$$\begin{array}{llll} x = x_1u_1 & y_2u_2 = x_3u_3 & ... & y_nu_n = vy. \\ y_1u_1 = x_2u_2 & y_3u_3 = x_4u_4 & ... & \end{array}$$

In first case $xy = xvy = xux = xx = x$. In the second case it can be seen that $w = vy$ belongs to S_γ. Thus $xy = x(vy)y = xvy = xux = x$ as required.

Sufficiency. Since S is an idempotent monoid, obviously it is regular. Now let $x, y, z, s, t \in S$, and $sz = tz$, then $xs\rho(xs, yt)yt$ implies that $xst\rho(xs, yt)yt$. If $w = xsts = xst$ then $w \in Ss \cap St$ and $w\rho(xs, yt)xs$. Thus S satisfies Condition $(R_{(WF)'})$, and so every right S-act satisfies Condition (P'_E) by Theorem 3.10. □

Lemma 3.3. *Let S be a weakly right reversible regular monoid. Then every right Rees factor S-act satisfies Condition $(W_{(WF)'})$.*

Proof. Suppose that S is a weakly right reversible regular monoid and K_S is a right ideal of S. Then there are two cases as follow:

Case 1. $K_S = S$. Since S is weakly right reversible, $S/K_S \cong \Theta_S$ satisfies Condition $(W_{(WF)'})$ by Theorem 2.1.

Case 2. $K_S \neq S$. Since S is regular, S/K_S is principally weakly flat by ([11], IV, 6.6) and so K_S is left stabilizing. Let $[x]_{\rho_K} s = [y]_{\rho_K} t$ and $sz = tz$, for $x, y, z, s, t \in S$. Since $(xs)\rho_K(yt)$ then $xs = yt$ or $xs, yt \in K_S$. If $xs = yt$, set $u = x$ and $v = y$, then $x(\rho_K \vee ker\rho_s)u$ and $y(\rho_K \vee ker\rho_t)v$. If $xs, yt \in K_S$, then there exist $l_1, l_2 \in K_S$ such that $l_1xs = xs$ and $l_2yt = yt$. Thus, $(l_1x)ker\rho_s(x)$ and $(l_2y)ker\rho_t(y)$. Since S is weakly right reversible there exist $u', v' \in S$ such that $u's = v't$. If $u = l_1u', v = l_1v'$, then $(x)ker\rho_s(l_1x)\rho_K(u)$ and so $x(\rho_K \vee ker\rho_s)u$. Similarly, $y(\rho_K \vee ker\rho_t)v$, and $us = l_1u's = l_1v't = vt$. So by ([11], III, 10.6) in both cases $[x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s$, and $[y]_{\rho_K} \otimes t = [v]_{\rho_K} \otimes t$ in $S/\rho_K \otimes {}_S Ss$, and $S/\rho_K \otimes {}_S St$, respectively. Hence

$$[x]_{\rho_K} \otimes s = [u]_{\rho_K} \otimes s = [1]_{\rho_K} \otimes us = [1]_{\rho_K} \otimes vt = [v]_{\rho_K} \otimes t = [y]_{\rho_K} \otimes t,$$

in $S/\rho_K \otimes_S (Ss \cup St)$. Thus, there exist $s_1, ..., s_k, t_1, ..., t_k, b_1, ..., b_{k-1} \in S, u_1, ..., u_k \in Ss \cup St$, such that

$$\begin{array}{cc} & s_1u_1 = s \\ [x]_{\rho_K} s_1 = [b_1]_{\rho_K} t_1 & s_2u_2 = t_1u_1 \\ [b_1]_{\rho_K} s_2 = [b_2]_{\rho_K} t_2 & s_3u_3 = t_2u_2 \\ ... & ... \\ [b_{k-1}]_{\rho_K} s_k = [y]_{\rho_K} t_k & t = t_ku_k. \end{array}$$

Let j be the first index such that $u_j \in St$. If $j = 1$, then $s = s_1u_1 \in St$, and so $s = v_1t$, for some $v_1 \in S$, thus Condition $(W_{(WF)'})$ holds for $w = s$ and $a'' = x$. Suppose now that $j > 1$. Then $u_{j-1} \in Ss$, and since $s_ju_j = t_{j-1}u_{j-1}$, we have $w = s_ju_j \in Ss \cap St$ and so

$$[x]_{\rho_K} s = [x]_{\rho_K} s_1u_1 = [b_1]_{\rho_K} t_1u_1 = ... = [b_{j-1}]_{\rho_K} s_ju_j = [b_{j-1}]_{\rho_K} w.$$

Thus Condition $(W_{(WF)'})$ holds for $a'' = b_{j-1}$. □

Theorem 3.14 ([11]). *Let S be a left PP monoid. A right S-act A_S is principally weakly flat if and only if for every $a, a' \in A_S, s \in S, as = a's$ implies that there exists $e \in E(S)$ such that $es = s$ and $ae = a'e$.*

Theorem 3.15. *For any monoid S the following statements are equivalent:*

(1) *all right Rees factor S-acts satisfy Condition (P'_E);*

(2) *S is a weakly right reversible regular monoid.*

Proof. (1) ⇒ (2). Suppose that all right Rees factor S-acts satisfy Condition (P'_E), then by Theorem 2.1(1) all right Rees factor S-acts are principally weakly flat and so S is regular by ([11], IV, 6.6). Also by assumption the one-element right S-act Θ_S satisfies Condition (P'_E), and so S is weakly right reversible by Theorem 2.1.

(2) $\Rightarrow$ (1). Suppose that S is a weakly right reversible regular monoid and let K_S be a right ideal of S. Then there are two cases that can arise:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ satisfies Condition (P'_E) by Theorem 2.1.

Case 2. $K_S \neq S$. Then S/K_S is principally weakly flat by ([11], IV, 6.6). Let $[x]_{\rho_K} s = [y]_{\rho_K} t$, and $sz = tz$, for $x, y, z, s, t \in S$. By Lemma 3.3, S/K_S satisfies Condition $(W_{(WF)'})$, so there exist $w, u, v \in S$ such that $[x]_{\rho_K} s = [w]_{\rho_K} us, [y]_{\rho_K} t = [w]_{\rho_K} vt$ and $us = vt$. Since S is regular it is left PP and so there exist $e, f \in E(S)$ such that $es = s, ft = t, [x]_{\rho_K} e = [w]_{\rho_K} ue$, and $[y]_{\rho_K} f = [w]_{\rho_K} vf$ by Theorem 3.14. Thus S/K_S satisfies Condition (P'_E) as required. □

In what follows we give a characterization of monoids over which all right acts satisfying Condition (P'_E) have some other flatness properties and vise versa.

Theorem 3.16. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are free;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are free;*

(3) *all (mono)cyclic right S-acts satisfying Condition (P'_E) are free;*

(4) *all right S-acts satisfying Condition (P'_E) are projective generators;*

(5) *all finitely generated right S-acts satisfying Condition (P'_E) are projective generators;*

(6) *all (mono)cyclic right S-acts satisfying Condition (P'_E) are projective generators;*

(7) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), are obvious.

(6) $\Rightarrow$ (7). Suppose that all (mono)cyclic right S-acts satisfying Condition (P'_E) are projective generators. Then all (mono)cyclic right S-acts satisfying Condition (P) are projective generators and so $S = \{1\}$ by ([11], IV, 12.8).

(7) $\Rightarrow$ (1). It is obvious. □

Theorem 3.17 ([3]). *Let S be a left PP monoid and A_S be a right S-act. Then A_S is weakly flat if and only if it satisfies Condition (P_E).*

From Theorems 3.21, 3.17 and ([11], IV, 8.1) we have the following result:

Proposition 3.1. *Let S be a left PP right collapsible monoid. If S satisfies Condition*

$$(L') : \forall e, f \in E(S), \exists z \in eS \cap fS : z\lambda(e, f)e$$

then every right S-act satisfying Condition (P'_E) is flat.

We recall from [6] that a monoid S satisfies Condition (FP_2) if every left collapsible submonoid of $< E(S) >$ contains a left zero.

Theorem 3.18 ([6])**.** *Let S be a monoid with $E(S) \subseteq C(S)$ and let U be a property of S-acts implied by Condition (P). If all cyclic right S-acts with property U are regular then S is an idempotent monoid which satisfies Condition (FP_2).*

Theorem 3.19. *For any monoid S with $E(S) \subseteq C(S)$, the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are regular;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are regular;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are regular;*

(4) $S = \{0, 1\}$ *or* $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Theorem 3.18, S is an idempotent monoid, thus S is commutative and left PP, and so by Theorem 3.2 Condition (P_E) and weak flatness coincide. Thus by assumption all weakly flat cyclic right S-acts are regular, and so by [[6], Theorem 1.14] every element of $S \setminus \{1\}$ is a right zero. Since S is commutative every element of S different from 1 is a zero, and so $S = \{0, 1\}$ or $S = \{1\}$.

(4) $\Rightarrow$ (1). It follows from ([11], IV, 14.4). □

Theorem 3.20. *Let S be a simple monoid, then the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are regular;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are regular;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are regular;*

(4) $S = \{1\}$.

Proof. It follows from [[6], Theorem 4.1]. □

Theorem 3.21. *Let S be a monoid. If S is right collapsible, then every right S-act satisfying Condition (P'_E) satisfies Condition (P_E).*

Proof. Suppose that A_S satisfies Condition (P'_E), and let $as = a't$, for $a, a' \in A_S$, and $s, t \in S$. Since S is right collapsible, there exists $z \in S$ such that $sz = tz$. Thus by assumption there exist $a'' \in A_S$, $u, v \in S, e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, ft = t$ and $us = vt$, that is A_S satisfies Condition (P_E) as required. □

We recall from [13] that an act A_S is called *strongly torsion free* if for every $a, a' \in A_S$ and every $s \in S$, the equality $as = a's$ implies $a = a'$. By [[13], Proposition 2.1] S_S is strongly torsion free if and only if S is right cancellative, while for any monoid S, S_S satisfies Condition (P'_E), so Condition (P'_E) does not imply strong torsion freeness in general.

Proposition 3.2. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are strongly torsion free;*

(4) *S is right cancellative.*

Proof. It follows from [[13], Theorem 3.1]. □

Recall from [11] that a right S-act A_S is divisible if $Ac = A$ for every left cancellable element $c \in S$. By Lemma 2.1, S_S satisfies Condition (P'_E), while it is not divisible in general. Now it is natural to ask for which monoids Condition (P'_E) implies divisibility.

Proposition 3.3. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts satisfying Condition (P'_E) are divisible;*

(2) *all finitely generated right S-acts satisfying Condition (P'_E) are divisible;*

(3) *all cyclic right S-acts satisfying Condition (P'_E) are divisible;*

(4) *all left cancellable elements of S are left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Lemma 2.1(3), S_S satisfies Condition (P'_E), and so by assumption it is divisible. Thus $Sc = S$, for every left cancellable element $c \in S$. That is, there exists $x \in S$ such that $xc = 1$.

(4) $\Rightarrow$ (1). It is clear from ([11], III, 2.2). □

We recall from [5] that a right S-act A_S satisfies *Condition* (EP) if for all $a \in A_S, s, t \in S$, $as = at$ implies that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also we recall from [4] that a right S-act A_S satisfies *Condition* $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

Notice that Condition (E) does not imply Condition (P'_E), otherwise Condition (E) implies principal weak flatness, which is not the case.

Theorem 3.22. *For any monoid S, the following statements are equivalent:*

(1) *S is regular;*

(2) *all right S-acts satisfying Condition $(E'P)$ satisfy Condition (P'_E);*

(3) *all right S-acts satisfying Condition (EP) satisfy Condition (P'_E);*

(4) *all right S-acts satisfying Condition (E') satisfy Condition (P'_E);*

(5) *all right S-acts satisfying Condition (E) satisfy Condition (P'_E).*

Proof. Implications $(2) \Rightarrow (3) \Rightarrow (5)$, $(2) \Rightarrow (4) \Rightarrow (5)$ are obvious.

$(1) \Rightarrow (2)$. Suppose that the right S-act A_S satisfies Condition $(E'P)$ and let $as = a't$, $sz = tz$, for $a, a' \in A_S$, $s, t, z \in S$. Since S is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$, thus $a't = ass's = a'ts's$ and $ts' = ts'ss'$. Since A_S satisfies Condition $(E'P)$, there exist $a'' \in A_S$, $u, v \in S$, such that $a' = a''u = a''v$ and $ut = vts's$. If $w = vts', e = ss', f = 1$, then $ae = a(ss')e = (a'ts')e = (a''vts')e = a''we$, $a'f = a''uf$, $es = s, ft = t$ and $ws = ut$. Thus A_S satisfies Condition (P'_E) as required.

$(5) \Rightarrow (1)$. Let $s \in S$. If $sS = S$ then obviously s is regular. Otherwise sS is a proper right ideal of S and the right S-act $A(sS) = S_S \coprod^{sS} S_S$ satisfies Condition (E) by ([11], III, 14.3(3)). Thus by assumption it satisfies Condition (P'_E) and so sS is left stabilizing by Lemma 3.1. That is, there exists $t \in sS$ such that $s = ts$ which implies that s is regular. □

Remark 3.23. By Theorem 2.1, the one-element right S-act Θ_S satisfies Condition (P'_E), if and only if S is a weakly right reversible monoid, but as we know for any monoid S, Θ_S satisfies Condition (PWP).

Theorem 3.24. *Let S be a right cancellative monoid. Then every right S-act satisfying Condition (PWP) satisfies Condition (P'_E).*

Proof. Suppose that the right S-act A_S satisfies Condition (PWP) and let $as = a't$, $sz = tz$, for $a, a' \in A_S, s, t, z \in S$. Since S is right cancellative $s = t$, and so $as = a's$. Thus by assumption there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs$. We take $e = f = 1$ and then A_S satisfies Condition (P'_E) as required. □

Recall from [8] that a right S-act A_S satisfies *Condition* (PWP_E) if for all $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $a'' \in A_S$, and $u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s = fs$ and $us = vs$.

It is obvious that Condition $(P'_E) \Rightarrow$ Condition (PWP_E).

Theorem 3.25 ([8]). *Let S be a monoid and A_S be a right S-act. If A_S satisfies Condition (PWP_E), then it is principally weakly flat. If S is a left PP monoid, then A_S is principally weakly flat if and only if it satisfies Condition (PWP_E).*

Theorem 3.26. *For any monoid S the following statements are equivalent:*

(1) *all torsion free right S-acts satisfy Condition (P'_E);*

(2) *all torsion free right S-acts are principally weakly flat and satisfy Condition $(W_{(WF)'})$;*

(3) *all torsion free right S-acts satisfy Condition $(W_{(WF)'})$ and S is left almost regular;*

(4) *all principally weakly flat right S-acts satisfy Condition (P'_E) and S is left almost regular;*

(5) *all principally weakly flat right S-acts satisfy Condition $(W_{(WF)'})$ and S is left almost regular;*

(6) *all right S-acts satisfying Condition (PWP_E) satisfy Condition (P'_E) and S is left almost regular;*

(7) *all right S-acts satisfying Condition (PWP_E) satisfy Condition $(W_{(WF)'})$ and S is left almost regular.*

Proof. Implications (4) $\Rightarrow$ (6), (5) $\Rightarrow$ (7) follow from Theorem 3.25. Implications (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) follow from Theorem 2.1.

(2) $\Rightarrow$ (3). By assumption all torsion free right S-acts are principally weakly flat, and so by ([11], IV, 6.5) S is left almost regular.

(3) $\Rightarrow$ (4). Suppose that A_S is a principally weakly flat right S-act and let $as = a't, sz = tz$ for $a, a' \in A_S, s, t, z \in S$. Since by assumption A_S satisfies Condition $(W_{(WF)'})$, there exist $a'' \in A_S, u, v \in S$ such that $as = a''us, a't = a''vt$ and $us = vt$. Since also S is left almost regular it is left PP and so by Theorem 3.14 there exist $e, f \in E(S)$ such that $es = s, ft = t, ae = a''ue$, and $a'f = a''vf$. Thus A_S satisfies Condition (P'_E) as required.

(7) $\Rightarrow$ (1). Since S is left almost regular, all torsion free right S-acts are principally weakly flat by ([11], IV, 6.5). Also S is left PP and so by Theorem 3.25 all principally weakly flat right S-acts satisfy Condition (PWP_E), thus by assumption all torsion free right S-acts satisfy Condition $(W_{(WF)'})$. The result follows similar to the proof of (3) $\Rightarrow$ (4). □

By [[13], Proposition 2.1] Θ_S is strongly torsion free for any monoid S, while it satisfies Condition (P'_E) if and only if S is weakly right reversible. So strong torsion freeness does not imply Condition (P'_E) in general.

Theorem 3.27. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) *all strongly torsion free right S-acts satisfy Condition (P'_E);*

(2) *all strongly torsion free right S-acts satisfy Condition ($W_{(WF)'}$);*

(3) *all finitely generated strongly torsion free right S-acts satisfy Condition (P'_E);*

(4) *all finitely generated strongly torsion free right S-acts satisfy Condition ($W_{(WF)'}$);*

(5) *all cyclic strongly torsion free right S-acts satisfy Condition (P'_E);*

(6) *all cyclic strongly torsion free right S-acts satisfy Condition ($W_{(WF)'}$);*

(7) *S is weakly right reversible.*

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (6) are obvious. Implications (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) follow from Theorem 2.1.

(6) $\Rightarrow$ (7). The one-element right S-act Θ_S is strongly torsion free by [[13], Proposition 2.1], and so by assumption it satisfies Condition ($W_{(WF)'}$). Thus S is weakly right reversible by Theorem 2.1.

(7) $\Rightarrow$ (1). Suppose that the right S-act A_S is strongly torsion free. Then the equality $ae = (ae)e$, for $a \in A_S$ and $e^2 = e \in S$, implies that $ae = a$. Hence $aS = \{a\}$, for every $a \in A_S$. Now let $as = a't$, $sz = tz$, for $a, a' \in A_S$, and $s, t, z \in S$. Then $a = a'$. Since S is weakly right reversible, there exist $u, v \in S$ such that $us = vt$. Let $a'' = a$, then $ae = a''ue, a'f = a''vf, es = s$ and $ft = t$ for $e = f = 1$, and so A_S satisfies Condition (P'_E) as required. □

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