On primary subgroups and the structure of finite groups

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Abstract. If P is a p-group for some prime p we shall write $\mathcal{U}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{U}_d(P) = \{P_1, \ldots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible. In this paper, the structure of a finite group G under some assumptions on the c-normal or ss-quasinormal subgroups in $\mathcal{U}_d(P)$, for each prime p, and Sylow p-subgroups P of G is researched. Some known results are generalized.

Keywords: c-normal subgroup, ss-quasinormal subgroup, supersolvable groups.

1. Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathcal{U}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G. A interesting topic in group theory is to study the influence of the elements of $\mathcal{U}(G)$ on the structure of G. A typical result in this direction is due to Srinivasan [1]. He proved that G is supersolvable provided that every member of $\mathcal{U}(G)$ is normal in G. This result has been widely generalized.

A subgroup H of G is called s-quasinormal in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [2] and has been studied extensively by Deskins [3] and Schmidt [4].

More recently, Li et al. [5] generalized s-quasinormal subgroups to ss-quasinormal subgroups. A subgroup H of G is said to be ss-quasinormal subgroup of G, if there exists a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. In [5], They showed that, let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is p-nilpotent. Furthermore, they showed that, for every prime p dividing the order of G and $P \in \mathrm{Syl}_p(G)$, if every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is supersolvable.

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As another generalization of the normality, Wang [6] introduced the following concept: A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G. In [6], Wang showed that G is supersolvable if every member of $\mathcal{U}(G)$ is c-normal. Wang′s result has been generalized by some authors(see [7-11], etc). For example, Guo and Shum showed in [7] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathcal{U}(P)$ is c-normal, then G is p-nilpotent. The research on c-normal subgroups has formed a series, which is similar to the series of s-quasinormal subgroups. However, the two series are independent of each other. The aim of this article is to unify and improve the results of [1], [5], [6] and some of [7].

If P is a p-group for some prime p we shall write $\mathcal{U}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{U}_d(P) = \{P_1, \ldots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible.

Such subset $\mathcal{U}_d(P)$ is not unique for a fixed P in general. We know that

$$|\mathcal{U}(P)| = \frac{p^d - 1}{p - 1}, |\mathcal{U}_d(P)| = d, \lim_{d\to\infty} \frac{p^d - 1}{(p-1)d} = \infty,$$

so $|\mathcal{U}(P)| \gg |\mathcal{U}_d(P)|$.

In this paper, we study the influence of the members of some fixed $\mathcal{U}_d(P)$ on the structure of group G. Our results are more general.

2. Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1 ([5]). *Suppose that H is an ss-quasinormal subgroup of G, $K \leq G$ and N is a normal subgroup of G. Then, we have the following:*

(1) *If $H \leq K$, then H is an ss-quasinormal subgroup of K;*

(2) *HN/N is an ss-quasinormal subgroup of G/N.*

Lemma 2.2 ([6]). *Let $X \leq H \leq G$ and $N \trianglelefteq G$. Then:*

(a) *If X is c-normal in G, then X is also c-normal in H;*

(b) *If X is c-normal in G, then XN/N is c-normal in G/N.*

Lemma 2.3 ([12]). *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \bigcap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.4 ([13]). *A group G is superslovable if and only if there exists a subgroup of order dividing $|H|$ for every subgroup H of G.*

Lemma 2.5 ([14]). *Let p_1 be the minimal prime dividing $|G|$ and p_s the maximal prime dividing $|G|$. If G possesses two supersolvable subgroups H and K with $|G : H| = p_1$ and $|G : K| = p_s$, then G is superslovable.*

Lemma 2.6 ([5]). *Let H be a p-subgroup of G. Then, the following statements are equivalent:*

(a) *H is s-quasinormal in G;*

(b) *$H \leq O_p(G)$, and H is ss-quasinormal in G.*

Lemma 2.7 ([15]). *Let G be a group and let P_0 be a maximal subgroup of P. Then the following two statements are equivalent:*

(a) *P_0 is normal in G;*

(b) *P_0 is s-quasinormal in G.*

3. Main results

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that every member of some fixed $\mathcal{U}_d(P)$ is either c-normal or ss-quasinormal in G. Then G is p-nilpotent.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathcal{U}_d(P) = \{P_1, \dots, P_d\}$. By hypothesis, each P_i is either c-normal or ss-quasinormal in G. Without loss of generality, let I_1 be the subset of $\{1, ..., d\}$ such that every $P_i(i \in I_1)$ is c-normal in G and I_2 is the subset such that every $P_i(i \in I_2)$ is ss-quasinormal in G. We prove the theorem by the following claims:

(1) $O_{p'}(G) = 1$.

Set $N = O_{p'}(G)$. Consider the quotient group G/N. We know that PN/N is a Sylow p-subgroup of G/N, $N_{G/N}(PN/N) = N_G(P)N/N$ and $\mathcal{U}(PN/N) = \{P_1N/N, \dots, P_mN/N\}$. Now, by Lemma 2.1 and Lemma 2.2, we see easily that G/N satisfies the condition. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent and hence G itself is p-nilpotent, a contradiction. Thus claim (1) holds.

(2) G/P_{iG} is p-nilpotent for all $i \in I_1$, where P_{iG} is the core of P_i in G.

In this case, P_i is a c-normal subgroup of G. We know that there exists a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i = P_{iG}$. Hence,

$$G/P_{iG} = P_i/P_{iG} \cdot K_i/P_{iG}, P_i \cap K_i = P_{iG}.$$

Therefore,

$$|K_i/P_{iG}|_p = |G : P_i|_p = |P : P_i| = p.$$

As p is the smallest prime dividing $|G|$, we know that K_i/P_{iG} is p-nilpotent by Burnside′s theorem. Therefore, K_i/P_{iG} has a normal Hall p'-subgroup H/P_{iG}. We see that H/P_{iG} is also a normal Hall p'-subgroup of G/P_{iG} because K_i/P_{iG} is normal in G/P_{iG}. It follows that G/P_{iG} is p-nilpotent for all $i \in I_1$.

(3) For every $P_i(i \in I_2)$, there exists a normal subgroup H_i of G such that G/H_i is p-nilpotent.

By the condition, there is a subgroup $B \leq G$ such that $G = P_iB$ and and P_i permutes with every Sylow subgroup of B. From $G = P_iB$, we obtain

$$|B : P_i \cap B|_p = |G : P_i|_p = p,$$

and hence, $P_i \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $P_i \cap B$. Thus, $S \nsubseteq P_i$ for all $S \in \mathrm{Syl}_p(B)$, and $P_iS = SP_i$ is a Sylow p-subgroup of G. In view of $|P : P_i| = p$ and by comparison of orders, $S \cap P_i = B \cap P_i$ for all $S \in \mathrm{Syl}_p(B)$. Therefore,

$$B \cap P_i = \bigcap_{b \in B} (S^b \cap P_i) \leq \bigcap_{b \in B} S^b = O_p(B).$$

We claim that B has a Hall p'-subgroup. In fact, because $|O_p(B) : B \cap P_i| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As p is the smallest prime dividing $|G|$, by a well-known theorem of Burnside, $B/O_p(B)$ is p-nilpotent, and hence, B is p-solvable. Therefore, B has a Hall p'-subgroup K. Now, set $\pi(K) = \{p_2, ..., p_s\}$ and $S_j \in \mathrm{Syl}_{p_j}(K)$. By the condition, P_i permutes with every S_j, and so, P_i permutes with the subgroup K. Thus, $P_iK \leq G$. It is easy to see that K is a Hall p'-subgroup of G, and P_iK is a subgroup of index p in G. Let $H_i = P_iK$, where H_i is a subgroup of index p in G. As p is the smallest prime dividing $|G|$, it follows that $H_i \trianglelefteq G$, and hence, G/H_i is a p-nilpotent.

Let

$$N = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_i).$$

(4) N is p-nilpotent.

First, as all P_{iG} and H_i are normal in G, we get $N \trianglelefteq G$. Second, we consider the subgroup $P \cap N$. Recall that P_i is a Sylow p-subgroup of H_i and $P_i \leq P$, so $P \cap H_i \leq P_i$. Moreover, $P_i \leq P \cap H_i$. We have $P \cap H_i = P_i$. Therefore,

$$P \cap N = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} H_i \cap P) = (\bigcap_{i \in I_1} P_{iG}) \bigcap (\bigcap_{i \in I_2} P_i) = \Phi(P).$$

Applying Lemma 2.3, we know that N is p-nilpotent.

(5) Final contradiction. □

Now, N possesses a Hall p'-normal subgroup $N_{p'}$ such that $N = N_pN_{p'}$, where N_p is a Sylow p-subgroup of N. Then, $N_{p'}$ char $N \trianglelefteq G$, so $N_{p'}$ is normal in G, and hence, $N_{p'} \leq O_{p'}(G)$. It follows by $O_{p'}(G) = 1$ that $N_{p'} = 1$. Consequently, N is a normal p-subgroup of G, and so, $N = P \cap N = \Phi(P)$. Also, note that the class of p-nilpotent groups is a formation, by steps (2) and (3), we have G/N must be p-nilpotent. It follows that $G/\Phi(P)$ is p-nilpotent. Moreover, by III, 3.3 Hilfs-Satz in [16], $\Phi(P) \leq \Phi(G)$, so $G/\Phi(G)$ is p-nilpotent. It follows that G would be p-nilpotent, contrary to the choice of G.

The following corollaries are immediate from Theorem 3.1.

Corollary 3.2 ([6]). *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of $\mathcal{U}(P)$ is c-normal, then G is p-nilpotent.*

Corollary 3.3 ([5]). *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is p-nilpotent.*

Theorem 3.4. *Let G be a group. If there exists a normal subgroup H of G such that G/H is supersolvable, and for each Sylow subgroup P of H, every member in some fixed $\mathcal{U}_d(P)$ is either c-normal or ss-quasinormal in G, then G is supersolvable.*

Proof. Suppose that the theorem is false so that there exists a counterexample G of minimal order. We shall finish the proof by the following claims.

(1) H is a q-group for some prime q.

By hypothesis and Theorem 3.1, we have that H has a Sylow tower of supersolvable type. Let q be the largest prime dividing $|H|$, and let Q be a Sylow q-subgroup of H. The property that H possesses an order Sylow tower property implies that Q is normal in H. Now, Q char H and $H \trianglelefteq G$, so $Q \trianglelefteq G$. Furthermore, $(G/Q)/(H/Q) \cong G/H$, and Lemmas 2.1 and 2.2 show that G/Q satisfies the condition of the theorem, by the choice of G, G/Q is supersolvable. Hence, $H = Q$ by the choice of H.

(2) Q is a Sylow q-subgroup of G.

Suppose that Q is not a Sylow q-subgroup of G. Let p be the smallest prime dividing $|G/Q|$ and r the largest prime dividing $|G/Q|$. By (1), G/Q is supersolvable. By Lemma 2.4, G/Q contains two subgroups M_1/Q and M_2/Q with $|G : M_1| = p$ and $|G : M_2| = r$. By Lemmas 2.1 and 2.2, $(M_i, Q)(i = 1, 2)$ satisfy the condition of the theorem. By the choice of G, M_1 and M_2 are supersolvable. Now, by Lemma 2.5, G would be supersolvable, which is a contradiction. Thus, (2) holds.

(3) $\Phi(Q) = 1$.

Otherwise, by Lemmas 2.1 and 2.2, $G/\Phi(Q)$ satisfies the hypothesis, applying induction, we have $G/\Phi(Q)$ is supersolvable. Furthermore, $\Phi(Q) \leq \Phi(G)$ by III, 3.3 Hilfs-satz in [16], so $G/\Phi(G)$ is supersolvable. It follows that G is supersolvable, which is a contradiction.

(4) Q is a minimal normal subgroup of G.

Let N be a minimal normal subgroup of G contained in Q. Clearly the quotient group $(G/N, Q/N)$ satisfies the condition, so G/N is supersolvable. As the class of supersolvable groups is a formation, N must be the unique minimal normal subgroup of G which is contained in Q and $N \nsubseteq (G)$. So there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Thus $Q = N(Q \cap M)$. As $G = QM$ and Q is normal abelian in G, we know that $Q \cap M$ is normal in G. If $Q \cap M > 1$, let N_1 be a minimal normal subgroup of

G such that $N_1 \leq Q \cap M$, then $N_1 \leq Q$ and $N \neq N_1$, this is a contradiction. Hence $Q \cap M = 1$, which implies $Q = N$.

(5) Every $Q_i \in \mathcal{U}_d(Q) = \{Q_1, \ldots, Q_d\}$ is ss-quasinormal in G.

Assume that there is a Q_i in $\mathcal{U}_d(Q)$ such that Q_i is c-normal in G. By definition, there is a normal subgroup K_i of G such that $G = Q_iK_i$ and $Q_i \cap K_i = Q_{iG}$ is a normal subgroup of G. By (4), $Q_i \cap K_i = 1$ or Q. If $Q_i \cap K_i = Q$, then $Q_i = Q$, a contradiction. If $Q_i \cap K_i = 1$, then $Q = Q_i(Q \cap Ki)$. But then $Q \cap K_i$ is a normal subgroup of order q of G. So $Q = Q \cap K_i$ by (4). As the class of supersolvable groups is a formation, thus G is supersolvable, contrary to the choice of G.

(6) $Q_i (i = 1, 2, ..., d)$ are normal subgroups of G.

Lemmas 2.6 and 2.7 imply that $Q_i (i = 1, 2, ..., d)$ are normal subgroups of G.

(7) The final contradiction. □

Now,

$$(G/Q_i)/(Q/Q_i) \cong G/Q,$$

by (1), G/Q is supersolvable. As Q/Q_i is cyclic of order q, it follows that G/Q_i is supersolvable. Set

$$N = \bigcap_{i=1}^{d} Q_i.$$

By the definition of $\mathcal{U}_d(Q)$,

$$\bigcap_{i=1}^{d} Q_i = \Phi(Q),$$

so $N = \Phi(Q)$. Now, by the class of supersolvable groups is a formation, $G/\Phi(Q)$ is supersolvable. It follows that $G/\Phi(G)$ is supersolvable, and hence, G is supersolvable. which is a final contradiction. The proof is now completed.

The following corollaries are immediate from Theorem 3.4.

Corollary 3.5 ([7])**.** *Let G be a group. If every member of $\mathcal{U}(G)$ is c-normal, then G is supersolvable.*

Corollary 3.6 ([5])**.** *Let G be a group. For every prime p dividing the order of G and $P \in Syl_p(G)$, if every member of some fixed $\mathcal{U}_d(P)$ is ss-quasinormal in G, then G is supersolvable.*

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