On the p-supersolvability of finite groups

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Abstract. A subgroup H of a finite group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. Obviously, the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups. In this paper, we investigate the influence of CSS-subgroups on the p-supersolvability of finite groups. Some further results are obtained.
Keywords: CSS-subgroups, p-solvable groups, $p-$supersolvable groups.

1. Introduction

All groups considered in this paper are finite.

Recall a subgroup H of a group G is said to be $c-$normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G = Core_G(H)$(see [6]); The other fundamental concept deriving from [4] is that of $SS-$quasinormality: a subgroup H of G is said to be an SS-quasinormal subgroup of G if there is a supplement B of H in G such that H permutes with every Sylow subgroup of B. Combining these two concepts, the authors in [8] introduce the concept of the $CSS-$ subgroups: A subgroup H of a group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. In this case, K is called a normal CSS-supplement of H in G.

Obviously, a $c-$normal subgroup must be a $CSS-$subgroup, while a $SS-$ quasinormal subgroup also can be regarded as a $CSS-$subgroup if we take the

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normal subgroup $K = G$ in the definition of the $CSS-$subgroups, which indicates that the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups.

It is noteworthy that many authors have studied the structure of a group G by using the $c-$normality and the $SS-$quasinormality, for example, [2, 7, 5, 1, 4, 6].

As for $c-$normal subgroups, the author in [6] asserts that G is suppersolvable if every maximal subgroup of Sylow subgroup of G is $c-$normal in G; In [5], the authors generalize the result and prove that G is supersolvable if G/N is supersolvable and every maximal subgroup of Sylow subgroup of N is $c-$normal in G, where N is a normal subgroup of G; Referring to $SS-$quasinormal subgroups, the authors show that a group G is supersolvable if some maximal subgroups of Sylow subgroups are $SS-$quasinormal in G, which are unified in [8] by using the conception of the $CSS-$subgroup and minimizing the number of maximal subgroups. In this paper, we proceed with the study about $CSS-$subgroups and obtain the $p-$supersolvability of G for a fixed prime p by considering some maximal subgroups of some Sylow $p-$subgroup:

Theorem 3.1. *Let G be a p-solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Based on this result, we also consider the case of quotient group:

Theorem 3.2. *Let G be a p-solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

It is necessary to point out that, in this article, the notation $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$ be a set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, where d be the smallest generator number of a p-group P; Apart from this, all unexplained notations and terminology are standard.

2. Preliminaries

In this section we list some known results which are needed in the sequel.

Lemma 2.1 ([4, Lemma 2.1])**.** *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

(3) *If $N \leq K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.*

(4) *If K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2 ([8, Lemma 2.3]). *Let H be a subgroup of a group G. We have:*

(1) *If H is a CSS-subgroup of G and $H \leq M \leq G$, then H is a CSS-subgroup of M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$, then H is a CSS-subgroup of G if and only if H/N is a CSS-subgroup of G/N.*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is a CSS-subgroup of G, then HN/N is a CSS-subgroup of G/N.*

Lemma 2.3 ([8, Lemma 2.5]). *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.4 ([4, Lemma 2.5]). *If a p-subgroup P of G is SS-quasinormal, where p is a prime. Then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.5 ([3, A. 1.2]). *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1). $U \cap VW = (U \cap V)(U \cap W)$.

(2). $UV \cap UW = U(V \cap W)$.

Lemma 2.6. *Let P be a Sylow $p-$subgroup of G, and N a normal subgroup of G, then $\Phi(PN/N) = \Phi(P)N/N$.*

Proof. Keeping in mind that $\Phi(P) = P'P^p$, so

$$\begin{aligned}\Phi(PN/N) &= (PN/N)'(PN/N)^p \\ &= P'N/N \cdot P^pN/N \\ &= P'P^pN/N \\ &= \Phi(P)N/N.\end{aligned}$$

Lemma 2.7. *Let H be a subgroup of G, N contained in H be a normal subgroup of G and $P \in Syl_p(H)$. Assume that some $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$, if every member in $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then:*
(1) $\mathcal{M}_d(PN/N) = \{P_iN/N \mid P_iN/N \lessdot PN/N, 1 \leq i \leq d\}$.
(2) Every member in $\mathcal{M}_d(PN/N)$ is a CSS-subgroup of G/N.

Proof. (1) It is of course by Lemma 2.6.

(2) For every $P_iN/N \in \mathcal{M}_d(PN/N)$, since $P_i \in \mathcal{M}_d(P)$, the hypotheses show that there exists a normal subgroup K such that $G = P_iK$ and $P_i \cap K$ is $SS-$quasinormal in G, so $G/N = P_iN/N \cdot KN/N$. Now, our task is to show $(P_iN \cap KN)/N$ is $SS-$quasinormal in G/N. In fact, note that $P_iN/N \lessdot PN/N$, so we can conclude that $P_i \cap N = P \cap N$; On the other hand, assume that N_q is a Sylow $q-$subgroup of N, where q is a prime with $q \neq p$, there exists a

Sylow $q-$subgroup Q with $Q \leq K$ since $G = P_1K$ and $K \trianglelefteq G$, so $N_q \leq N \cap K$. Therefore $N = (P_i \cap N) \cdot (K \cap N)$, hence $P_iK \cap N = N = (P_i \cap N) \cdot (K \cap N)$, which leads to $P_iN \cap KN = (P_i \cap K)N$ by Lemma 2.5, Lemma 2.1(2) shows that $(P_i \cap K)N/N$ is $SS-$quasinormal, so the conclusion (2) holds.

3. Main results

Now we are equipped to prove the main results.

Theorem 3.1. *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Proof. Suppose that the theorem is not true, and let G be a counter-example of minimal order with $\mathcal{M}_d(P) = \{P_1, \cdots P_d\}$ and $\bigcap_{i=1}^{d} P_i = \Phi(P)$. We proceed in the following steps:

Step 1. *$O_{p'}(G) = 1$.*

Otherwise, considering the quotient group $G/O_{p'}(G)$. In view of the minimal choice of G, Lemma 2.2 shows that $G/O_{p'}(G)$ is $p-$supersolvable, and so is G, a contradiction.

Step 2. *G has a unique minimal normal subgroup N, and $\Phi(G) = 1$.*

Let N be a minimal normal subgroup of G. By Lemma 2.7, the factor group $\overline{G} = G/N$ satisfies the hypotheses of the theorem. The choice of G implies that $\overline{G}$ is p-supersolvable. Notice that the class of p-supersolvable groups is a saturated formation, so we may assume that N is a unique minimal normal subgroup of G. On the other hand, if $\Phi(G) > 1$, then $N \leq \Phi(G)$ by the above, therefore the supersolvability of G/N indicates that G is supersolvable, and so is G, a contradiction.

Step 3. *$N = O_p(G)$.*

Keeping in mind that G is $p-$solvable, so N is a (normal) p-subgroup by Step 1. By employing $\Phi(G) = 1$ in Step 2, we have that there exists a maximal subgroup M of G such that $G = NM$. Also, $\Phi(N) \leq \Phi(G) = 1$, it is followed that N is an elementary abelian $p-$group, which yields to $N \cap M = 1$ by the minimal normality of N. On the other hand, it is clear that $G = O_p(G)M$, we are now in a position to apply Lemma 2.3 concluding that $O_p(G) \cap M \trianglelefteq G$, so $N \leq O_p(G) \cap M$ if $O_p(G) \cap M > 1$, which leads to $N \leq M$, a contradiction. Therefore $O_p(G) \cap M = 1$. By the above that $G = NM$ and $N \cap M = 1$, which forces us conclude that $N = O_p(G)$.

Step 4. *Without loss of generality, we may assume that $P = NP_1$. Set $N_1 = N \cap P_1$, then $N_1 \lessdot N$.*

If $N \leq \Phi(P)$, then $N \leq \Phi(G)$ contrary to the Step 1. Thus $N \nleq \Phi(P) = \bigcap_{i=1}^{d} P_i$, we may assume that $N \nleq P_1$, so $P = NP_1$, and

$$|N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p.$$

Step 5. *Let* K_1 *be a* $CSS-$*supplement of* P_1 *in* G*, then* $P \cap K_1 = (P_1 \cap K_1)N$ *and* $P_1 \cap K_1 \lessdot P \cap K_1$.

By the hypotheses of the theorem that P_1 is a CSS-subgroup of G, so there exists a normal subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is SS-quasinormal in G. Also $G = PK_1$, by computing the order of G, we have that $P_1 \cap K_1 \lessdot P \cap K_1$.

On the other hand, the normality of K_1 asserts that $P \cap K_1$ is a Sylow p-subgroup of K_1, and

$$P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N.$$

Step 6. *Let* K_{1_q} *be a Sylow* q*-subgroup of* K_1*, where* $q \neq p$*, then* $N_1 = N \cap (P_1 \cap K_1)K_{1_q}$.

Let K_{1_q} be a Sylow q-subgroup of K_1, where $q \neq p$. Clearly, K_{1_q} is also a Sylow q-subgroup of G. Since $P_1 \cap K_1$ is SS-quasinormal in G, by Lemma 2.4, we have $(P_1 \cap K_1)K_{1_q}$ is a subgroup, and so is $(P_1 \cap K_1)K_{1_q}N$. Concentrate on the group $(P_1 \cap K_1)K_{1_q}N$, the following task is to show that $N_1 = N \cap (P_1 \cap K_1)K_{1_q}$.

Firstly, we show $N_1 \leq N \cap (P_1 \cap K_1)K_{1_q}$. Notice that $N_1 = P_1 \cap N$ and $N \leq K_1$, so $N_1 \leq P_1 \cap K_1$, and hence $N_1 \leq N \cap (P_1 \cap K_1)K_{1_q}$.

Secondly, we are required to prove that $|N_1| = |N \cap (P_1 \cap K_1)K_{1_q}|$. Considering $|(P_1 \cap K_1)K_{1_q}N|$. On the one hand, it is clear that

$$|(P_1 \cap K_1)K_{1_q}N| = |(P_1 \cap K_1)NK_{1_q}| = |(P_1 \cap K_1)N||K_{1_q}|,$$

according to Step 5, $(P_1 \cap K_1)N = P \cap K_1$, so

$$|(P_1 \cap K_1)K_{1_q}N| = |(P \cap K_1)||K_{1_q}|.$$

On the other hand,

$$|(P_1 \cap K_1)K_{1_q}N| = \frac{|(P_1 \cap K_1)K_{1_q}| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1_q}|} = \frac{|(P_1 \cap K_1)| \cdot |K_{1_q}| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1_q}|}.$$

Comparing the two equalities, we are allowed to conclude that

$$|P \cap K_1| = \frac{|(P_1 \cap K_1)| \cdot |N|}{|N \cap (P_1 \cap K_1)K_{1_q}|},$$

by the Step 5 that $P_1 \cap K_1 \lessdot P \cap K_1$, we are forced to deduce that $N \cap (P_1 \cap K_1)K_{1_q} \lessdot N$, in view of $N_1 \lessdot N$ by Step 4, while $N_1 \leq N \cap (P_1 \cap K_1)K_{1_q}$, and therefore $N_1 = N \cap (P_1 \cap K_1)K_{1_q}$.

Step 7. *The final contradiction.*

It is clear that $N_1 = N \cap (P_1 \cap K_1)K_{1q} \trianglelefteq (P_1 \cap K_1)K_{1q}$. Therefore N_1 is normal in the subgroup $\langle N, (P_1 \cap K_1)K_{1q} | q \in \pi(G), q \neq p\rangle = K_1$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq P_1$, then $N_1 \trianglelefteq P_1K_1 = G$. The minimality of N yields $N_1 = 1$. Consequently, N is a cyclic subgroup of order p. Since G/N is p-supersolvable, G is p-supersolvable, a contradiction.

Theorem 3.2. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a CSS-subgroup of G, then G is p-supersolvable.*

Proof. Let G be a minimal counter-example. We claim:

Step 1. *G has a unique minimal normal subgroup N contained in H, G/N is p-supersolvable and $N \not\leq \Phi(G)$.*

Let N contained in H be a minimal normal subgroup of G, Lemma 2.7 shows that G/N is $p-$supersolvable. Notice that the $p-$supersolvable class is saturated, so we may assume that $N \not\leq \Phi(G)$ and N is a unique minimal normal subgroup contained in H.

Step 2. *$O_{p'}(G) = 1$ and $\Phi(G) = 1$.*

It is clear that $O_{p'}(G) = 1$ by Lemma 2.7. Now, we show $\Phi(G) = 1$. Otherwise, concentrating on the quotient group $G/H\Phi(G)$. Obviously,

$$(G/\Phi(G))/(H\Phi(G)/\Phi(G)) \cong G/H\Phi(G) \cong (G/H)/(H\Phi(G)/H),$$

we have that $(G/\Phi(G))/(H\Phi(G)/\Phi(G))$ is $p-$supersolvable, Lemma 2.7 shows that $H\Phi(G)/\Phi(G)$ satisfies the hypotheses of the theorem, so the choice of G indicates that $G/\Phi(G)$ is $p-$supersolvable, and so is G, a contradiction.

Step 3. *$N = O_p(H)$*

As G is p-solvable, we have N is a p-subgroup by Step 2, $\Phi(N) \leq \Phi(G) = 1$ by Step 1. It follows that there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Applying Lemma 2.3, we have $O_p(H) \cap M \trianglelefteq G$, so $O_p(H) \cap M = 1$, thus $N = O_p(H)$.

Step 4. *Without loss of generality, we may assume that $P = NP_1$. Set $N_1 = N \cap P_1$, we have $N_1 \lessdot N$.*

Since $N \not\leq \Phi(P)$, then there exists a $P_1 \in \mathcal{M}_d(P)$ such $N \not\leq P_1$, so $P = NP_1$, set $N_1 = P_1 \cap N$, clearly $N_1 \lessdot N$.

Step 5. *Let K_1 be a $CSS-$supplement of P_1 in G, then $N \leq K_1$, $P \cap K_1 = (P_1 \cap K_1)N$ and $P_1 \cap K_1 \lessdot P \cap K_1$.*

Notice that P_1 is a $CSS-$subgroup of G, so there exists a normal subgroup K_1 such that $G = P_1K_1$, also, $G = PK_1$, therefore $P_1 \cap K_1 \lessdot P \cap K_1$.

Next, we shall show that $N \leq K_1$. Since $H \cap K_1 \trianglelefteq G$, then either $N \leq H \cap K_1$ or $H \cap K_1 = 1$ by Step 1. If $H \cap K_1 = 1$, then $P_1 = P$ as $G = P_1K_1 = PK_1$, a contradiction, which shows that $N \leq H \cap K_1 \leq K_1$.

Therefore, $P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N$.

Step 6. *The final conclusion.*

By using a similar argument to the proof of Theorem 3.1, we have the similar conclusions to Step 6 and Step 7, so G is $p-$supersolvable.

Notice that the conception of $CSS-$ subgroups is a generalization and unification of $c-$normal and $SS-$quasinormal subgroups, so we have the following:

Corollary 3.3. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a $c-$normal subgroup of G, then G is p-supersolvable.*

Corollary 3.4. *Let G be a $p-$solvable group for some prime p, H a normal subgroup of G such that G/H is p-supersolvable, and P a Sylow p-subgroup of H. If every member of $\mathcal{M}_d(P)$ is a $SS-$quasinormal subgroup of G, then G is p-supersolvable.*

In Corollary 3.3 and Corollary 3.4, if we take $H = G$, we have that:

Theorem 3.5. *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a $c-$ normal subgroup of G, then G is p-supersolvable.*

Theorem 3.6 ([4, Theorem 1.3])**.** *Let G be a $p-$solvable group for some prime p and P a Sylow p-subgroup of G. If every member of $\mathcal{M}_d(P)$ is a $SS-$quasinormal subgroup of G, then G is p-supersolvable.*

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