A note on $(\mathbf{m}, \mathbf{n})$-full stability Banach algebra modules relative to an ideal H of $A_{m\times n}$

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Abstract. In this paper the concept of (m, n)- fully stable Banach Algebra-module relative to ideal ($F-(m,n)-S-B-A$-module relative to ideal) is introducing, we study some properties of $F-(m,n)-S-B-A$-module relative to ideal and another characterization is given.

Keywords: fully stable Banach A-module relative to ideal, fully (m, n)-stable Banach A-module relative to ideal, multiplication (m, n)- A-module relative to ideal.

1. Introduction

"A non-empty set A is an algebra if, $(A, +, \cdot)$ is a vector space over a field F, $(A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in F$, for every $a, b \in A$" [1]. In [2] "a ring R is an algebra $< R, +, \cdot, -, 0 >$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $< R, +, -, 0 >$ is an abelian group, $< R, \cdot >$ is a semigroup and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y + z)$. "Let A be an algebra, recall that a Banach space E is a Banach left A-module (B-A- module) if E is a left A-module, and $\|a \cdot x\| \leq \|a\| \|x\|$ $(a \in A, x \in E)$"[1]. Following [3] "a map from a left B-A-module X into a left Banach A-module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a \cdot x) = a \cdot Tx$ for all $a \in A, x \in X$". In [4], "a submodule N of an R-module M is said to be stable, if $f(N) \subseteq N$ for each R-homomorphism $f : N \longrightarrow M$. M is called a fully stable module, each submodule of M is stable". " Let X be Banach A - module, X is called fully stable Banach A-module relative to ideal K of A, if for every submodule N of X and for each multiplier $\theta : N \longrightarrow X$ satisfy $\theta(N) \subseteq N + KX$" [5]. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of N. Let M be a right Banach Algebra-module and N be a left R-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in M^{m\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well defined element in $M^{1\times n}$

(resp. $N^{n\times k}$). "If $X \subseteq M^{1\times m}$, $S \subseteq R^{n\times n}$ and $Y \subseteq N^{n\times k}$ define

$$\begin{aligned}\ell_{M^{l\times m}}(S) &= \Big\{u \in M^{l\times m} | us = 0; \forall s \in S\Big\}\\ r_{N^{n\times k}}(S) &= \Big\{v \in N^{n\times k} | sv = 0; \forall s \in S\Big\}\\ \ell_{R^{m\times n}}(Y) &= \big\{s \in R^{m\times n} | sy = 0; \forall y \in Y\big\}\\ r_{R^{m\times n}}(X) &= \big\{s \in R^{m\times n} | xs = 0; \forall x \in X\big\}\end{aligned}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$" [6]. In this paper for two fixed positive integers n, m the concept of (m, n)- full stable and (m, n)-fully pseudo stable Banach Algebra modules relative to ideal have been introduced.

2. Fully (m, n)-stable Banach algebra modules relative to ideal

"A left B-A-module X is n-generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k \cdot x_k$ for some $a_1, \dots, a_n \in A$. A module which is 1-generated is called a cyclic module" [7]. A right R-module M is called fully (m, n)-stable relative to ideal A of $R^{n\times m}$, if $\theta(N) \subseteq N + M^n A$ for each n-generated submodule of M^m and R-homomorphism $\theta : N \longrightarrow M$ [8].

Definition 2.1. Let K be $B-A$ - module, K is called (m, n)-fully stable $B-A$-module relative to ideal H of $A^{m\times n}$, if for every m-generated submodule L of K^n and for each multiplier $\theta : L \longrightarrow K^n$ wsatisfy $\theta(L) \subseteq L + K^m H$ for two fixed positive integers n, m.

In [5] "for a nonempty subset M in a left $B - A$-module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A : a \cdot x = 0$ for all $x \in M\}$".

Notetion 2.2. Let X be a $B - A$-module

1. $L_{x_1,x_2,\dots,x_n} = \{\oplus l_{x_i} | n \in N, x_i \in X, i = 1, 2, \dots, n\}$

   $K_{y_1,y_2,\dots,y_n} = \{\oplus k_{y_i} | k \in K, y_i \in X, i = 1, 2, , n\}$

2. $\ell_{A^{m\times n}} L_{x_1,x_2,\dots,x_n} = \{a \in A^{m\times n}, a \cdot (\oplus l_{x_i}) = 0, \forall l_{x_i} \in L_{x_1,x_2,\dots,x_n}\}$

   $\ell_{A^{m\times n}} K_{y_1,y_2,\dots,y_n} = \{a \in A^{m\times n}, a \cdot (\oplus k_{y_i}) = 0, \forall k_{y_i} \in K_{y_1,y_2,\dots,y_n}\}$

Proposition 2.3. *A $B - A$-module X is (m, n)-fully stable, if and only if any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \dots, L_{x_1,x_2,\dots,x_m}\}$ and*

$$\{K_{y_1}, K_{y_1,y_2}, \dots, K_{y_1,y_2,\dots,y_m}\}$$

of X^n, if $\beta_j \notin \sum_{i=1}^{n} \alpha_i A + X^m H$, for each $j = 1, \ldots, m$, $i = 1, \ldots, n$, $\alpha_i \in \{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\beta_j \in \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ implies

$$\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \not\subseteq \mathfrak{r}_{An}\left(\{K_{x_1}, K_{x_1,x_2}, \ldots, K_{x_1,x_2,\ldots,x_m}\}\right).$$

where H of $A^{m\times n}$.

Proof. Assume that K is $F-(m,n)-S-B-A$-module and there exist two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n such that if $K_{y_j} \notin \sum_{i=1}^{n} A\alpha_i + X^m H$, for each $j = 1, \ldots, m$ and

$$\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}\right).$$

Define $f : \sum_{i=1}^{n} \alpha_i A \longrightarrow X^n$ by $f\left(\sum_{i=1}^{n} a_i L_{x_i}\right) = \sum_{i=1}^{n} a_i K_{y_i}$.

Let $L_{x_i} = (k_{1i}, k_{2i}, \ldots, k_{ni})$. If $\sum_{i=1}^{n} a_i L_{x_i} = 0$, then $\sum_{i=1}^{n} a_i k_{ij} = 0$, $j = 1, 2, \ldots, m$, implies that $rL_{x_j} = 0$ where $r = (r_1, \ldots, rn)$ and hence

$$r^T \in \mathfrak{r}A_n\left\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\right\}.$$

By assumption $rK_{y_j} = 0$ $j = 1, \ldots, m$ so $\sum_{i=1}^{n} r_i K_{y_i} = 0$. This shows that f is well defined. It is an easy matter to see that f is multiplier. (m,n)-fully stability of M implies that there exists $t = (t_1, \ldots, t_n) \in A^n$ such that $f\left(\sum_{i=1}^{n} r_i L_{x_i}\right) = \sum_{k=1}^{n} t_k \left(\sum_{i=1}^{n} r_i L_{x_i}\right) + b = \sum_{k=1}^{n} \sum_{i=1}^{n} (t_k r_i) L_{x_i} + b$ for each $\sum_{i=1}^{n} r_i L_{x_i} \in \sum_{i=1}^{n} L_{x_i} A$ and $b \in X^m H$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = f(L_{x_i}) = \sum_{k=1}^{n} t_k L_{x_i} + b \in \sum_{i=1}^{n} L_{x_i} A + X^m H$, which is contradiction.

Conversely assume that there exists m-generated $B-A$-submodule of K^n and multiplier $\mu : \sum_{i=1}^{n} L_{x_i} A \longrightarrow K^n$ such that $\mu\left(\sum_{i=1}^{n} L_{x_i} A\right) \not\subset \sum_{i=1}^{n} L_{x_i} A + X^m H$. Then, there exists an element $\beta \left(= \sum_{i=1}^{n} r_i L_{x_i}\right) \in \sum_{i=1}^{n} L_{x_i} A$ such that $\mu(K_y) \notin \sum_{i=1}^{n} L_{x_i} A + X^m H$. Take $K_{y_i} = K_y$, $j = 1, \ldots, m$, then we have m-element subset $\{\mu(K_y), \ldots, \mu(K_y)\}$, such that $\mu(K_y) \notin \sum_{i=1}^{n} L_{x_i} A + X^m H$, $j = 1, \ldots, m$. Let $\eta = (t_1, \ldots, t_n) \in \mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right)$, then $\eta\alpha_j = 0$, i.e $\sum_{i=1}^{n} t_i a_{ij} = 0$, for each $j = 1, \ldots, m$, $L_{x_j} = (a_{1j}, a_{2j}, \ldots, a_{nj})$ and $\{\mu(K_y), \ldots, \mu(K_y)\}$

$$\eta = \sum_{k=1}^{n} t_k \mu(K_y) = \sum_{k=1}^{n} t_k \mu\left(\sum_{i=1}^{n} r_i L_{x_i}\right) = \sum_{k=1}^{n} \mu\left(\sum_{i=1}^{n} t_k r_i L_{x_i}\right) = 0$$

hence $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{\mu(K_y), \ldots, \mu(K_y)\}\right)$, thus $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \cdot, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{\mu(K_{y_1}), \ldots, \mu(K_{y_1,y_2,\ldots,y_m})\}\right)$ which is a contradiction. Thus X is $F-(m,n)-S-B-A$-module relative to ideal H of $A^{m\times n}$. □

Corollary 2.4. *Let X be an $F-(m,n)-S-B-A$-module, then for any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of X^n, $\mathfrak{r}_{An}\left(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}\right) \subseteq \mathfrak{r}_{An}\left(\{K_{x_1}, K_{x_1,x_2}, \ldots, K_{x_1,x_2,\ldots,x_m}\}\right)$ implies that $L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_m} + AX^m H = K_{y_1}A + K_{y_1,y_2}A + K_{y_1,y_2,\ldots,y_m}A + X^m H$, H of $A^{m\times n}$.*

In [9], $AB{-}A$ - module X is said to satisfy Baer criterion if each submodule of X satisfies Baer criterion, that is for every submodule N of X and $A-$ multiplier $\theta : N \longrightarrow X$, there exists an element a in A such that $\theta(n) = an$ for all $n \in N$".

Definition 2.5. A $B - A$- module X is said to satisfy Baer (m, n)-criterion relative to an ideal H of $A^{m\times n}$ if each submodule of X satisfies Baer (m, n)-criterion relative to an ideal H, that is for every m-generated submodule L of X^n and A - multiplier $\theta : L \longrightarrow X^n$, there exists an element a in A such that $\theta(l) - al \in X^m H$ for all $l \in L$.

Proposition 2.6. *If X satisfies Baer $(m, 1)$-criterion and $\mathfrak{r}_A(L\cap M) = \mathfrak{r}_A(L)+ \mathfrak{r}_A(M)$ for each m-generated submodules of X^n, then X satisfies Baer (m, n)-criterion. ,H of $A^{m\times n}$.*

Proof. Let $P = Ax_1 + Ax_2 + \ldots + Ax_m$ be an m-generated submodule of X^n and $f : P \longrightarrow X^n$ a multiplier. We use induction on m. It is clear that M satisfies Bear (m, n)- criterion, if $m = 1$. Suppose that X satisfies Bear (m, n)-criterion for all k-generated submodule of X_n, for $k \le n - 1$. Write $L = Ax_1$, $M = Ax_2 + \ldots + Ax_m$, then for each $w_1 \in L$ and $w_2 \in M$ $f|_L(w_1) = y_1w_1$, $f|_M(w_2) = y_2w_2$ for some $y_1, y_2 \in A$. It is clear $y_1 - y_2 \in \mathfrak{r}_A(L \cap M) = \mathfrak{r}_A(L) + \mathfrak{r}_A(M)$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in \mathfrak{r}_A(L)$, $z_2 \in \mathfrak{r}_A(M)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $w = w_1 + w_2 \in P$ with $w_1 \in L$ and $w_2 \in M$, $f(w) = f(w_1) + f(w_2) = w_1y_1 + w_2y_2 = w_1(y_1 - z_1) + w_2(y_2 + z_2) = w_1y + w_2y = (w_1 + w_2)y = wy$. □

Proposition 2.7. *Let X be a $B - A$- module. Then X satisfies Baer (m, n) criterion if and only if $\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A) \subseteq L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A + X^m H$ for n-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ of X^n, H of $A^{m\times n}$.*

Proof. Suppose that Baer (m, n)-criterion holds for m-generated submodule of X^n , let $L_{x_i} = (k_{i1}; k_{i2}, \ldots, k_{im})$, for each $i = 1, \ldots, n$ and
$K_y = \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\} \in$
$\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A)$, $K_{y_i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$.

Define $\mu : L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A \longrightarrow X^n$ by $\mu\left(\sum_{i=1}^n L_{x_i}a_i\right) = \sum_{i=1}^n K_{y_i}a_i$. If $\sum_{i=1}^n L_{x_i}a_i$, then $\sum_{i=1}^n k_{ij}a_i = 0$. $j = 1, \ldots, m$, this implies that $L_{x_i}r = 0$ where $r = (r_1, \ldots, r_n)$ and hence

$$r \in \mathfrak{r}_{An}(L_{x_1}A + L_{x_1,x_2}A + \ldots + L_{x_1,x_2,\ldots,x_n}A).$$

By assumption $rL_{x_i} = 0$, $i = 1, \ldots, n$ so $\sum_{i=1}^n K_{y_i}a_i = 0$. This show that f is well defined. It is an easy matter to see that μ is an multiplier. By assumption there exists $t \in A$ such that $\mu\left(\sum_{i=1}^n L_{x_i}a_i\right) = t\left(\sum_{i=1}^n K_{y_i}a_i\right) = \sum_{i=1}^n K_{y_i}(ta_i)$ for each $\sum_{i=1}^n L_{x_i}a_i \in \sum_{i=1}^n L_{x_i}A$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = \mu\left(\sum_{i=1}^n L_{x_i}\right) = \sum_{i=1}^n L_{x_i}t \in \sum_{i=1}^n L_{x_i}A$ which is contradiction.

This implies that $\ell_{X^n}\mathfrak{r}_{An}\left(L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A\right)\subseteq L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A+X^mH$.

Conversely, assume that $\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A)=L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A+X^mH$, for each $\{L_{x_1},L_{x_1,x_2},\ldots,L_{x_1,x_2,\ldots,x_m}\}$ in X^n. Then, for each multiplier $f:AL_{x_1}+AL_{x_1,x_2}+\ldots+AL_{x_1,x_2,\ldots,x_n}\longrightarrow X^n$ and

$$s=(s_1,\ldots,s_n)\in r_{An}(L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A),\sum_{k=1}^{n}s_k(\sum_{i=1}^{n}L_{x_i}t_i)=0,$$

for each $\sum_{i=1}^{n}L_{x_i}t_i\in\sum_{i=1}^{n}L_{x_i}A$, hence

$$\sum_{k=1}^{n}s_kf(\sum_{i=1}^{n}L_{x_i}t_i)=\sum_{k=1}^{n}f(\sum_{i=1}^{n}L_{x_i}s_kt_i)=0,$$

thus $f(\sum_{i=1}^{n}L_{x_i}t_i)\in\ell_{X^n}\mathfrak{r}_{An}(L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A)\subseteq L_{x_1}A+L_{x_1,x_2}A+\ldots+L_{x_1,x_2,\ldots,x_n}A+X^mH$. Then X satisfies Baer (m,n)-criterion. □

Corollary 2.8. *Let X be a $B-A$- module. Then X is $F-(m,n)-S-B-A$-module if and only if*

$$\begin{aligned}&\mathfrak{r}X_n\ell_{A^n}\left(AL_{x_1}+AL_{x_1,x_2}+\ldots+AL_{x_1,x_2,\ldots,x_n}\right)\\&\subseteq AL_{x_1}+AL_{x_1,x_2}+\ldots+AL_{x_1,x_2,\ldots,x_n}+X^mH\end{aligned}$$

for n-element subset $\{L_{x_1},L_{x_1,x_2},\ldots,L_{x_1,x_2,\ldots,x_m}\}$ of X_n, H of $A^{m\times n}$.

Following [5] "Let A be a unital Banach algebra and let $\alpha>1$. A-module X is called quasi -injective if, $\phi:N\longrightarrow X$ is A-module homomorphisms such that $\|\phi\|\leq 1$, there exists A-module homomorphism $\theta:X\longrightarrow X$, such that $\theta\circ i=\phi$ and $\|\theta\|\leq\alpha$ where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α".

Following [5], "Let A be a unital Banach algebra and let $\alpha>1$. A-module X is called quasi-α-injective relative to ideal H of A if, $\phi:N\longrightarrow X$ is A-module homomorphisms such that $\|\phi\|\leq 1$, there exists A-module homomorphism $\theta:X\longrightarrow X$, such that $(\theta\circ i)(n)\phi(n)\in XH$ and $\|\theta\|\leq\alpha$ where i is an isometry from submodule N of X".

The concepts quasi $(m,n)-\alpha-B-A$ - module relative to ideal for some α is introducing.

Definition 2.9. Let A be a unital Banach algebra and let $\alpha>1$. A-module X is called quasi $(m,n)-\alpha$-injective relative to ideal H of $A^{m\times n}$ if, $\phi:N\longrightarrow X^n$ is A-module homomorphisms such that $\|\phi\|\leq 1$, there exists A-module homomorphism $\theta:X^n\longrightarrow X^n$, such that $(\theta\circ i)(n)\phi(n)\in X^mH$ and $\|\phi\|\leq\alpha$ where i is an isometry from m-generated submodule N of X. We shall say that X is quasi (m,n)- injective relative to ideal if it is quasi $(m,n)-\alpha$ injective relative to ideal for some α. , H of $A^{m\times n}$.

Proposition 2.10. *If X is $F-(m,n)-S-B-A$-module relative to ideal then X is quasi $(m,n)-B-A$-module relative to ideal H of $A^{m\times n}$.*

Proof. Let $N = \alpha_1 A + \ldots + \alpha_n A$ be m-generated submodule of X^n, $\alpha_i \in X^n$ let $\alpha > 1$ and f be any A-module homomorphism from N to X^n such that $\|f\| \leq 1$. Since X is $F-(m,n)-S-B-A$-module relative to ideal, then $f(\alpha 1A + \ldots + \alpha_n A) \subseteq \alpha_1 A + \ldots + \alpha_n A + X^m H$, then there exist $t = (t_1, \ldots, t_n) \in A_n$ and $wX^m H$. Let $a_i = (0, \ldots, 1, 0, \ldots, 0)$ such that $f\left(\sum_{i=1}^n \alpha_i\right) = t\left(\sum_{i=1}^n \alpha_i\right) + w$. Define $g : X^n \longrightarrow X$ by $g(\alpha_i) = tT\alpha_i$ it is clear that g is well defined A-module homomorphism. Now $f\left(\sum_{i=1}^n \alpha_i\right) - g\left(\sum_{i=1}^n \alpha_i\right) = t\left(\sum_{i=1}^n \alpha_i\right) + w - t\left(\sum_{i=1}^n \alpha_i\right) = w \in X^m H$ and since for each $y \in \alpha_1 A + \ldots + \alpha_n A$, $y = \sum_{i=1}^n \alpha_i s_i$ for some $s = \{()\, s_1, \ldots, s_n) \in A$, $f(y)g(y) = f(\sum_{i=1}^n \alpha_i s_i) - g\left(\sum_{i=1}^n \alpha_i s_i\right) = f\left(\left(\sum_{i=1}^n \alpha_i\right) s\right) - g\left(\left(\sum_{i=1}^n \alpha_i\right) s\right) = \left(f\left(\sum_{i=1}^n \alpha_i\right) g\left(\sum_{i=1}^n \alpha_i\right)\right) s \in X^m H$, therefore X is quasi $(m,n)-B-A$ -module relative to ideal. □

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