On Laplacian eigenvalues of ℕ-sum graphs and ℤ-sum graphs and few more properties

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Abstract. The concept of sum graphs was introduced by Harary [4]. A graph G is a sum graph if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. Harary extended the concept to allow any integers and called them as integral sum graphs. To distinguish between the two types, we call graphs that use only positive integers as ℕ-*sum graphs* and those with any integers as ℤ-*sum graphs* [9]. In this paper we investigate the Laplacian eigenvalues of ℕ-*Sum Graphs and* ℤ-*Sum Graphs* and its *anti-sum graphs*. Also, we obtain a few more properties of ℕ-sum graph.

Keywords: ℕ-sum graph, ℤ-sum graph, Laplacian matrix, operation on graphs, isomorphism of two graphs.

1. Introduction

Harary [4] introduced the concepts of sum and integral sum graphs. A graph G is a sum graph if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. The graphs that use only positive integers for labeling are called ℕ-*sum graphs* and those with any integers are called ℤ-*sum graphs* [9]. If G is a sum or integral sum graph with respect to a label set S, then graph G can be denoted by $G^+(S)$. The complement of a sum graph G satisfies the property that $e = uv$ is an edge of G^c if and only if the sum of the labels on vertices u and v is not a vertex label in G^c [9]. Harary [4] introduced a family of sum graphs G_n over a set of positive integers, $\{1, 2, ..., n\}$. In 1994, Harary [4] also introduced a family of integral sum graphs $G_{n,n}$ over a set of integers, $\{-n, -n+1, ..., n\}$ and specified the structure of these graphs in terms of the graphs G_n as $G_{n,n} = K_1\nabla(G_n\nabla G_m)$, where ∇ denotes join of graphs. Vilfred [8] generalized integral sum graph $G_{n,n}$ to $G_{m,n}$, $m, n \in \mathbb{N} \cup \{0\}$.

In this paper, we investigate Laplacian eigenvalues and Laplacian eigenvectors of G_n, $G_{m,n}$ and G_n^c. The terms that are not defined in this paper, one may refer [3].

We use the following known results in this paper.

Theorem 1.1 ([8]). *For $n \geq 2$, the degree of the vertex with label i in G_n is*

$$deg(i) = \begin{cases} n-i-1, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ n-i, & \lfloor \frac{n}{2} \rfloor \leq i \leq n. \end{cases}$$

Theorem 1.2 ([7]). *For all $n \in \mathbb{N}$, $G_{n+1} \cong_{wvl} G_n^c \cup \{v_{n+1}\}$, i.e G_{n+1} is isomorphic to $G_n^c \cup \{v_{n+1}\}$ without the vertex labels , where v_{n+1} is an isolated vertex with label $n+1$.*

Theorem 1.3 ([5]). *Let G and H be graphs with Laplacian eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$, respectively. Then Laplacian eigenvalues of the join $G \nabla H$ is $n+k, k+\alpha_1, k+\alpha_2, ..., k+\alpha_{n-1}, n+\beta_1, n+\beta_2, ..., n+\beta_{k-1}, 0$.*

Theorem 1.4 ([8]). *For $m, n \in \mathbb{N} \cup \{0\}$, $G_{m,n} \cong_{wvl} K_1 \nabla G_m \nabla G_n$*

Theorem 1.5 ([8]). *For $m, n > 0$, $K_1 \nabla G_m \cong_{wvl} G_{0,m} \cong_{wvl} G^c_{m+1}$.*

2. Results on sum graphs

2.1 Laplacian eigenvalues of G_n and G_n^c

One may observe that in Laplacian matrix of G_n, $L(G_n) = (a_{i,j})_{n \times n}$, the entries $a_{i,i} = a_{i+1,i+1} + 1$, except for the $a_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ and $a_{\lfloor n/2 \rfloor+1, \lfloor n/2 \rfloor+1}$. Also it is interesting to observe that for the matrix $L(G_n)$, $(1,1,...,1,1)$ and $(0,0,...,0,1)$ are eigenvectors each with eigenvalue 0. Thus, $L(G_n)$ has 0 as an eigenvalue with multiplicity at least 2.

Theorem 2.1. *For any $n \in \mathbb{N}$($n \geq 4$), there exist a G_n with Laplacian eigenvalues 1 and $n-1$ with the corresponding eigenvectors $(0, -\frac{1}{n-3}, -\frac{1}{n-3}, ..., -\frac{1}{n-3}, 1, 0)$ and $(-(n-2), 1, ..., 1, 0)$.*

Proof. For convenience, the odd and even cases on the order of the graph G_n are considered separately, and because the two cases are so similar, we prove only the even case here. Invoking Theorem 1.1, we have the degree sequence of G_{2k} is $\{2k-2, 2k-3, ..., k-1, k-1, ..., 1, 0\}$, while that of G_{2k+1} is $\{2k-1, 2k-2, ..., k, k, ..., 1, 0\}$. Let the graph be G_{2n}. Then for $n \geq 2$, we observe the following:

i) $d(v_1) = 2n-2$ in G_{2n}.

ii) $a_{i,i} = a_{i+1,i+1} + 1$ except for $a_{n,n}$ and $a_{n+1,n+1}$ in $L(G_{2n})$, $1 \leq i \leq 2n-1$.

iii) $d(v_n) = d(v_{n+1}) = n-1$ in G_{2n}.

Let $V = (0, -\frac{1}{2n-3}, -\frac{1}{2n-3}, ..., -\frac{1}{2n-3}, 1, 0)$ and R_i be the i^{th} row of $L(G_{2n})$, $i = 1, 2, ..., 2n$.

Claim: V is an eigenvector with eigenvalue 1 for $L(G_{2n})$.

Consider the product $R_k \times V^T$ of dimension 1×1. If the resultant element is the k^{th} element of V, then 1 is an eigenvalue. Clearly, $R_1 \times V^T = 0 = R_{2n} \times V^T$. Correspondingly we get 0 as the first and last element in V. Using (i) and (ii), for $2 \leq k \leq n$, $d(v_k) = 2n - (k+1)$ and the row is of the form $R_k = [-1, -1, ..., -1, 2n - (k+1), -1, .., -1, 0, .., 0]$. In the sum graph G_{2n}, v_k is non-adjacent to vertices v_{2n+m-k} for $m = 1, 2, 3, ..., k$. So the last k entries of the row R_k are zero and -1 appears in this row $2n-(k+1)$ times. Thus we get, $R_k \times V^T = -\frac{1}{2n-3}$, which is the k^{th} term of V, $2 \leq k \leq n$. For $n+1 \leq k \leq 2n-2$, say $k = n+a, a = 1, 2, ..., n-2$, then using (i) and (ii) $d(v_k) = n - a$ and the row $R_k = [-1, -1, ..., -1, 0, .., 0, n-a, 0, ..., 0]$ and -1 appears in the row $(n-a)$ times. This implies, $R_k \times V^T = -\frac{1}{2n-3}$, for $n = 1 \leq k \leq 2n-2$. $R_{2n-1} = [-1, 0, ..., 0, 1, 0]$ and clearly $R_{2n-1} \times V^T = 1$.

Similarly, we get $(n-1)$ as an eigenvalue for eigenvector $(-(2n-2), 1, ..., 1, 0)$. Hence the result is true for the graph G_{2n}. □

Theorem 2.2. *Let $\lambda \neq 0$ be an eigen value of $L(G_n)$ and $(x_1, x_2, ..., x_n)$ be the corresponding eigenvector. Then $(0, x_1, x_2, ..., x_n, 0)$ is an eigenvector for $L(G_{n+2})$ with eigenvalue $(\lambda + 1)$.*

Proof. Let R_i denote the i^{th} row of $L(G_n)$, $1 \leq i \leq n$. Let $V = (x_1, x_2, ..., x_n)$ be an eigenvector of $L(G_n)$, with eigenvalue $\lambda \neq 0$. Then, $L(G_n).V^T = \lambda.V^T$, $n > 2$.

$$ie, \begin{pmatrix} n-2 & -1 & -1 & . & . & . & -1 & -1 & 0 \\ -1 & n-3 & -1 & . & . & . & -1 & 0 & 0 \\ \\ -1 & -1 & . & . & \lfloor n/2 \rfloor - 1 & . & . & 0 & 0 \\ -1 & -1 & . & . & . & \lfloor n/2 \rfloor - 1 & . & 0 & 0 \\ \\ -1 & 0 & . & . & . & . & . & 1 & 0 \\ 0 & 0 & . & . & . & . & . & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$= \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix}$$

Expanding the above we get,

$$\begin{array}{rcll} (n-2)x_1 - x_2 - \cdot \cdot \cdot - x_{n-1} & = & \lambda x_1 & (1) \\ -1.x_1 + (n-3)x_2 - \cdot \cdot \cdot - x_{n-2} & = & \lambda x_2 & (2) \\ & \vdots & & \\ -1.x_1 - 0 - \cdot \cdot \cdot - 0 + x_{n-1} & = & \lambda x_{n-1} & (n-1) \\ 0.x_1 + 0.x_2 \cdot \cdot \cdot + 0.x_{n-1} + 0.x_n & = & \lambda x_n & (n) \end{array}$$

Let $U = (0, x_1, x_2, ..., x_n, 0)$, $n > 2$.

Claim: $L(G_{n+2}) \times U^T = (\lambda + 1)U^T$.

Consider each row R_i of $L(G_{n+2})$, $1 \leq i \leq n+2$, then $R_1 \times U^T = -x_1 - x_2 - \cdots - x_n = -1(1, 1, ..., 1).(x_1, x_2, ..., x_{n-1}, x_n) = 0$, since it is the dot product of two eigenvectors of symmetric matrix $L(G_n)$.

Expanding $L(G_{n+2}) \times U^T$ and using equations from (1) to (n), for each $R_i \times U^T$, we get $(\lambda + 1)x_{i-1}$ as common term, $2 \leq i \leq n + 1$. Also every entires in R_{n+2} is zero and so $R_{n+2} \times U^T = 0 = (\lambda + 1).0$.

Thus, whenever $\lambda \neq 0$ is an eigenvalue for $L(G_n)$, $(\lambda + 1)$ is an eigenvalue for $L(G_{n+2})$. □

We deonte the set of all Laplacian eigenvalues of a graph G by $\mu(G)$.

Theorem 2.3. *For $n \geq 2$, Laplacian eigenvalues of G_n are $\{0, 0, 1, 2, ..., n-2, n-1\} \setminus \{\lfloor n/2 \rfloor\}$ and of G_n^c are $\{0, 1, 2, ..., n-1, n\} \setminus \{\lceil n/2 \rceil\}$.*

Moreover, $\mu(G_{n+1}) = \{0\} \cup \mu(G_n^c)$.

Proof. We use principle of mathematical induction on n to prove the Theorem. Since we are using Theorem 2.2 in the proof, we consider even and odd case separetly. Consider the Laplacian matrix $L(G_{2n})$. For $n = 1$, $L(G_2)$ is zero matrix and hence the Theorem holds. For $L(G_4)$, we get $(1, 1, 1, 1)$ and $(0, 0, 0, 1)$ as the eigenvectors corresponding to eigenvalue 0 and using Theorem 2.1, we get $(-2, 1, 1, 0)$ and $(0, -1, 1, 0)$ as the eigenvectors corresponding to eigenvalues 3 and 1, respectively. Thus $\mu(G_4) = \{0, 0, 1, 3\}$. Note that when $n = 4$, $\lfloor n/2 \rfloor = 2$ is not an eigenvalue for $L(G_4)$. Thus the result is true for $n = 4$. Now consider $L(G_6)$. Clearly, 0 is an eigenvalue with multiplicity two and using Theorem 2.1, 1 and 7 are eigenvalues of $L(G_6)$. Since 1 and 3 are the eigenvalues of $L(G_4)$, 2 and 4 are eigenvalues of $L(G_6)$ using Theorem 2.2. Thus $\mu(G_6) = \{0, 0, 1, 2, 4, 5\}$. Note that 3 is not an eigenvalue of $L(G_6)$. Assume that Theorem 2.3 is true for $n = k$. ie. $\mu(G_{2k}) = \{0, 0, 1, 2..., k-1, k+1, ..., 2k-2, 2k-1\}$. Consider the graph $G_{2(k+1)}$. Using Theorem 2.1 and Theorem 2.2 we get $\mu(G_{2(k+1)}) = \{0, 0, 1, 2, ..., k, k+2, ..., 2k, 2k+1\}$. Thus by Induction, the Theorem holds for every $n \in \mathbb{N}$, i.e $\mu(G_n) = \{0, 0, 1, 2, ..., n-2, n-1\} \setminus \{\lfloor n/2 \rfloor\}$. Similarly we can prove the result for $L(G_{2n+1})$. Hence the result is true for all G_n, $n \in \mathbb{N}$ and $n \geq 2$.

To prove for anti-sum graph we use Theorem 1.2. An isolated vertex give an empty row and column. Thus $\mu(G_{n+1}) = \mu(G_n^c) \cup \{0\}$. i.e, eigenvalues of G_{n+1} are the set of all eigenvalues of G_n^c and 0. Now we have $\mu(G_n) = \{0, 0, 1, 2, ..., n-2, n-1\} \setminus \{\lfloor n/2 \rfloor\}$. Thus we get eigenvalues of G_n^c is $\{0, 1, 2, ..., n-1, n\} \setminus \lceil n/2 \rceil$, since $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. □

From Theorem 2.3 and Theorem 1.5 we have the following results.

Corollary 2.1. *Characteristic polynomial of G_n and G_n^c are*

$$\Phi(G_n, x) = (-1)^n x^2 (x-1)(x-2)...(x-(n-1))/(x - \lfloor n/2 \rfloor),$$
$$\Phi(G_n^c, x) = (-1)^n x(x-1)(x-2)...(x-n))/(x - \lceil n/2 \rceil).$$

Corollary 2.2. $\mu(G_{0,m}) = \mu(G^c_{m+1})$.

Corollary 2.3. *Let $d(v_i)$ denote the degree of vertex v_i, then for a sum graph G_n and for $v_i \in G_n$, $\Sigma_i d(v_i) = \Sigma_i \mu_i(G_n) = 2\lfloor n/2\rfloor(\lceil n/2\rceil - 1)$, $\Sigma_i d(v_i) = \Sigma_i \mu_i(G^c_n) = 2\lfloor n/2\rfloor\lceil n/2\rceil$.*

2.2 Properties of sum graph

Theorem 2.4. *For $n > 0$, $n \in \mathbb{N}$, $G_n - \{v_{\lfloor n/2\rfloor}\} \cong_{wvl} G_{n-1}$.*

Proof. For $n = 1, 2$, the case is trivial. Consider odd and even cases on the order of the graph G_n separately. We prove only even case here, because the two cases are so similar. Let graph be G_{2r}, $r \geq 2$. Our aim is to prove $G_{2r} - \{v_r\} \cong_{wvl} G_{2r-1}$, $r > 0$. Let $V(G_{2r}) = \{v_1, v_2, ..., v_{2r}\}$ and $V(G_{2r-1}) = \{w_1, w_2, ..., w_{2r-1}\}$, i be the sum label of the vertex v_i in G_{2r} and j be the sum label of the vertex w_j in G_{2r-1}, $i = 1, 2, ..., 2r$ and $j = 1, 2, ..., 2r - 1$. Clearly, the set $K = \{v_1, v_2, ..., v_r\}$ is a clique and set $L = \{v_{r+1}, v_{r+2}, ..., v_{2r}\}$ is an independent set in G_{2r}. Also it is clear that the set $K_1 = \{v_1, v_2, ..., v_{r-1}, v_{r+1}\}$ is a clique and the set $L_1 = \{v_{r+2}, v_{r+3}, ..., v_{2r}\}$ is an independent set in $G_{2r} - \{v_r\}$ and the set $K_2 = \{w_1, w_2, ..., w_r\}$ is a clique and the set $L_2 = \{w_{r+1}, w_{r+2}, ..., w_{2r-1}\}$ is an independent set in (G_{2r-1}).

Define the mapping $f : V(G_{2r} - \{v_r\}) \to V(G_{2r-1})$ by $f(v_i) = w_i$, $1 \leq i \leq r - 1$ and $f(v_j) = w_{j-1}$, $r + 1 \leq j \leq 2r$. Clearly, this mapping preserves the adjacency between clique set K_1 and K_2. We now show that this map preserves adjacency between Independent sets L_1 and L_2. Suppose $v_j \in L_1$ and $v_i \in K_1$, then v_j is adjacent to v_i if and only if $1 \leq i + j \leq 2r$, by definition of G_n. And this is same as $3 \leq i + j \leq 2r$, since $i, j \neq 0$ and $r > 1$. Clearly, $f(v_j) = w_{j-1} \in L_2$ and $f(v_i) = w_i \in K_2$. And w_j is adjacent to w_i if and only if $2 \leq i + j - 1 \leq 2r - 1$, ie. $3 \leq i + j \leq 2r$. This implies v_j is adjacent to v_i if and only if $f(v_j)$ is adjacent to $f(v_i)$. Hence the adjacency is preserved.

Similarly we can prove that $G_{2r+1} - \{v_{\lfloor (2r+1)/2\rfloor}\} \cong_{wvl} G_{2r}$. □

Theorem 2.5. *For $n > 0$, $G_{0,n} - \{v_{\lfloor n/2\rfloor}\} \cong_{wvl} G^c_n$.*

Proof. For convenience, the odd and even cases on the order of the graph G_n are considered separetely and because the two cases are so similar, we prove only the even case here. Let graph be $G_{0,2r}$, $r > 0$. We prove $G_{0,2r} - \{v_r\} \cong_{wvl} G^c_{2r}$. Let $V(G_{0,2r} - \{v_r\}) = \{v_0, v_1, ..., v_{r-1}, v_{r+1}, ..., v_{2r}\}$ and $V(G^c_{2r}) = \{w_1, w_2, ..., w_{2r}\}$, both subscript-labeled. Clearly, in $G_{0,2r} - \{v_r\}$, the set $K_1 = \{v_0, v_1, ..., v_{r-1}, v_{r+1}\}$ induces a clique and the set $L_1 = \{v_{r+2}, v_{r+3}, ..., v_{2r}\}$ induces an independent set. For (G^c_{2r}), clearly, the set $K_2 = \{w_{2r}, w_{2r-1}, ..., w_r\}$ induces a clique and the set $L_2 = \{w_{r-1}, w_{r-2}, ..., w_1\}$ induces an independent set.

Define the mapping $f : V(G_{0,2r} - \{v_r\}) \to V(G^c_{2r})$ by $f(v_i) = w_{2r-i}$ for $i = 0, 1, 2, ..., r - 1$ and $f(v_j) = w_{2r-j+1}$ for $j = r + 1, r + 2, ..., 2r$. Clearly, this mapping preserves the adjacency between clique set K_1 and K_2. We now show that this map preserves adjacency between Independent sets L_1 and L_2. Suppose

$v_j \in L_1$ and $v_i \in K_1$, then v_j is adjacent to v_i if and only if $1 \le i + j \le 2r$. Clearly, $f(v_j) = w_{2r-j+1} \in L_2$ and $f(v_i) = w_{2r-i} \in K_2$. And w_{2r-j+1} is adjacent to w_{2r-i} if and only if $2r - j + 1 + 2r - i \ge 2r + 1$, ie. $i + j \le 2r$ and clearly $0 \le i + j$. This implies, v_i adjacent to v_j if and only if $f(v_i)$ is adjacent to $f(v_j)$. Hence the adjacency is preserved in this mapping.

Similarly we can prove that $G_{0,2r+1} - \{v_{\lfloor (2r+1)/2 \rfloor}\} \cong_{wvl} G^c_{2r+1}$. □

3. Laplacian eigenvalues of $G_{m,n}$

From the above discusions on Laplacian eigenvalues of sum graph and invoking Theorem 1.4 and Theorem 1.5 one may conclude that

Lemma 3.1. *For $m, n \in \mathbb{N} \cup \{0\}$, $G_{m,n} \cong_{wvl} G^c_{m+1} \nabla G_n$.*

Also considering Theorem 1.3 and Theorem 2.3 we get the Laplacian eigenvalues of $G_{m,n}$ as follows

Theorem 3.1. *For $m, n \in \mathbb{N} \cup \{0\}$, $\mu(G_{m,n}) = \{n + m + 1, m + 1 + n - 1, m + 1 + n - 2, .., m + 1 + 2, m + 1 + 1, m + 1 + 0, n + m + 1, n + m, ..., n + 2, n + 1, 0\} \backslash \{n + \lceil m + 1/2 \rceil, m + 1 + \lfloor n/2 \rfloor\}$.*

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