Lyapunov-type inequalities for ψ-Laplace equations

Xu Guo
School of Information Science & Technology
Southwest Jiaotong University
Chengdu, Sichuan, 611756
P.R. China
guoxu@my.swjtu.edu.cn

Jun Zheng*
School of Mathematics
Southwest Jiaotong University
Chengdu, Sichuan, 611756
P.R. China
zhengjun@swjtu.edu.cn

Abstract. In this paper, we establish several Lyapunov-type inequalities for a class of ψ-Laplace equations

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0$$

with Dirichlet boundary conditions, where ψ, f are nonlinear functions defined on $\mathbb{R}$ and one of them is imposed on structural conditions of Tolksdorf type. The obtained Lyapunov-type inequalities are extensions and complements of the known results in the sense that compared with the existing literature, neither sub-multiplicative property of ψ nor convexity of $\frac{1}{\psi(t)}$ (or $\psi(t)t$) is required in this paper.
Keywords: Lyapunov inequality; ψ-Laplace, nonlinear differential equation.

1. Introduction

For the Hill's equation

$$u''(x) + r(x)u(x) = 0, \quad x \in (a, b), \tag{1a}$$

$$u(a) = u(b) = 0, \tag{1b}$$

where r is a continuous and nonnegative function defined on $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, if there exists a nontrivial solution u of (1), then the inequality

$$\int_a^b r(x)\mathrm{d}x \geq \frac{4}{b-a} \tag{2}$$

holds. This result is originally due to Lyapunov [12], and is known as "Lyapunov inequality". Lyapunov inequality and many of its generalizations have proved

*. Corresponding author

to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations, and also in time scales. In the past few years, independent works appeared generalizing Lyapunov inequality for the p-Laplacian, by using Hölder, Jensen or Cauchy-Schwarz inequalities. A thorough literature review of Lyapunov-type inequalities and their applications can be found in the survey articles by Brown and Hinton [3], Cheng [6] and Tiryaki [18]. Some other related topics can be found in the resent articles [1, 2, 4, 5, 8, 9, 10, 13, 14, 15, 17, 19, 21, 22, 23, 24, 25] and the references therein.

It should be mentioned that De Nápoli and Pinasco considered Lyapunov-type inequalities for certain nonlinear differential equations (ψ-Laplace equations) in [7]. The main result they obtained is the following theorem.

Theorem A. Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function such that $t\psi(t)$ is a convex function in t. Besides, suppose that there exists a constant $k > 0$ such that $\psi(2t) \leq k\psi(t)$ for any $t \geq 0$. If r is a positive integrable function and the following equation with Dirichlet boundary conditions

$$(\psi(u'))' + r\psi(u) = 0 \quad \text{in } (a, b), \tag{3a}$$

$$u(a) = u(b) = 0, \tag{3b}$$

admits a nontrivial solution, then

$$2\left(\frac{k}{2}\right)^{[1-\log_2(b-a)]} \leq \int_a^b r(x)\mathrm{d}x,$$

where $[v]$ is the largest integer less than or equal to v.

In 2011, Sáchez and Vergara extended (3) to the following equations having a general nonlinear form (see [16]):

$$(\psi(u'))' + \lambda r f(u) = 0 \quad \text{in } (a, b), \tag{4a}$$

$$u(a) = u(b) = 0, \tag{4b}$$

where $\lambda > 0$ is a constant and
($\boldsymbol{A_1}$) $f \in C(\mathbb{R}; \mathbb{R})$ is odd and satisfies $tf(t) > 0$ for $t \neq 0$;
($\boldsymbol{A_2}$) $r \in C([a, b]; (0, +\infty))$;
($\boldsymbol{A_3}$) ψ is odd, increasing, and sub-multiplicative on $[0, +\infty)$, and $\frac{1}{\psi(t)}$ is convex on $(0, +\infty)$.

Under the above assumptions, the authors obtained the following theorem.

Theorem B. If u is a nontrivial solution of (4), satisfying $u(x) \neq 0$ for all $x \in (a, b)$, then the following inequality holds:

$$\frac{2}{\psi\left(\frac{b-a}{2}\right)} \leq \lambda \int_a^b \frac{f(u(x))}{\psi(u(x))}\mathrm{d}x,$$

whenever the integral exists.

It should be noticed that the convexity of $t\psi(t)$ proposed in [7], the convexity of $\frac{1}{\psi(t)}$ and the sub-multiplicative property of ψ proposed in [16] play essential roles in the establishment of a Lyapunov-type inequality for ψ-Laplace equations, which motivate us to consider the problem of a large class of nonlinear equations. More precisely, in this paper, we establish Lyapunov-type inequalities for ψ-Laplace equations governed by (5) (or (4)) with structural conditions of Tolksdorf type. With these conditions, the function ψ is allowed to permit much more nonlinearities than those in [7, 16], e.g., there is no a convexity assumption on $t\psi(t)$ or $\frac{1}{\psi(t)}$, and no a sub-multiplicative assumption on ψ (see Remark 2.2 in Section 1 and examples in Section 4). Moreover, in view of ($\boldsymbol{H_4}$) in Section 2, we do not impose any odd-even assumptions on f, and require less sign conditions of f than those in [16].

The rest of this paper is organized as follows. Section 2 presents the considered problem and the main results on Lyapunov-type inequalities. Some remarks on the structural conditions are also provided in this section. Detailed proofs of Lyapunov-type inequalities are given in Section 3. Additional examples satisfying our structural conditions are provided in Section 4.

2. Problem setting and main results

In this paper, we establish Lyapunov-type inequalities for the following equation

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0, \qquad x \in (a,b), \tag{5a}$$

$$u(a) = u(b) = 0, \tag{5b}$$

where ψ and f satisfy the following structural conditions:

($\boldsymbol{H_1}$) $\psi, f \in C(-\infty,\infty) \cap C^1(0,\infty)$ with $f \not\equiv 0$ on $(-\infty,\infty)$;

($\boldsymbol{H_2}$) ψ is odd on $(-\infty,\infty)$;

($\boldsymbol{H_3}$) $f(t) \geq 0$ for all $t \in [0,\infty)$;

($\boldsymbol{H_4}$) There exists $k_0 > 0$ such that $|f(t)| \leq k_0\psi(|t|)$ for all $t \in (-\infty,\infty)$.

Moreover, ψ or f satisfies the structural condition of Tolksdorf type:

($\boldsymbol{H_\psi}$) There exist constants $\delta_0, \delta_1 \geq 0$ such that

$$\delta_0\psi(t) \leq t\psi'(t) \leq \delta_1\psi(t), \quad \forall t > 0;$$

or,

($\boldsymbol{H_f}$) There exist constants $\theta_0, \theta_1 \geq 0$ such that

$$\theta_0 f(t) \leq tf'(t) \leq \theta_1 f(t), \quad \forall t > 0.$$

Throughout this paper, we always assume that $r \in L^1(a,b)$ with $r \not\equiv 0$ on (a,b), and conditions $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ are satisfied. Moreover, we always assume that (5) has a non-trivial solution u in the sense that $u \in C^1(a,b) \cap C([a,b])$, $\psi(u'(x))$ is absolutely continuous in x, and u satisfies (5) almost everywhere in (a,b).

The first main result is stated as follows, which can be seen as a complement of the work of [7] and [16] in the setting of functions satisfying the structural condition $(\boldsymbol{H_\psi})$ or $(\boldsymbol{H_f})$.

Theorem 2.1. *The following statements hold true:*

(*i*) *If* ψ *satisfies* $(\boldsymbol{H_\psi})$, *then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\delta_0}{1+\delta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$;

(*ii*) *If* f *satisfies* $(\boldsymbol{H_f})$, *then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\theta_0}{1+\theta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

Corollary 2.1. *The following statements hold true:*

(*i*) *If* $\psi(t) = f(t) = |t|^{p-2}t$ $(p > 1)$, *then* $\int_a^b |r(x)|dx \geq \frac{2^p}{(b-a)^{p-1}}$, *which is the same as one of the results obtained in [10, 13];*

(*ii*) *If* $\psi(t) = f(t) = |t|^{a-1}t \log_c(b|t| + d),\ a, b > 0, c, d > 1$, *then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{1+(1+a)\ln d} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a+\frac{1}{\ln d}}\right\}$;

(*iii*) *If* $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)},\ b > 0, c, d > 1, a > \frac{1}{\ln d}$, *then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{(1+a)\ln d - 1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a-\frac{1}{\ln d}}\right\}$.

If we make a further assumption that $\psi(t)t$ (or $f(t)t$) is convex on $[0+\infty)$, we can obtain the following stronger results, which can be seen as extensions of the work in [7].

Theorem 2.2. *The following statements hold true:*

(*i*) *If* ψ *satisfies* $(\boldsymbol{H_\psi})$ *and* $\psi(t), t$ *is convex in* $t \in [0, +\infty)$, *then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$;

(*ii*) *If* f *satisfies* $(\boldsymbol{H_f})$ *and* $f(t), t$ *is convex in* $t \in [0, +\infty)$, *then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

We provide some remarks on the structural conditions.

Remark 2.1. (*i*) We point out that a slight version of the structural condition $(\boldsymbol{H_\psi})$ (or $(\boldsymbol{H_f})$) was firstly introduced by Tolksdorf in [20], i.e., $0 < \delta_0 \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1, \forall t > 0$, which is a generalization of the natural conditions of Ladyzhenskaya and Uralľtseva for elliptic equations (see [11]). It should be noticed that $\frac{t\psi'(t)}{\psi(t)}$ is always required to be positive in [11, 20], while in this paper $\psi'(t)$ in $(\boldsymbol{H_\psi})$ (or $f'(t)$ in $(\boldsymbol{H_f})$) can be zero at some point $t_0 > 0$, i.e., $\delta_0 = 0$ (or $\theta_0 = 0$). Indeed, considering $\psi(t) = \begin{cases} \frac{7}{8} \cdot (2t)^{\frac{3}{7}}, \ 0 < t < \frac{1}{2}, \\ (t-1)^3 + 1, \ \frac{1}{2} \le t \le \frac{3}{2}, \\ \frac{3}{4}t, \ t > \frac{3}{2}, \end{cases}$ we have $\psi \in C^1((0,+\infty))$ and $0 \le \frac{t\psi'(t)}{\psi(t)} \le 3$ for all $t > 0$. The lower boundedness of $\frac{t\psi'(t)}{\psi(t)}$ can be achieved when $\psi'(1) = 3(t-1)^2\big|_{t=1} = 0$.

(*ii*) By ($\boldsymbol{H_1}$)-($\boldsymbol{H_4}$), it follows that $\psi(0) = f(0) = 0$ and $\psi(t) \ge 0$ for any $t \ge 0$. Furthermore, if ψ (or f) satisfies ($\boldsymbol{H_\psi}$) (or ($\boldsymbol{H_f}$)), then $\psi'(t) \ge 0$ (or $f'(t) \ge 0$), which guarantees the increasing monotonicity of $\psi(t)$ (or $f(t)$) in $t \ge 0$.

Remark 2.2. The following two examples show that our assumptions on ψ (or f) are much weaker than those in [7, 16] (see Theorem A and B) in a certain sense. More examples of functions satisfying ($\boldsymbol{H_\psi}$) or ($\boldsymbol{H_f}$) are provided in Section 4.

(i) Let $\psi_0(t) = t + \frac{1}{t}$ with $t \in [\frac{6}{5}, \sqrt{3}]$. Due to the continuities of ψ_0 and ψ_0', there exist $\widehat{\delta_0}, \widehat{\delta_1} > 0$ such that $\widehat{\delta_0} \le \frac{t\psi_0'(t)}{\psi_0(t)} \le \widehat{\delta_1}$ for all $t \in [\frac{6}{5}, \sqrt{3}]$. Let $p = \frac{11}{61}, a = \left(\frac{6}{5} + \frac{5}{6}\right)\left(\frac{6}{5}\right)^{-p}$ and $q = \frac{1}{2}, b = \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)(\sqrt{3})^{-q}$. Let $\psi(t) = \begin{cases} at^p, \ 0 < t < \frac{6}{5} \\ \psi_0(t), \ \frac{6}{5} \le t \le \sqrt{3} \\ bt^q, \ t > \sqrt{3} \end{cases}$. By direct computations, we have $\left(\frac{1}{\psi(t)}\right)'' = \left(\frac{1}{\psi_0(t)}\right)'' = \frac{-2}{(t+1)^3} < 0$ for $t \in (\frac{6}{5}, \sqrt{3})$. Thus $\frac{1}{\psi}$ is not convex on $[0, +\infty)$, i.e, ψ does not satisfy the condition ($\boldsymbol{A_3}$). However, ψ defined as above satisfies ($\boldsymbol{H_\psi}$) with $\delta_0 = \min\{\widehat{\delta_0}, p, q\}$ and $\delta_1 = \max\{\widehat{\delta_1}, p, q\}$ in this paper.

(ii) Let $\psi_0(t) = \sin t$ with $t \in (0, \frac{\pi}{2})$. Then $(\psi_0(t)t)'' = -t\sin t + 2\cos t \to -\frac{\pi}{2} < 0$ as $t \to \left(\frac{\pi}{2}\right)^-$. Thus there exists $[t_0, t_1] \subset (0, \frac{\pi}{2})$ such that $(\psi_0(t)t)'' < 0$ for all $t \in [t_0, t_1]$. Let $\psi(t) = \begin{cases} at^p, \ 0 < t < t_0 \\ \psi_0(t), \ t_0 \le t \le t_1 \\ bt^q, \ t > t_1 \end{cases}$, where $p = t_0 \cot t_0 > 0, a = t_0^{-p} \sin t_0 > 0$ and $q = t_1 \cot t_1 > 0, b = t_1^{-q} \sin t_1 > 0$. Note that there exist $\widehat{\delta_0}, \widehat{\delta_1} > 0$ such that $\widehat{\delta_0} \le \frac{t\psi_0'(t)}{\psi_0(t)} \le \widehat{\delta_1}$ for all $t \in [t_0, t_1]$. It's easy to see that ψ satisfies ($\boldsymbol{H_\psi}$) with $\delta_0 = \min\{\widehat{\delta_0}, p, q\}$ and $\delta_1 = \max\{\widehat{\delta_1}, p, q\}$. However, $(\psi(t)t)'' < 0$ for $t \in [t_0, t_1]$. That is to say such a function ψ does not satisfy the convexity condition in [7] while it still satisfies ($\boldsymbol{H_\psi}$) in this paper.

3. Proof of main results

We present first some auxiliary results needed for the main proof. Let $\Psi(t) = \int_0^t \psi(s)\mathrm{d}s$ for $t \geq 0$.

Lemma 3.1. *Assume that ψ satisfies $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ and $(\boldsymbol{H_\psi})$. The following statements hold true:*

(i) $\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t),\ \forall s, t \geq 0$;

(ii) Ψ *is C^2-contiuous on* $(0, +\infty)$ *and convex on* $[0, +\infty)$;

(iii) $\frac{t\psi(t)}{1+\delta_1} \leq \Psi(t) \leq \frac{t\psi(t)}{1+\delta_0},\ \forall t \geq 0$.

Proof. Let $h_0(t) = \frac{\psi(t)}{t^{\delta_0}}, h_1(t) = \frac{\psi(t)}{t^{\delta_1}}$ for $t > 0$. By $(\boldsymbol{H_\psi})$, it follows that

$$h_0'(t) = \frac{\psi'(t)t^{\delta_0} - \psi(t)\delta_0 t^{\delta_0 - 1}}{t^{2\delta_0}} = \frac{t\psi'(t) - \psi(t)\delta_0}{t^{\delta_0+1}} \geq 0,$$

which implies that $h_0(t)$ is increasing in $t > 0$. Therefore $h_0(st) \leq h_0(t)$ for $0 \leq s \leq 1$. It follows that

$$\psi(st) \leq s^{\delta_0}\psi(t),\ \forall t > 0, 0 \leq s \leq 1. \tag{6}$$

Similarly, one may prove that $h_1(t)$ is decreasing in $t > 0$. Then $h_1(st) \leq h_1(t)$ for $s \geq 1$. It follows that

$$\psi(st) \leq s^{\delta_1}\psi(t),\ \forall t > 0, s \geq 1. \tag{7}$$

By (6) and (7), we have

$$\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t),\ \forall t > 0, s \geq 0,$$

which and the continuity of ψ in $t = 0$ yield (i).

(ii) is obvious since $\Psi''(t) = \psi'(t) \geq 0$ for $t > 0$ (see Remark 2.1) and $\Psi(t)$ is continuous in $t = 0$.

To conclude (iii), let $\Psi_0(t) = (1+\delta_0)\Psi(t) - t\psi(t)$ and $\Psi_1(t) = (1+\delta_1)\Psi(t) - t\psi(t)$ for $t \geq 0$. It is easy to see that $\Psi_0'(t) \leq 0$ and $\Psi_1'(t) \geq 0$ for $t > 0$. Then $\Psi_0(t) \leq \Psi_0(0) = 0$ and $\Psi_1(t) \geq \Psi_1(0) = 0$, which and the continuities of Ψ_0, Ψ_1 yield (iii). □

Remark 3.1. Let $F(t) = \int_0^t f(s)\mathrm{d}s$ for $t \geq 0$. For the function f satisfying $(\boldsymbol{H_1})$-$(\boldsymbol{H_4})$ and $(\boldsymbol{H_f})$, and the function F, one may obtain similar properties as above.

Proof of Theorem 2.1. Without loss of generality, assume that $|u(c)| = \max_{x\in[a,b]} |u(x)| > 0$ with $c \in (a,b)$. Note that

$$|u(c)| = \frac{1}{2}\Big(\big|\int_a^c u'(x)\mathrm{d}x\big| + \big|\int_c^b u'(x)\mathrm{d}x\big|\Big) \le \frac{1}{2}\int_a^b |u'(x)|\mathrm{d}x.$$

Firstly, we prove Theorem 2.1 (i) under the assumption that ψ satisfies the structural condition ($\boldsymbol{H_\psi}$). Indeed, by the monotonicity of ψ and Lemma 3.1 (i) (iii), we get

$$\begin{aligned}
&\psi(|u(c)|)|u(c)|\\
\le&\frac{1}{2}\cdot\psi\Big(\frac{1}{2}\int_a^b |u'(x)|\mathrm{d}x\Big)\cdot\int_a^b |u'(x)|\mathrm{d}x\\
=&\frac{b-a}{2}\cdot\psi\Big(\frac{b-a}{2}\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\Big)\cdot\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\\
\le&\max\Big\{\Big(\frac{b-a}{2}\Big)^{1+\delta_0},\Big(\frac{b-a}{2}\Big)^{1+\delta_1}\Big\}\cdot\psi\Big(\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\Big)\cdot\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\\
\le&\max\Big\{\Big(\frac{b-a}{2}\Big)^{1+\delta_0},\Big(\frac{b-a}{2}\Big)^{1+\delta_1}\Big\}\cdot(1+\delta_1)\cdot\Psi\Big(\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\Big)\\
\le&\max\Big\{\Big(\frac{b-a}{2}\Big)^{1+\delta_0},\Big(\frac{b-a}{2}\Big)^{1+\delta_1}\Big\}\cdot(1+\delta_1)\cdot\frac{1}{b-a}\cdot\int_a^b \Psi(|u'(x)|)\mathrm{d}x,
\end{aligned}$$

where in the last inequality we used the convexity of Ψ (see Lemma 3.1 (ii)). It follows that

$$\psi(|u(c)|)|u(c)| \le (1+\delta_1)\cdot\max\Big\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\Big\}\cdot\int_a^b \Psi(|u'(x)|)\mathrm{d}x. \tag{8}$$

Multiplying (5) by u, integrating over (a,b), using Lemma 3.1 (iii), ($\boldsymbol{H_2}$)-($\boldsymbol{H_4}$) and (8), we get

$$\begin{aligned}
\int_a^b \Psi(|u'|)\mathrm{d}x \le&\frac{1}{1+\delta_0}\int_a^b \psi(|u'|)|u'|\mathrm{d}x\\
=&\frac{1}{1+\delta_0}\int_a^b \psi(u')u'\mathrm{d}x\\
=&\frac{1}{1+\delta_0}\int_a^b r(x)f(u)u\mathrm{d}x\\
\le&\frac{1}{1+\delta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|\mathrm{d}x\\
\le&\frac{k_0}{1+\delta_0}\max_{x\in[a,b]}(\psi(|u|)|u|)\int_a^b |r(x)|\mathrm{d}x\\
\le&\frac{k_0(1+\delta_1)}{1+\delta_0}\cdot\max\Big\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\Big\}\cdot\int_a^b\Psi(|u'|)\mathrm{d}x\cdot\int_a^b |r(x)|\mathrm{d}x.
\end{aligned}$$

Note that $\int_a^b \Psi(|u'|)\mathrm{d}x > 0$, otherwise, $\int_a^b \Psi(|u'|)\mathrm{d}x = 0$. By (8) and Lemma 3.1 (i), we have,

$$\begin{aligned}0 \leq& \psi(t)|u(c)| = \psi\left(\frac{t}{u(c)}u(c)\right)|u(c)|\\ \leq& \max\left\{\left(\frac{t}{u(c)}\right)^{\delta_0}, \left(\frac{t}{u(c)}\right)^{\delta_1}\right\}\psi(u(c))|u(c)|\\ \leq& 0, \quad \forall t \geq 0,\end{aligned}$$

which implies $\psi \equiv 0$ for all $t \in [0, +\infty)$. Then by the odd property of ψ, we have $\psi \equiv 0$ for all $t \in (-\infty, +\infty)$. Due to $(\boldsymbol{H_4})$, $f \equiv 0$ for all $t \in (-\infty, +\infty)$, which is a contradiction with the assumption $(\boldsymbol{H_1})$.

Then we get

$$\int_a^b |r(x)|\mathrm{d}x \geq \frac{1+\delta_0}{k_0(1+\delta_1)} \cdot \min\left\{\frac{2^{1+\delta_0}}{(b-a)^{\delta_0}}, \frac{2^{1+\delta_1}}{(b-a)^{\delta_1}}\right\}.$$

Theorem 2.1 (i) has been proved.

Now for Theorem 2.1 (ii), we proceed in a similar way as above. Indeed, if f satisfies the structural condition $(\boldsymbol{H_f})$, proceeding as in (8), we get

$$f(|u(c)|)|u(c)| \leq (1+\theta_1) \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'(x)|)\mathrm{d}x. \tag{9}$$

Multiplying (5) by u, integrating over (a, b), using Lemma 3.1 (iii) and $(\boldsymbol{H_2})$-$(\boldsymbol{H_4})$, we get

$$\begin{aligned}\int_a^b F(|u'|)\mathrm{d}x \leq& \frac{1}{1+\theta_0}\int_a^b f(|u'|)|u'|\mathrm{d}x\\ \leq& \frac{k_0}{1+\theta_0}\int_a^b \psi(|u'|)|u'|\mathrm{d}x\\ =& \frac{k_0}{1+\theta_0}\int_a^b \psi(u')u'\mathrm{d}x\\ =& \frac{k_0}{1+\theta_0}\int_a^b r(x)f(u)u\mathrm{d}x\\ \leq& \frac{k_0}{1+\theta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|\mathrm{d}x\\ \leq& \frac{k_0}{1+\theta_0}f(|u(c)|)|u(c)|\int_a^b |r(x)|\mathrm{d}x,\end{aligned}$$

which and (9) give

$$\int_a^b F(|u'|)\mathrm{d}x \leq \frac{k_0(1+\theta_1)}{1+\theta_0} \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x. \tag{10}$$

Note that $\int_a^b F(|u'|)\mathrm{d}x > 0$, otherwise, we can argue as in the proof of (i) to conclude $f(t) \equiv 0$ for any $t \in (-\infty, +\infty)$. Finally, (10) implies the desired result. □

Proof of Theorem 2.2. The proof of Theorem 2.2 is a slight modification of the proof of Theorem 2.1. Indeed, let $\Phi(t) = \psi(t)t$ for $t \geq 0$, and let $c, u(c)$ be defined as in the proof of Theorem 2.1. If ψ satisfies the structural condition $(\boldsymbol{H_\psi})$, arguing as in (8), we get

$$\begin{aligned}&\psi(|u(c)|)|u(c)|\\ \leq&\max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \psi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right) \cdot \frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\\ =&\max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \Phi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right),\end{aligned}$$

which and the convexity of Φ give

$$\psi(|u(c)|)|u(c)| \leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \frac{1}{b-a}\int_a^b \Phi(|u'|)\mathrm{d}x. \tag{11}$$

Concerning with (5) and (11), we have

$$\begin{aligned}\int_a^b \Phi(|u'|)\mathrm{d}x &= \int_a^b \psi(|u'|)|u'|\mathrm{d}x\\ &= \int_a^b \psi(u')u'\mathrm{d}x\\ &= \int_a^b r(x)f(u)u\mathrm{d}x\\ &\leq k_0\psi(|u(c)|)|u(c)|\int_a^b |r(x)|\mathrm{d}x\\ &\leq \frac{k_0}{2}\max\left\{\left(\frac{b-a}{2}\right)^{\delta_0}, \left(\frac{b-a}{2}\right)^{\delta_1}\right\} \cdot \int_a^b \Phi(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x,\end{aligned}$$

which yields the desired result in Theorem 2.2 (i). The desired result in Theorem 2.2 (ii) can be proved in a similar way. □

Proof of Corollary 2.1. For (i), it should be noticed that $\delta_0 = \theta_0 = \delta_1 = \theta_1 = p - 1$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

For (ii), it should be noticed that for $t \geq 0$,

$$\psi(t) = f(t) = t^a \log_c(bt + d),\ \ a, b > 0, c, d > 1.$$

Then we have

$$\begin{aligned}a \leq&\frac{t\psi'(t)}{\psi(t)} = a + \frac{bt}{(bt+d)\ln c \cdot \log_c(bt+d)}\\ \leq&a + \frac{bt}{(bt+d)\ln c \cdot \log_c d}\\ \leq&a + \frac{1}{\ln d},\quad \forall t > 0.\end{aligned}$$

Thus $\delta_0 = \theta_0 = a > 0, \delta_1 = \theta_1 = a + \frac{1}{\ln d} > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

For (iii), it should be noticed that for $t \geq 0$,

$$\psi(t) = f(t) = \frac{t^a}{\log_c(bt+d)}, \ b > 0, c, d > 1, a > \frac{1}{\ln d}.$$

Then we have

$$a - \frac{1}{\ln d} \leq \frac{t\psi'(t)}{\psi(t)} = a - \frac{bt}{(bt+d)\ln(bt+d)} \leq a, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a - \frac{1}{\ln d} > 0, \delta_1 = \theta_1 = a > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$. □

4. Examples

In this section, we provide more examples of $\psi(t)$ (or $f(t)$) satisfying $(\boldsymbol{H_\psi})$ (or $(\boldsymbol{H_f})$). For simplicity, we only restrict $\psi(t)$ (or $f(t)$) to the case $t \in [0, +\infty)$, since one may construct functions by odd or even extensions to $t \in (-\infty, +\infty)$.

Example 4.1.

$$\psi(t) = f(t) = \ln(1+at) + bt, \quad \forall a > 0, b > 0. \tag{12}$$

For (12), we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{at}{1+at}\frac{1}{\ln(1+at)+bt} + \frac{bt}{\ln(1+at)+bt} \leq \frac{a}{b} + 1, \quad \forall t > 0.$$

Note that $\ln(1+at) \leq at$ for all $t > 0$, it follows that

$$\frac{b}{a+b} \leq \frac{t\psi'(t)}{\psi(t)} \leq \frac{a}{b} + 1, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = \frac{b}{a+b} > 0, \delta_1 = \theta_1 = \frac{a}{b} + 1 > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.2.

$$\psi(t) = f(t) = (1+t)\ln(1+t) - t. \tag{13}$$

For (13), firstly, note that $\psi'(t) = \ln(1+t) \geq 0$ for any $t \geq 0$. Thus $\psi(t) \geq \psi(0) = 0$. By direct computations, we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{t\ln(1+t)}{(1+t)\ln(1+t) - t} = \frac{t}{(1+t) - \frac{t}{\ln(1+t)}}, \quad \forall t > 0.$$

Since $\ln(1+t) \leq t$ for all $t > 0$, it follows that

$$\frac{t}{(1+t) - \frac{t}{\ln(1+t)}} \geq \frac{t}{(1+t) - \frac{t}{t}} = 1, \quad \forall t > 0.$$

Now we prove that for any $t > 0$, there holds

$$\frac{t\ln(1+t)}{(1+t)\ln(1+t)-t} \le 2. \tag{14}$$

Indeed, let $h_1(t) = t\ln(1+t) - 2((1+t)\ln(1+t)-t) = 2t - t\ln(1+t) - 2\ln(1+t)$. Then $h_1'(t) = 1 - \ln(1+t) - \frac{1}{1+t}$. Let $h_2(t) = (1+t) - (1+t)\ln(1+t) - 1 = t - (1+t)\ln(1+t)$. It is easy to check that $h_2'(t) = -\ln(1+t) < 0$ for any $t > 0$. Thus $h_2(t) \le h_2(0) = 0$, which leads to $h_1'(t) \le 0$ for any $t > 0$. Therefore $h_1(t) \le h_1(0) = 0$. As a consequence, (14) holds true for any $t > 0$. Finally $\delta_0 = \theta_0 = 1, \delta_1 = \theta_1 = 2$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.3.

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \le t < t_0, \\ bt^q + c, t \ge t_0, \end{cases} \tag{15}$$

where $a, b, c, p, q, t_0 > 0$ satisfying $at_0^p = bt_0^q + c$ and $apt_0^{p-1} = bqt_0^{q-1}$.

For (15), we have $\psi = f \in C^1((0,+\infty))$ and $\min\{p,q\} \le \frac{t\psi'(t)}{\psi(t)} \le \max\{p,q\}$. Thus $\delta_0 = \theta_0 = \min\{p,q\} > 0, \delta_1 = \theta_1 = \max\{p,q\} > 0$ in $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$.

Example 4.4. The following example is interesting since ψ or f is with a variable exponent:

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \le t < t_0, \\ bt^{g(t)-1}, t \ge t_0, \end{cases}$$

where $t_0 > 1$, $a, b, p > 0$ and $g \in C^1([t_0, +\infty)$ satisfying

$$\begin{cases} c \le g'(t)t\ln t + g(t) - 1 \le d, \forall t \ge t_0, \\ p = g'(t_0)t_0\ln t_0 + g(t_0) - 1, \\ a = bt_0^{g(t_0)-1-p}, \end{cases}$$

with some constants c, d satisfying $d \ge c > 0$. Note that $\frac{t(bt^{g(t)-1})'}{bt^{g(t)-1}} = tg'(t)\ln t + g(t) - 1$. By direct computations, one may verify that $\psi = f \in C^1((0,+\infty))$, and satisfy $(\boldsymbol{H_\psi})$ and $(\boldsymbol{H_f})$ with $\delta_0 = \theta_0 = \min\{p,c\} = c, \delta_1 = \theta_1 = \max\{p,d\} = d$.

Example 4.5. In [16], the authors provided two examples of $\psi(t)$, i.e,

(*i*) $\psi(t) = |t|^a\varphi_p(t)$ with $a > 1-p$, and

(*ii*) $\psi(t) = (\ln(|t|+a))^b\varphi_p(t)$ with $a \ge e, b > 0$,

showing that $\frac{1}{\psi(t)}$ is convex in $t > 0$, where $\varphi_p(t) = |t|^{p-2}t$ $(p > 1)$. We point out that $\psi(t)$ given by (i) or (ii) also satisfies the structural condition $(\boldsymbol{H_\psi})$. Indeed, for (i), it is easy to see that $\frac{t\psi'(t)}{\psi(t)} = a+p-1 > 0$ for $t > 0$. Thus $\delta_0 = \delta_1 = a+p-1$ in $(\boldsymbol{H_\psi})$. For (ii), by direct computations, we have $\frac{t\psi'(t)}{\psi(t)} = p + \frac{bt}{(t+a)\ln(t+a)}$ for $t > 0$. Note that $0 \le \frac{bt}{(t+a)\ln(t+a)} \le \frac{b}{a\ln a}$ for $t > 0$. Then $\delta_0 = p, \delta_1 = p + \frac{b}{a\ln a}$ in $(\boldsymbol{H_\psi})$.

References

[1] R. P. Agarwal, A. Özbekler, *Lyapunov type inequalities for second order sub and super-half-linear differential equations*, Dyn. Syst. Appl., 24 (2015), 211-220.

[2] P. Almenar, L. Jódar, *A Lyapunov inequality for a second order nonlinear differential equation*, Appl. Math. Lett., 24 (2011), 524-527.

[3] R. C. Brown, D. B. Hinton, *Lyapunov inequalities and their applications, in: T. M. Rassias (Ed.), Survey on Classical Inequalities*, Kluwer AcademicPublishers, Dordrecht, The Netherlands, 2000.

[4] D. Çakmak, *On Lyapunov-type inequality for a class of nonlinear systems*, Electron. J. Qual. Theory Differ. Equ., 16 (2013), 101-108.

[5] D. Çakmak, *On Lyapunov-type inequality for a class of quasilinear systems*, Electron. J. Qual. Theory Differ. Equ., (9)(2014), 1-10.

[6] S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math., 23(23)(1991), 25-41.

[7] P. L. De Nápoli, J. P. Pinasco, *A Lyapunov inequality for monotone quasilinear operators*, Differ. Integral Equ., 18 (2005), 1193-1200.

[8] O. Dosly, P. Rehak, *Half-linear differential equations, in: Math. Stud.*, volume 202. Amsterdam: Elsevier, North-Holland, 2005.

[9] G. Guseinov, B. Kaymakcalan, *Lyapunov inequalities for discrete linear Hamiltonian systems*, Comput. Math. Appl., 45 (2003), 1399-1416.

[10] C. Lee, C. Yeh, C. Hong, R. P. Agarwal, *Lyapunov and Wirtinger inequalities*, Appl. Math. Lett., 17 (2004), 847-853.

[11] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Urallˇtseva for elliptic equations*, Comm. Partial Differ. Equ., 16 (1991), 311-361.

[12] A. Lyapunov, *Probleme general de la stabilite du mouvement*, in: Ann. Math. Studies, volume 17. Princeton Univ. Press, 1948. Reprinted from Ann. Fac. Sci. Toulouse, 9 (1907), 203-474, Translation of the original paper published in Comm. Soc. Math. Kharkow, 1892.

[13] J. P. Pinasco, *Lower bounds for eigenvalues of the one-dimensional p-Laplacian*, Abstr. Appl. Anal., 2004 (2004), 147-153.

[14] J. P. Pinasco, *Comparison of eigenvalues for the p-Laplacian with integral inequalities*, Appl. Math. Comput., 182 (2006), 1399-1404.

[15] J. P. Pinasco, *Lyapunov-type inequalities: with applications to eigenvalue problems*, in: Springer Briefs in Mathematics, Springer, New York-Heidelberg-Dordrecht-London, 2013.

[16] J. Sánchez, V. Vergara, *A Lyapunov-type inequality for a ψ-Laplacian operator*, Nonlinear Anal., 74(18)(2011), 7071-7077.

[17] X. Tang, M. Zhang, *Lyapunov inequalities and stability for linear Hamiltonian systems*, J. Differential Equations, 252 (2012), 358-381.

[18] A. Tiryaki, *Recent development of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl., 5 (2010), 231-248.

[19] A. Tiryaki, M. Ünal, D. Çakmak, *Lyapunov-type inequalities for nonlinear systems*, J. Math. Anal. Appl, 332 (2007), 497-511.

[20] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differ. Equ., 8 (1983), 773-817.

[21] X. Yang, *On Liapunov-type inequality for certain higher-order differential equations*, Appl. Math. Comput., 134 (2003), 307-317.

[22] X. Yang, Y. Kim, K. Lo, *Lyapunov-type inequality for quasilinear systems*, Appl. Math. Comput., 219 (2012), 1670-1673.

[23] X. Yang, K. Lo, *Lyapunov-type inequality for a class of even-order differential equations*, Appl. Math. Comput., 215 (2010), 3884-3890.

[24] Q. Zhang, X. He, *Lyapunov-type inequalities for a class of even-order differential equations*, J. Inequal. Appl., 2012 (2012), DOI: 10.1186/1029-242X-2012-5.

[25] X. Zuo, W. Yang, *Lyapunov-type inequalities for higher-order differential equations with one-dimensional p-Laplacian*, J. Math. Inequal., 8 (2014), 737-746.

Accepted: 24.10.2019