The isomorphic factorization of complete equipartite graphs $K_n(m)$

Jiangdong Liao
School of Mathematics and Statistics
Yangtze Normal University
Fuling 408100, Chongqing
P.R. China
ljq6949@126.com

Abstract. Harary, Robinson and Wormald suggest that in an international conference on combinational theory (Canberra 1977), a conjecture for a complete equipartite graph $G = K_n(m)$, the divisibility condition $t|\frac{1}{2}n(n-1)m^2$ is a sufficiency condition to $t|G$ [1][2], which were proved by Jianfang Wang [3] and S.J. Quinn [4] respectively. The first author of the paper also proved the conjecture with a new way,which is completely different from ones in [3] and [4] at the same time. Here we devote the proof. And in this paper, we give a new isomorphic factorization of K_n as well, which are usefull to the decomposition of $K_n(m)$. We proved that when t is odd or $t < \frac{n}{2}$, the t's factor of K_n is an union of a complete graph and a 3-colorable graphs; and when $t < \frac{n}{2}$ is even, the factor is an union of two complete graphs and 4-colorable graphs.

Keywords: isomorphic factorization, decomposition, complete graphs.

1. Introduction

A partition $\{E_1, E_2, \cdots, E_t\}$ of the edge set $E = E(G)$ of a graph $G = (V, E)$ is called an isomorphic factorization of the graph G, if the subgraphs $G_i = (V, E_i)$ spanned by E_i, $i = 1, 2, \cdots, t$ are isomorphic to each other. In this case, we say that t divides G, denote by $t|G$. If $G_i \cong F$, we also say that F divides G, denoted by $F|G$. An obvious necessary condition for $t|G$ is that $t||E(G)|$.

For convenience, we do not distinguish the edge set $E = E(G)$ of a graph $G = (V, E)$ from the graph itself, especially, if $E^* \subset E = E(G)$, we usually use E^* to denote the subgraph $G^* = (V, E^*)$ spanned by E^*. We denote, by (A, B), the edge set of the complete bipartite graph $K(A, B)$ with independent vertex sets A and B.Let $G = (V, E)$ be a graph, $A, B \subset V = V(G)$ are two disjoint nonempty subsets of the vertex set of the graph G, A and B are said to be completely adjacent, adjacent, and nonadjacent, if and only if, $(A, B) \subset E(G), (A, B) \cap E(G) \neq \emptyset$, and $(A, B) \cap E(G) = \emptyset$, respectively.We set $S = \bigcup_{i=1}^{t} S_i, S_i \cap S_j = \emptyset, S_i \neq \emptyset, (|S_i| = \frac{|S|}{t})$ to mean that the family $\{S_1, S_2, \cdots, S_t\}$ of nonempty sets is a partition(equipartition) of a set S.

For a graph $G = (V, E)$, a graph $G^* = (V^*, E^*)$ ia slit to be a slip of the graph G, if there is a map $\sigma : V \to \{V_1, V_2, \cdots, V_n\}$ from the vertex set

$V = V(G) = \{v_1, v_2, \cdots v_n\}$ into a partition $\{V_1, V_2, \cdots, V_n\}$ of the vertex set of G^* defined by $\sigma(v_i) = V_i$, such that V_i and V_j are completely adjacent, if and only if v_i is adjacent to v_j, else $(V_i, V_j) \cap E^* = \emptyset$. Furthermore, we say that the map σ slits G into G^*, or G^* is the graph obtained from G by slitting σ.A graph $G = (V, E)$ is said to be a quasi complete multipartite graph, if there is a partition $\{V_1, V_2, \cdots V_n\}$ of $V = V(G)$, such that

$$(V_i, V_j) \cap E(G) = \begin{cases} (V_i, V_j) \\ \emptyset \end{cases}$$

$i \neq j, i, j = 1, 2, \cdots, n$. It is easy to see that, a graph G is a quasi complete multipartite graph, if and only if, G is a slit of some graph.

Remark. An complete multipartite graph, denoted by $K(n_1, n_2, \cdots n_r)$, is a slit of K_r, where $n_i = |V_i|, i = 1, 2, \cdots, r$.

Two quasi-complete multipartite graphs G_1 and G_2 are said to be homeomorphic, if both are slits of the same graph F. Furthermore, G_1 is isomorphic to G_2, whenever, the corresponding vertex sets in G_1 and G_2, denoted by $\sigma_1(x)$ and $\sigma_2(x)$ respectively, have the same cardinality, i.e $|\sigma_1(x)| = |\sigma_2(x)|$ for every $x \in V(F)$. For integers $a, b, m > 1$, we denote $a + b (\mathrm{mod}\, m)$ to mean that $a, b \in \{1, 2, \cdots, m\}$[5] and if $a + b > m$, then , replace $a + b$ by $a + b - m$. The notation and terminology that are not defined in this paper follow [6]. For complete graphs K_n [1] and some special case of tripartite graphs $K(m, n, s)$ [2], Harary, Robinson and Wormad have proved the sufficiency of the divisibility condition and suggested two conjectures, if t is even and $t|(mn + ms + ns)$ then $t|K(m, n, s)$, which was proved by Shihui Yang [7], if $t|\frac{1}{2}n(n-1)m^2$, then $t|K_n(m)$, which was proved by Jianfang Wang [3] and S.J. Quinn [4] and Jiangdong Liao independently, may proof is different from theirs in [3] and [4], here we devote the proof to the conjecture. Shihui Yang get some sufficient conditions for complete tripartite graph $K(A, B, C)$ to have isomorphic factorization[8].

Recently, some papers have investigated the gregarious decomposition problem in a multipartite graph. [5,9,14]considered it for the path and cycle decompositions of complete equipartite graphs; and results about kite decomposition can be found in [10]; [11]proved that the equipartite graphs have a characterization; [12]considered it for the group divisible designs with two associate classes; [13]main result is to prove that there is a k-semi-perfect 1-factorization of Q_d for all k and all d, except for one possible exception when $k = 3$ and $d = 6$, the most obvious question: is it possible to find a 1-factorization M of Q_6 such that $G[M] = K_{3,3}$? In this paper, we give a new isomorphic factorization of K_n as well, which is useful to the decomposition of $K_n(m)$, we proved that when t is odd, or $t \geq \frac{n}{2}$, the t's factor of K_n is a union of a complete graph and a 3-colorable graphs, when $t < \frac{n}{2}$ is even, the factor is an union of two complete graphs and 4-colorable graphs.

2. The isomorphic factorization of K_n

Theorem 2.1. *Assume $G = K_n, t|\frac{1}{2}n(n-1)$ then $t|G$, and each $t's$ factor is a 3-colorable or a disconnected graph whose components are 3-colorable graphs or 4-colorable complete graphs.*

Proof. We will prove the proposition by four case:
Case 1. [1] $t > n$, then $\frac{1}{2}n(n-1) = tr, r < \frac{n-1}{2}$. We can label the edges by using $1, 2, \cdots, \frac{1}{2}n(n-1)$ such that the edges labeled by $i+1, i+2, \cdots, i+r(\text{mod}\,\frac{1}{2}n(n-1))$ is a matching.
Case 2. [1] The $t's$ factor is a union of some K_3 and K_2 with a vertex in common, when $t = [\frac{n}{2}]$, $n = 2t$ and $[\frac{n}{2}] < t \leq n$. Denote $V(K_n) = \{1, 2, \cdots, n\}$, K_n can be decompose into t factors $F_i, i = 1, 2, \cdots, t$ as follows: $3, n, 3, n-1, \cdots, t-1, t+3, t, t+2, t+3$. F_i can obtained from F_1 by using the element plus $i-1(\text{mod}\, n)$ replace the corresponding element,

$F_2 : 4, 3, 1, 4, n, \cdots, t, t+4, t+1, t+3, t+4$

$F_3 : 5, 4, 2, 5, 1, \cdots, t+1, t+5, t+2, t+4, t+5$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$F_t : t+2, t+1, t-1, t+1, t-2, \cdots, n-2, n+2, n-1, n+1, n+2(\text{mod}\, n)$

and F_i has the property that $V(F_i) \cap \{i, i+t\} = \emptyset$. For $n = 2t+1$, denote $V(K_n) = \{0, 1, 2, \cdots, n\}$, let $\tilde{F_i} = F_i \cup \{(0, i+1), (0, i+1+t)\}, i = 1, 2, \cdots, t$. Each F_i is the same as in the case that $n = 2t$ and $i+1, i+1+t \in \{1, 2, \cdots, n\}$. We have $V(\tilde{F_i}) \cap \{i, i+t\} = \emptyset$.
Case 3. t is odd and $t < \frac{n}{2}$. Set $n = at+b, 0 \leq b < t, V(K_n) = \{1, 2, \cdots, n\}, V^i = \{a(i-1)+1, a(i-1)+2, \cdots, ai\}, i = 1, 2, \cdots, t, V^0 = \{at+1, at+2, \cdots, at+b\}$. Denote by $K(V^i)$ the complete graph with vertex set V^i, then K_n can be represented as $K_n = K(V^1, V^2, \cdots, V^t) \cup K(V^0, V^1 \cup V^2 \cup \cdots \cup V^t) \cup (\bigcup_{i=0}^{t} K(V^i))$. Set $F_i = K(V^i) \cup K(V^{i+1}, V^{i+2} \cup V^{i+3} \cup \cdots \cup V^{i+\frac{t-1}{2}} \cup V^0) \cup M^i, i = 1, 2, \cdots, t, i+j(\text{mod}\, t)$, where M^i is a t's factor of $K(V^0)$, having $r = \frac{b(b-1)}{2t}$ edges, as in case(1)(Notice that it is easy to see $t|\frac{1}{2}b(b-1)$), and the isomorphic factorization of K_a is obtained. And we have F_i is a union of complete graph K_b and a 3-color graph.
Case 4. t is even and $t < \frac{n}{2}$. Set $n = 2at + b, 0 \leq b < 2t, V(K_n) = \{1, 2, \cdots, n\}, V^i = \{a(i-1)+1, a(i-1)+2, \cdots, ai\}, i = 1, 2, \cdots, 2t$, $V^0 = \{2at+1, 2at+2, \cdots, 2at+b\}$, $K_n = K(V^1, V^2, \cdots V^{2t}) \cup K(V^0, V^1 \cup V^2 \cup \cdots \cup V^{2t}) \cup (\bigcup_{i=0}^{2t} K(V^i))$.

We describe a subgraph of $K(V^1, V^2, \cdots, V^{2t})$ by a set sequence as follows

$F_1^1 : V^3V^2V^{2t}V^3V^{2t-1} \cdots V^{t-1}V^{t+3}V^tV^{t+2}V^{t+3}$ which is similar as case 2, where two sets, which are neighbor in the sequence are completely adjacent. For any integer $i \geq 2$, F_1^i can be obtained by using the scripts plus $i-1(\text{mod}\ 2t)$ to replace the corresponding scripts, so

$F_1^2 : V^4V^3V^1V^4V^{2t} \cdots V^tV^{t+4}V^{t+1}V^{t+3}V^{t+4}$, where $V(F_1^i) \cap (V^i \cup V^{i+t}) = \emptyset$. Since $b < 2t$, by the case 2 $K(V^0)$ can be decomposed into t isomorphic factors, each factor F_0^i is a union of some K_3 and K_2 with a vertex in common.

Set $F^i = F_0^i \cup F_1^i \cup K(V^0, V^{i+1} \cup V^{i+t+1}) \cup K(V^i) \cup K(V^{i+t})$, $i+1, i+t+1 (\mathrm{mod}\, 2t), i = 1, 2, \cdots, 2t$, which is an isomorphic factorization of K_n, each factor is a union of two complete graphs and a 4-colorable graph, and if F_0^i has no K_3, then $F_i = F_0^i \cup F_1^i \cup K(V^0, V^{i+1} \cup V^{i+t+1})$ is 3-colorable. □

3. The isomorphic factorization of $K_n(m)$

3.1 Preliminary results

Lemma 3.1. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a 3-colorable quasi complete multipartite graph. If $q||V^i|, i = 1, 2, \cdots, n$, then $q^2|G$.*

Proof. Let β: $V(G) \to \{1, 2, 3\}$ be a 3-coloring of the graph G, such that $|\beta(V^i)| = 1$. And denote $\beta(i) = \beta(x)$, $x \in V_i$, $i = 1, 2, \cdots, n$.

$V^i = \bigcup_{r=1}^q V_r^i$, $|V_r^i| = \frac{1}{q}|V^i|$, $i = 1, 2, \cdots, n$. Set $A_r = \bigcup_{\beta(i)=1} V_r^i$, $B_r = \bigcup_{\beta(i)=2} V_r^i$, $C_r = \bigcup_{\beta(i)=3} V_r^i$. $A = \bigcup_{r=1}^q A_r$, $B = \bigcup_{r=1}^q B_r$, $C = \bigcup_{r=1}^q C_r$. Thus we have $G \subset K(A, B, C), q||A|, q||B|, q||C|$, and so $q^2|K(A, B, C)$, and $K(A_r, B_s, C_{r+s}), r, s = 1, 2, \cdots, q$, $r + s (\mathrm{mod} q)$ are q^2 isomorphic factors of complete tripartite graph $K(A, B, C)$[7]. Let $G^{rs} = G \cap K(A_r, B_s, C_{r+s})$, then, it is clear, that each G^{rs} is a q^2's factor of G, and the proof is complete. □

Lemma 3.2. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a 3-color quasi complete multipartite graph. If $q||V^i|, d|q, i = 1, 2, \cdots, n$, then $q|G$. Furthermore, if $d|q$, then $dq|G$.*

Proof. Set $V^i = \bigcup_{r=1}^q V_r^i, |V_r^i = \frac{1}{q}|V^i|$, and $E^r = \bigcup_{(i,j) \in (I,J)} (V^i, V_r^j)$,

$$(I, J) = \{(i, j) | 1 \le i < j \le n, (V^i, V^j) \subset E(G)\}, r = 1, 2, \cdots, q.$$

It is easy to see that $\{E^1, E^2, \cdots, E^q\}$ is an isomorphic factorization of G. For the case that $d|q$, let $q = dq_1$, then $dq = d^2 q$, by Lemma 3.1, $d^2|G$, and then decompose each d^2 s-factor into q_1 factors and the proof is complete. □

3.2 Main theorems

Theorem 3.1. *Assume $G = K(V^1, V^2, \cdots, V^n)$ is a complete multipartite graph, $q||V^i|, i = 1, 2 \cdots, n$ and $q \ge 2n - 3$, then $q^2|G$.*

Theorem 3.2. *Assume $G = G(V^1, V^2, \cdots, V^n)$ is a quasi-complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \ge n$, then $q^2|G$.*

Theorem 3.3. *Let $G = G(V^1, V^2, \cdots, V^n)$ be a quasi complete multipartite graph, $|V^i| = m, q|m, i = 1, 2, \cdots, n$, and $q \ge 2[\frac{n+2}{3}] - 1$, then $q^2|G$.*

Theorem 3.4. *Assume $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ and $q \ge [\frac{n}{2}]$, then $q^2|G$.*

Theorem 3.5. *Assume* $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ *and* $1 < q < \frac{n}{2}$ *then* $q^2|G$.

Theorem 3.6. *Let* $G = K_n(m) = K(V^1, V^2, \cdots, V^n)$ *be a complete equipartite graph. If* $t|\frac{1}{2}n(n-1)m^2$ *then* $t|G$.

3.3 Proof of the main theorems

Theorem 3.1. *Assume* $G = K(V^1, V^2, \cdots, V^n)$ *is a complete multipartite graph,* $q||V^i|, i = 1, 2 \cdots, n$ *and* $q \geq 2n-3$, *then* $q^2|G$.

Proof. Set $V^i{=}\bigcup_{r=1}^{q} V_r^i, q||V^i|, i{=}1, 2, \cdots, n$, and $E^{rs}{=}\bigcup_{(i,j)\in(I,J)}(V^i_{\varphi_s(j)+r-1}, V^j_{\varphi_s(j)+r+s-2}) = (\bigcup_{j=2}^{n}((\bigcup_{i=1}^{j-1}(V^i_{\varphi_s(j)+r-1}), V^j_{\varphi_s(j)+r+s-2})$, $\varphi_s(j) + r - 1(\text{mod } q)$, $\varphi_s(j) + r + s - 2(\text{mod } q)$, $r, s = 1, 2, \cdots, q$. $\varphi_s(j)$ is defined as follows:

(i) $\varphi_s(2) = 1$, and

$$(ii)\varphi_s(j) = \begin{cases} \varphi_s(j-1)+1, & j{\geq}3, \varphi(j-1){+}1{\neq}\varphi_s(t){+}s, 2{\leq}t{\leq}j-1 \\ \varphi_s(j-1)+s(\text{mod}\, q), & \text{else.} \end{cases}$$

So, for instance, $E^{1s}, s = 1, 2, \cdots, q$ as follows

$E^{11} = (V_1^1, V_1^2) \cup (V_2^1 \cup V_2^2, V_2^3) \cup (V_3^1 \cup V_3^2 \cup V_3^3, V_3^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V^i_{n-1}, V^n_{n-1})$

$E^{12} = (V_1^1, V_2^2) \cup (V_3^1 \cup V_3^2, V_4^3) \cup (V_5^1 \cup V_5^2 \cup V_5^3, V_6^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V^i_{2n-3}, V^n_{2n-2})$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$E^{1s} = (V_1^1, V_s^2) \cup (V_2^1 \cup V_2^2, V_{s+1}^3) \cup \cdots \cup (\bigcup_{i=1}^{s-1} V^i_{s-1}, V^s_{2s-2}) \cup (\bigcup_{i=1}^{s} V^i_{2s-1}, V^{s+1}_{3s-2}) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V^i_{\varphi_s(n)}, V^n_{\varphi_s(n)+s-1})$

$E^{1q} = (V_1^1, V_q^2) \cup (V_2^1 \cup V_2^2, V_1^3) \cup (V_3^1 \cup V_3^2 \cup V_3^3, V_2^4) \cup \cdots \cup (\bigcup_{i=1}^{n-1} V^i_{n-1}, V^n_{n-2})$

And E^{rs} can be obtained from E^{1s} by using the subscripts in E^{1s} plus $r-1$(mod q) to replace the corresponding subscripts respectively.

In order to prove that $\{E^{rs}|r, s = 1, 2, \cdots, q\}$ is an isomorphic factorization of G into q^2 factors. We first indicate that $\varphi_s(2) < \varphi_s(3) < \cdots < \varphi_s(n) \leq 2n-3$. From definition of the function, $\varphi_s(j)$ is increasing.

$s = 1, \varphi_1(2) = 1, \varphi_1(j) = \varphi_1(j-1) + 1 = j - 1, \varphi_1(n) = n - 1$, $s = 2, \varphi_2(j-1) + 1 = \varphi_2(j-1) + s - 1, \quad j \geq 3$, hence $\varphi_2(j) = \varphi_2(j-1) + s = \varphi_2(j-1) + 2$, thus $\varphi_2(2) = 1, \varphi_2(3) = 3, \cdots, \varphi_2(j) = 2j - 3, \cdots, \varphi_2(n) = 2n - 3$.

$s \geq n$, since $\varphi_s(t) + s - 1 \geq \varphi_s(2) + s - 1 = s \geq n, 2 \leq t \leq j - 1$, hence, $\varphi_s(2) = 1, \varphi_s(3) = 2, \cdots, \varphi_s(j-1) + 1 = j - 2 + 1 = j - 1 = \varphi_s(j), \cdots, \varphi_s(n) = n-1$, $2 < s < n$, then $n = ks{+}h, 0 \leq h < s$, when $3 \leq j \leq s$, since $\varphi_s(t){+}s{-}1 \geq \varphi_s(2){+}s{-}1 = s$, hence $\varphi_s(2) = 1, \varphi_s(3) = 2, \cdots, \varphi_s(j) = j{-}1, \cdots, \varphi_s(s) = s{-}1$, $\varphi_s(2) + s - 1 = s = \varphi_s(s) + 1$, so we have $\varphi_s(s + 1) = \varphi_s(s) + s = 2s - 1$.

Set $t_1 = min\{j = \varphi_s(j{-}1){+}1 = \varphi_s(t){+}s{-}1\}$ for some t, $2 \leq t \leq j{-}1, 3 \leq j \leq n$. Thus we have, by the definition of $\varphi_s(j)$, $t_1 = s{+}1$, and $\varphi_s(t_1) = 2s{-}1 = 2t_1 - 3$ recurrently, we define $t_k = min\{j|j > t_{k-1}\varphi_s(j-1) + 1 = \varphi_s(t) + s - 1$, for some t, $2 \leq t \leq j - 3$.

Thus, we have $t_1 < t_2 < \cdots < t_k \leq n$.

We will prove, by induction, that $\varphi_s(t_k) = 2t_k - 3$. For $k = 1$, we are done. For $k \geq 1$, we suppose that $\varphi_s(t_k) = 2t_k - 3$, for $k+1$, since $1 \leq i \leq s-2$, $\varphi_s(t_k)+i < \varphi_s(t_k)+s-1)$ and, for $2 \leq t \leq t_k-1$, $\varphi_s(t)+s-1 \leq \varphi_s(t_k-1)+s-1 = \varphi_s(t_k) - 1 < \varphi_s(t_k)+i$, hence we have $\varphi_s(t_k+i) = \varphi_s(t_k+i-1)+1 = \varphi_s(t_k)+i-1+1 = \varphi_s(t_k)+i$, and so $\varphi_s(t_k+s-2)+1 = \varphi_s(t_k)+s-2+1 = \varphi_s(t_k)+s-1$, hence, by the definition we have $t_{k+1} = t_k+s-1$, $\varphi_s(t_{k+1}) = \varphi_s(t_k+s-1) = \varphi_s(t_k+s-2)+s= \varphi_s(t_k)+s-2+s= 2(t_k)+3+2s-2 = 2t_{k+1}-3$.

Denote $h = max\{t_1, t_2, \cdots, t_k, \cdots\}$, if $h = n$, we have $\varphi_s(n) = 2n-3$. If $h < n$, we have $\varphi_s(h+i) = \varphi_s(h+i-1)+1 = \varphi_s(h)+i-1+1 = \varphi_s(h)+i$, $\varphi_s(n) = \varphi_s(h)+n-h = 2h-3+n-h = n+h-3 < 2n-3$.

Hence, when $q \geq 2n-3$, $\varphi_s(2), \varphi_s(3), \cdots, \varphi_s(n)$ are $n-1$ different elements. Hence, there are no two elements, in the index set $\{\varphi_s(j)+r-1|2 \leq j \leq n\}$ (as well as in the set $\{\varphi_s(j)+r-1(\bmod q)|2 \leq j \leq n\}$, and $\{\varphi_s(j)+r+s-1(\text{mod}q)|2 \leq j \leq n\}$ are the same, and so E^{rs} is a union of $n-1$ complete bipartite graphs which and are isomorphic to each other.

Next, we prove that for any $e = (u, v) \in E(G)$, there is one and only one set $E^{r_1 s_1}$, such that $e \in E^{r_1 s_1}$. Since, for $e = (u, v)$, there is unique i, j, r_0, s_0, such that $u \in V^i_{r_0}$ and $v \in V^j_{s_0}$, since u, v are adjacent. We have $i \neq j$, and with out loss of generality, we assume that $i < j, e \in (V^i_{r_0}, V^j_{s_0})$, set $\varphi_s(j)+r-1 = r_0, \varphi_s(j)+r+s-2 = s_0, r_1 = r_0+1-\varphi_s(j)(\bmod q), s_1 = s_0+1-r_0(\bmod q)$, we have $e \in E^{r_1 s_1}$, which say that, there are no two sets in the family $\{E^{rs}|r, s = 1, 2, \cdots, q\}$ are intersect, and $E(G) = \bigcup_{r=1}^q \bigcup_{s=1}^q E^{rs}$. This completes the proof. □

Lemma 3.3. *Let $G = K(V^1, V^2, \cdots, V^n)$ be a complete multipartite graph, $q||V^i|, i = 1, 2 \cdots, n$ and $q \geq 2n-1$. Set $V^i = \bigcup_{r=1}^q V^i_r, |V^i_r| = \frac{1}{q}|V^i|, r = 1, 2, \cdots, q$, then there is a function $\tau(s)$ such that*

$$E^{rs} = \bigcup_{1 \leq i < j \leq n} (V^i_{\varphi_s(j)+r-1+\tau(s)}, V^j_{\varphi_s(j)+r+s-2+\tau(s)}), r, s = 1, 2, \cdots, q$$

is an isomorphic factorization of G, where $\varphi_s(j)$ is defined as in Theorem 3.1, and satisfies that $V(E^{rs}) \cap V^j_r = \emptyset, j = 1, 2, \cdots, n$.

Proof. Let $\tilde{E^{rs}} = \bigcup_{1 \leq i < j \leq n} (V^i_{\varphi_s(j)+r-1}, V^j_{\varphi_s(j)+r+s-2})$, $r, s = 1, 2, \cdots, q$. We known from Theorem 3.1 that $\{\tilde{E^{rs}}|r, s = 1, 2, \cdots, q\}$ is a factorization of G into q^2 factors.

Set $\tilde{u}(r, s) = \{\varphi_s(j)+r-1, \varphi_s(j)+r+s-2|j = 2, 3, \cdots, n\}(\text{mod } q)$, $Z_q = \{1, 2, \cdots, q\}$. Since $q \geq 2n-1$, $Z_q - \tilde{u}(r, s) \neq \emptyset$.

So, there is an element, denoted by $\tau_r(s)$, such that $\tau_{rs} \in Z_q - \tilde{u}(r, s)$, especially, we have $\tau_{1s} \in Z_q - \tilde{u}(1, s)$. Set $\tau(s) = q - \tau_1(s) + 1$. We will have prove that $\tau(s) = q - \tau_1(s) + 1$ is the index function that we want to find.

Set $u(r, s) = \{\varphi_s(j)+r-1+\tau(s), \varphi_s(j)+r+s-2+\tau(s)|j = 1, 2, \cdots, n\} modq$. We will prove that $r \notin u(r, s)$. By contrary, suppose that $r \in u(r, s)$, one of the following two cases would appear.

(1) $r=\varphi_s(j)+r-1+\tau(s)(\mathrm{mod}\, q)$, then $\varphi_s(j)-1+\tau(s)=\varphi_s(j)+q-1-\tau_1(s)+1=0(\mathrm{mod}\, q)$, thus we have $\tau_1(s)=\varphi_s(j)=\varphi_s(j)+1-1\in\tilde{u}(1,s)$.

(2) $r=\varphi_s(j)+r+s-2+\tau(s)(\mathrm{mod}\, q)$, then $\varphi_s(j)+s-2+q-\tau_1(s)+1=0(\mathrm{mod}\, q)$, and we have $\tau_1(s)=\varphi_s(j)+s-1=\varphi_s(j)+1+s-2\in\tilde{u}(1,s)$.

The two cases contradict the hypotheses that $\tau_1(s)\notin\tilde{u}(1,s)$, which proves that $r\notin u(r,s)$, and $V(E^{rs})\cap V_r^j=\emptyset$, $j=1,2,\cdots,n,r,s=1,2,\cdots,q$, E^{rs} are obviously isomorphic to each other. The rest is to prove that it is a partition of $E(G)$ as well, and for which we need only to prove that $(\varphi_{s_1}(j)+r_1-1+\tau(s_1),\varphi_{s_1}(j)+r_1+s_1-2+\tau(s_1))$ =$(\varphi_{s_2}(j)+r_2-1+\tau(s_2),\varphi_{s_2}(j)+r_2+s_2-2+\tau(s_2))$, whenever $(r_1,s_1)\neq(r_2,s_2)$.

If $\varphi_{s_1}(j)+r_1-1+\tau(s_1)=\varphi_{s_2}(j)+r_2-1+\tau(s_2)$, and $\varphi_{s_1}(j)+r_1-2+\tau(s_1)=\varphi_{s_2}(j)+r_2-2+\tau(s_2)$, then we have $s_1=s_2$, and so $\tau(s_1)=\tau(s_2)$, $\varphi_{s_1}(j)=\varphi_{s_2}(j)$ and, thus $r_1=r_2$, which contradict that $(r_1,s_1)\neq(r_2,s_2)$. □

Theorem 3.2. *Assume $G=G(V^1,V^2,\cdots,V^n)$ is a quasi-complete multipartite graph, $|V^i|=m,q|m,i=1,2,\cdots,n$, and $q\geq n$, then $q^2|G$.*

Proof. Set $K_n(m)=K(V^1,V^2,\cdots,V^n)$ be the complete n-partite graph with the same independent vertex sets as the graph G.

Denote $n_1=[\frac{n+1}{2}],n_2=[\frac{n}{2}],n=n_1+n_2$, and $A=V^1\cup V^2\cup\cdots\cup V^{n_1},B=V^{n_1+1}\cup V^{n_2+1}\cup\cdots\cup V^n$. Set $V^i=\bigcup_{r=1}^q V_r^i,|V_r^i|=\frac{m}{q},i=1,2,\cdots,n$, $A_r=\bigcup_{i=1}^m V_r^i$, $B_r=\bigcup_{r=n_1+1}^n V_r^i,r=1,2,\cdots,q$, and we have

$$A=\bigcup_{r=1}^q A_r,B=\bigcup_{r=1}^q B_r.$$

Since $q\geq n\geq 2[\frac{n+1}{2}]-1\geq[\frac{n}{2}]-1$, according to Lemma 3.3, we construct the q^2 isomorphic factors G_1^{rs} and G_2^{rs} of $G_1=K(V^1,V^2,\cdots,V^{n_1})$ and $G_2=K(V^{n_1},\cdots,V^n)$, $r,s=1,2,\cdots,q$ respectively, such that $V(G_1^{rs})\cap A_r=\emptyset$, $V(G_2^{rs})\cap B_r=\emptyset$.

Set $G^{rs}=G_1^{sr}\cup G_2^{rs}\cup K(A_r,B_s)$, $r,s=1,2,\cdots,q$, and so $\tilde{G^{rs}}=G^{rs}\cap G$, $r,s=1,2,\cdots,q$ are q^2 isomorphic factors of G. □

Theorem 3.3. *Let $G=G(V^1,V^2,\cdots,V^n)$ be a quasi complete multipartite graph, $|V^i|=m,q|m,i=1,2,\cdots,n$, and $q\geq 2[\frac{n+2}{3}]-1$, then $q^2|G$.*

Proof. Denote $n_1=[\frac{n+2}{3}],n_2=[\frac{n+1}{3}],n_3=[\frac{n}{3}],n=n_1+n_2+n_3$. Set $V^i=\bigcup_{r=1}^q V_r^i,|V_r^i|=\frac{1}{q}|V^i|,i=1,2,\cdots,n$, $A=\bigcup_{i=1}^{n_1}V^i$, $B=\bigcup_{i=1}^{n_2}V^{i+n_1}$, $C=\bigcup_{i=1}^{n_3}V^{i+n_1+n_2}$. Then, $K(V^1,V^2,\cdots,V^n)=G_1\cup G_2\cup G_3\cup K(A,B,C),G_1=K(V^1,\cdots,V^{n_1}),G_2=K(V^{n_1+1},\cdots,V^{n_1+n_2}),G_3=K(V^{n_1+n_2+1},\cdots,V^n)$,

$$A_r=\bigcup_{i=1}^{n_1}V_r^i,B_r=\bigcup_{i=1}^{n_2}V_r^{i+n_1},C_r=\bigcup_{i=1}^{n_3}V_r^{i+n_1+n_2}$$

decompose G_1, G_2, G_3 into q^2 factors, as Lemma 3.3, $G_1^{rs}, G_2^{rs}, G_3^{rs}$ respectively, such that

$$V(G_1^{rs}) \cap A_r = \emptyset, V(G_2^{rs}) \cap B_r = \emptyset, V(G_3^{rs}) \cap C_r = \emptyset$$

and decompose $K(A, B, C)$ into $K(A_r, B_s, C_{r+s}), r, s = 1, 2, \cdots, q$. Set

$$G^{rs} = G_1^{sr} \cup G_2^{rs} \cup G_3^{r+s,s} \cup K(A_r, B_s, C_{r+s})$$

$\tilde{G^{rs}} = G^{rs} \cap G, r, s = 1, 2, \cdots, q$, then $\{\tilde{G^{rs}} | r, s = 1, 2, \cdots, q\}$ is an isomorphic factorization of G into q^2 factors. □

Theorem 3.4. *Assume* $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ *and* $q \geq [\frac{n}{2}]$, *then* $q^2|G$.

Proof. When $q \geq n-1, n \geq 4$, we have $q \geq n-1 \geq 2[\frac{n+2}{3}] - 1$, the proof is obtained from Theorem 3.3, so we assume that $[\frac{n}{2}] \leq q \leq n-2$. Set $n_1 = [\frac{n-q+1}{2}], n_2 = [\frac{n-q}{2}]$, then we have $n = n_1 + n_2 + q$. Set

$$V^i = \bigcup_{r=1}^{q} V_r^i, |V_r^i| = \frac{1}{q} m, 1 = 1, 2, \cdots, n,$$

$$A = \bigcup_{i=1}^{n_1} V^i, B = \bigcup_{i=1}^{n_2} V^{i+n_1}, A_r = \bigcup_{i=1}^{n_1} V_r^i, B_r \bigcup_{i=1}^{n_2} V_r^{i+n_1}.$$

Denote $C^i = V^{i+n_1+n_2}, i = 1, 2, \cdots, q, C = \bigcup_{i=1}^{q} C^i, C_r = \bigcup_{i=1}^{q} C_r^i$.

Case 1. q is odd. Decompose $K(C^1, C^2, \cdots, C^q)$ into $G_3^{rs} = \bigcup_{i=1}^{q_1} (C_s^r, C^{r+i}), r, s = 1, 2, \cdots, q, r+i (\operatorname{mod} q), q_1 = \frac{q-1}{2}$. Since $q \geq [\frac{n}{2}]$, $q \geq 2n_1 - 1 \geq 2n_2 - 1$. By the same way as we do in Lemma 3.3. Set G_1^{rs} and G_2^{rs}, $r, s = 1, 2, \cdots, q$ are q^2 isomorphic factors of $K(V^1, V^2, \cdots, V^{n_1})$ and $K(V^{n_1+1}, V^{n_2} + 2, \cdots V^{n_1+n_2})$ respectively, and satisfy that $V(G_1^{rs}) \cap A_r = \emptyset, V(G_2^{rs}) \cap B_r = \emptyset$. Thus $G^{rs} = G_1^{sr} \cup G_2^{rs} \cup G_3^{r+s,s} \cup K(A_r, B_s, C_{r+s})$, $r, s = 1, 2 \cdots, q, r+s (\operatorname{mod} q)$ is an isomorphic factorization of $G = K_n(m)$.

Case 2. q is even, from Theorem 2.1, let $F_1, F_2, \cdots F_{q_1}$ is an isomorphic factorization of K_q with $V(F_i) \cap (i, i+q_1) = \emptyset, q_1 = \frac{q}{2}, i + q_1 (\operatorname{mod}\ q)$. Each F_i is a 3-colorable graph. Let $\tilde{F_i}$ be a slit of F_i, which is obtained by using C^r to replace r in F_i. So $\tilde{F_i}, i = 1, 2, \cdots, q_1$ is an isomorphic factorization of $K(C^1, C^2, \cdots, C^q)$, and each factor is a 3-color quasi-complete graph. Since $q = 2q_1, q|m$, so from Lemma 2 we can decompose each $\tilde{F_i}$ into $2q$ isomorphic factors, denoted by G_3^{is} and $G_3^{i+q_1,s}, i, s = 1, 2, \cdots, q$.

Thus, we have that $V(G_3^{rs}) \cap (C^r \cup C^{r+q_1}) = \emptyset, r + q_1 (\operatorname{mod} q)$. Set $G^{rs} = G_1^{r+s,s} \cup G_2^{sr} \cup G_3^{rs} \cup K(A_{r+s}, B_s, C_r)$, $r, s = 1, 2 \cdots, q, r+s (\operatorname{mod} q)$, then it is an isomorphic factorization of G and each factor has 4 components. □

Theorem 3.5. *Assume* $G = K_n(m) = K(V^1, V^2, \cdots, V^n), q|m$ *and* $1 < q < \frac{n}{2}$ *then* $q^2|G$.

Proof. Set $n = 2eq + f, 0 \le f \le 2q-1, V^i = \bigcup_{r=1}^{q} V_r^i, |V_r^i| = \frac{m}{q}, i = 1, 2, \cdots, n,$

$$B^i = V^{f+e(i-1)+1} \cup V^{f+e(i-1)+2} \cup \cdots \cup V^{f+ei}, i = 1, 2 \cdots, 2q,$$
$$G^i = K(V^{f+e(i-1)+1}, \cdots, V^{f+ei}) = G[B^i]$$

$G[V'] \subset G$ as the subgraph induced by $V' \subset V(G)$, $G^0 = K(V^1, V^2, \cdots, V^f)$, $A = \bigcup_{i=1}^{f} V^i, B = \bigcup_{i=1}^{2q} B^i$. Since $q > \frac{f}{2}$, we decompose G^0 into q^2 factors $G_0^{rs}, r, s = 1, 2, \cdots, q$.

Set $G = K(B^1, B^2, \cdots, B^{2q}) \cup (\bigcup_{i=1}^{2q} G^i) \cup G^0 \cup K(A, B)$.
By Theorem 2.1, let $F^r, r = 1, 2, \cdots, q$ be q isomorphic factors of K_{2q}, where $V(K_{2q}) = \{1, 2, \cdots, 2q\}$ and $V(F^r) \cap \{r, r+q\} = \emptyset$, $\tilde{F^r}$ is a slit of F^r by using B^i replace i, F^r is a quasi-complete multipartite graph decompose each $\tilde{F^r}$ into q isomorphic factors, denote by $\tilde{F^{rs}}, r, s = 1, 2, \cdots, q$, decompose each G^i into q factors, denoted $G_s^i, i = 1, 2, \cdots, 2q, s = 1, 2, \cdots, q$, $q \ge \frac{f}{2}$, we can decompose. G^0 into q^2 factors, denote G_0^{rs}, and then set

$$G^{rs} = \tilde{F^{rs}} \cup G_s^r \cup G_s^{r+q} \cup G_0^{rs} \cup K(AB_s^r) \cup K(A, B_s^{r+q}),$$

where $B_s^i = V_s^{f+e(i-1)+1} \cup \cdots \cup V_s^{f+ei}, i = 1, 2, \cdots, 2q$. $G^{rs}, r, s = 1, 2, \cdots, q$ are q^2 isomorphic factor of G. □

Theorem 3.6. *Let $G = K_n(m) = K(V^1, V^2, \cdots, V^n)$ be a complete equipartite graph. If $t|\frac{1}{2}n(n-1)m^2$ then $t|G$.*

Proof. Let $F = K_n, V(F) = \{1, 2, \cdots, n\}$, then $G = K_n[\bar{K_m}]$ is a slit obtained from F by using V^i to replace i. By the condition, we have $t = t_1pq^2, t_1|\frac{1}{2}n(n-1), pq|m$. We first decompose $F = K_n$ into t_1 isomorphic factors F^i and then obtain t_1 isomorphic factors of G, by setting $G_i = F^i[\bar{K_m}]$, which is a quasi complete equipartite graph, and then decompose each G_i into q^2 factor G_i^{rs}, and then decompose each G_i^{rs} into p factors, and complete the decomposition. The key is to decompose G_i into G_i^{rs}. For t_1 is odd or $t_1 \ge [\frac{n}{2}]$, by Theorem 2.1 we known that F^i as well as G_i are both unions of some complete (equipartite) graphs and (quasi complete) 3-colourable (equipartite) graphs. So we can decompose G_i into q^2 isomorphic factors G_i^{rs} by using the preceeding lemmas and theorems. The rest we need only to consider the case that $t_1 < \frac{n}{2}$ is even. In this case, the t_1's factor of K_n, denoted by F_i is a union of two complete graphs and a 4-colorable graph F_i', as in the case 4 of Theorem 2.1. $G_i = G_i' \cup G_{i2} \cup G_{i3}$, where G_{i2}, G_{i3} both are complete equipartite graphs, we can decompose G_{i2} and G_{i3} into q^2 factors as we have done in the preceeding, so we need only to discuss the decomposition of G_i'.

If F_0^i as in the case 4 of Theorem 2.1, has no K_3, then both F_i' and G_i' are 3-colorable, so we have proved that $q^2|G_i'$. If F_0^i has a K_3 then G_i' has a corresponding complete tripartite subgraph $K(V^{i_1}, V^{i_2}, V^{i_3})$. For convenience, denote it by $K(A,B,C)$, then $G_i^{''} - G_i^{'} - (B,C)$ is 3-colorable. So we can decompose $G_i^{''}$ into q^2 factors, denoted by H^{rs}, such that $(A_{r+s}, B_r) \cup (A_{r+s}, C_r) \subset H^{rs}$, $r,s = 1,2,\cdots,q$, and then construct the factor $\tilde{H^{rs}}$ of $G_i^{'}$ from H^{rs} as follows.

(1) When $q \geq 3$, set

$$\tilde{H^{rs}} = \begin{cases} H^{rs} \cup (B_{r+1}, C_{r+s}), & s = 1,2,\cdots,q-1 \quad r+j (\mathrm{mod}\, q) \\ H^{rs} \cup (B_{r+2}, C_{r+1}), & s = q \end{cases}$$

Since $V(H^{rs}) \cap (B_{r+1} \cup B_{r+s}) = \emptyset, s \neq q$, and $V(H^{rq}) \cap (B_{r+2} \cup C_{r+1}) = \emptyset$, which guarantee the property that $\tilde{H^{rs}}$ are isomorphic to each other. The rest we need to see that each edge $e = (u,v) \in (B,C)$ is in some $\tilde{H^{rs}}$, see $u \in B_i, v \in C_j$ for some i,j. If $j - i = -1 = q - 1 (\mathrm{mod}\, q)$ then we have $s = q$, and $r = i - 2 (\mathrm{mod}\, q)$, thus $e \in \tilde{H}^{i-2,q}$. If $j - i \neq q - 1 (\mathrm{mod}\, q)$ then $r = i-1, s = j - r = j - i + 1 (\mathrm{mod}\, q)$ and so $e \in \tilde{H}^{i-1,j-i+1}$.

(2) $q = 2$, $(B,C) = (B_1,C_1) \cup (B_1,C_2) \cup (B_2,C_1) \cup (B_2,C_2)$, let

$$\begin{aligned} \tilde{H^{11}} &= (H^{11} - (A_2,B_1) - (A_2,C_1)) \cup (A_2,B)(B_1,C_1), \\ \tilde{H^{12}} &= (H^{12} - (A_1,B_2) - (A_1,C_2)) \cup (A_1,B)(B_1,C_2), \\ \tilde{H^{21}} &= (H^{21} - (A_1,B_1) - (A_1,C_1)) \cup (A_1,C)(B_2,C_1), \\ \tilde{H^{22}} &= (H^{22} - (A_2,B_2) - (A_2,C_2)) \cup (A_2,C)(B_2,C_2). \end{aligned}$$

It is easy to see that $H^{rs}, r,s = 1,2,\cdots,q$ are q^2 isomorphic factors of $G_i^{'}$.

If F_0^i has more than one K_3, then $G_i^{'}$ has corresponding complete tripartite graphs as $K(A,B^j,C^j), j = 1,2,\cdots,k$. And thus $G_i^{''} = G_i^{'} - \bigcup_{j=1}^{k}(B^j \cup C^j)$ is 3-colorable, and the vertex in the set $\bigcup_{j=1}^{k}(B^j \cup C^j)$ may have same color of a 3-coloring of $G_i^{''}$. So we can decompose $G_i^{''}$ into $H^{rs}, r,s = 1,2,\cdots,q$. Such that $\bigcup_{j=1}^{k}(A_{r+s}, B_r^j \cup C_r^j) \subset H^{rs}, j = 1,2,\cdots,k$. And then get the factors $\tilde{H^{rs}}$ of $G_i^{'}$ from H^{rs} as we have done as above, and the proof is complete. □

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