First reformulated Zagreb index of four graph operations

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Abstract. A generalization of classical Zagreb indices of chemical graph theory were introduced in 2004, which is called the reformulated Zagreb indices. The first reformulated Zagreb index EM_1 of any graph G is the sum of the squares of degree of edges. In this paper, we compute the first reformulated Zagreb index of four operations of graphs.
Keywords: Zagreb indices, reformulated Zagreb indices, degree, subdivision of graph, operation on graph.

1. Introduction

The physical and chemical properties of a compound depends on its structure, and topology of a structure can be easily studied with the help of graph theory. The chemical structure can be described graphically, in which edges denotes

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chemical bonding of atoms, where as vertices denote atoms. In 1965, Crum Brown and Fraser presented the first quantitative structure activity relationships (QSAR) in [3]. In this work only simple connected graphs are under consideration. For a graph G we denote vertex set by $V(G)$ and edge set by $E(G)$. The degree of a vertex $v \in V(G)$ is the number of vertices incident to G and the degree of an edge e in G is defined as if $e = uv$, where $u, v \in V(G)$ then $d(e) = d(u) + d(v) - 2$. Topological index is a numerical quantity which gives information about molecular structure. In the description of a chemical structure, topological index is one of the most powerful and useful tool. The oldest topological index is the Wiener index, which was introduced in 1947. Later on many topological indices were introduced, for example Wiener index Hyper-Wiener index, Estrada index, Rand$\acute{i}$c index, Zagreb index, and Szeged index. There are many intense applications of topological indices of a molecular graphs, for further details of applications see [9, 10, 15, 21, 22, 23, 24].

In 1972, one of the oldest graph invariant was introduced by Gutman and N. Trinajstič called Zagreb index, for more details of Zagreb index see [13]. They examined the dependence of total π-electron energy on molecular structure and it was further studied in [12]. For a molecular graph, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} d_G(u) + d_G(v),$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

They were further studied in [1, 2, 5, 6, 11, 14, 17, 18]. An edge of graph G corresponds to a vertex of its line graph $L(G)$. Motivated by this fact, in 2004 Miličević et al. introduced the reformulated Zagreb indices in [19], which is a generalization of classical Zagreb indices of chemical graph theory. They defined Zagreb indices in term of edge degree instead of vertex degree as:

$$EM_1(G) = \sum_{e \in E(G)} d_G^2(e),$$

$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f).$$

where if $e = uv$, where $u, v \in V(G)$ then $d(e) = d(u) + d(v) - 2$ and $e \sim f$ means edges e and f are adjacent, that is, they shares a common end vertex. Different mathematical properties of reformulated Zagreb indices have been studied in [25]. In [16], Ilić et al.,establish further mathematical properties of the reformulated Zagreb indices. Recently, the upper and lower bounds on $EM_1(G)$ and $EM_2(G)$ were presented in [25, 16, 7]; Su et al. [20] characterize the extremal graph properties on $EM_1(G)$ with respect to given connectivity. The motivation to study reformulated Zagreb indices is the property that the reformulated

Zagreb index of a graph G is equals to the Zagreb index of corresponding line graph $L(G)$. Where, line graph of a graph G is a graph whose vertices are the edges of G, that is, $V(L(G)) = E(G)$ and two vertices in line graph of G are connected if they are adjacent edges in G. In other words, $EM_1(G) = M_1(L(G))$ and $EM_2(G) = M_2(L(G))$.

In this work, we considered four operations on graphs, which were introduced in [8] and we compute the first reformulated Zagreb index for them. While first and second Zagrab indices of these operations are given in [8]. It is worth recalling those operations here. Let $G_1 \times G_2$ denotes the cartesian product of the graphs G_1 and G_2. Then $G_1 \times G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $G_1 \times G_2$ forms an edge if and only if $u_1 = u_2$ and v_1v_2 is an edge in G_2 or $v_1 = v_2$ and u_1u_2 is an edge in G_1.

For a simple connected graph G, the subdivision graph is denoted by $S(G)$ and formulated by inserting an additional vertex in each edge of G. In other words, each edge of G is replaced by path of length 2. Other three related graphs can be defined as follows:

1. For a simple connected graph G, $R(G)$ is be formulated by adding a new vertex corresponding to every edge of G, then joining each new vertex to the end vertices of the corresponding edge.

2. For a simple connected graph G, $Q(G)$ is be formulated by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on edges of G.

3. For a simple connected graph G, $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. The graphs $T(G)$ is called the total graph of G.

For more details on these operations we refer the reader to [4].

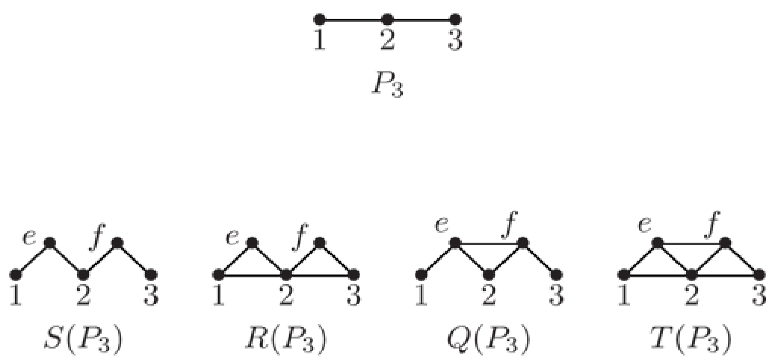

Figure 1: Four Operations on P_3.

In Figure 1, we have the examples of all four operations discussed above, where P_3 represent the path graph on 3 vertices. In general, a path graph on vertex set $\{v_1, v_2, \ldots, v_n\}$ is a graph whose edges are $\{v_i, v_{i+1}\}$, where $1 \leq i \leq n-1$. It is denoted by P_n.

Suppose that G_1 and G_2 are two connected graphs. Based on these operations above, Eliasi and Taeri [9] introduced four new operations on these graphs in the following: Let $F \in S, R, Q, T$. The F-sum of G_1 and G_2, denoted by $G_1 +_F G_2$ is defined by $F(G_1) \times G_2 - E^*$, where $E^* = \{(u, v_1)(u, v_2) \in E(F(G_1) \times (G_2)) : \ u \in V(F(G_1)) - V(G_1), \ v_1v_2 \in E(G_2)\}$ i.e., $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \bigcup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))]$.

2. The first reformulated Zagreb index of F-sums of graphs

In this section we compute the first reformulated Zagreb index of four operations of graphs discussed earlier. First of all we consider the case when the operation is S. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_S G_2$ is shown in Figure 2. Where $G_1 +_S G_2 = P_4 +_S P_2$ is defined by $S(P_4) \times G_{P_2} - E^*$, where $E^* = \{(u, v_1)(u, v_2) \in E(S(P_4) \times (P_2)) : \ u \in V(S(P_4)) - V(P_4), \ v_1v_2 \in E(P_2)\}$ i.e., $P_4 +_S P_2$ is a graph with the set of vertices $V(P_4 +_S P_2) = (V(P_4) \bigcup E(P_4)) \times V(P_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $P_4 +_S P_2$ are adjacent if and only if $[u_1 = v_1 \in V(P_4)$ and $u_2v_2 \in E(P_2)]$ or $[u_2 = v_2 \in V(P_2)$ and $u_1v_1 \in E(S(P_4))]$.

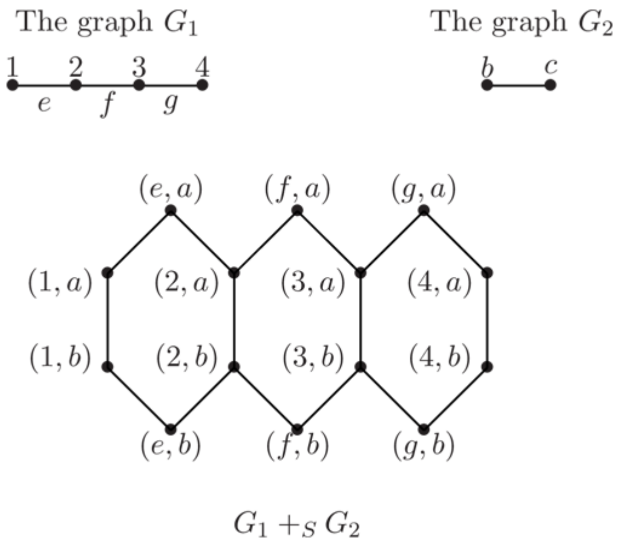

Figure 2: Graph G_1, G_2 and $G_1 +_S G_2$.

In the following theorem we compute $EM_1(G_1 +_S G_2)$.

Theorem 2.1. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1 +_S G_2) = 8e_2M_1(G_1)+10e_1M_1(G_2)+n_1EM_1(G_2)+n_2EM_1(S(G_1))-16e_1e_2$, *where* $n_i = |V(G_i)|$ *and* $e_i = |E(G_i)|, i = 1, 2$.

Proof. Let $d(u, v) = d_{G_1+_SG_2}(u, v)$ and

$$d((u_1, v_1)(u_2, v_2)) = d_{G_1+_SG_2}((u_1, v_1)(u_2, v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1+_SG_2$ respectively

$$\begin{aligned}
EM_1(G_1 +_S G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_SG_2)} d^2((u_1, v_1)(u_2, v_2)) \\
&= S_1 + S_2 \\
\text{where, } S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u, v_1)(u, v_2)) \\
\text{and } S_2 &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} d^2((u_1, v)(u_2, v)) \\
S1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u, v_1) + d(u, v_2) - 2]^2 \\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [2d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\
&= 4e_2M_1(G_1) + n_1EM_1(G_2) + 8e_1M_1(G_2) - 16e_1e_2. \qquad (1)
\end{aligned}$$

In the similar way,

$$\begin{aligned}
S_2 &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d(u_1, v) + d(u_2, v) - 2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d_{S(G_1)}(u_1) + d_{(G_2)}(v) + d_{S(G_1)}(u_2) - 2]^2.
\end{aligned}$$

Since $u_1 \in V(G_1)$, $u_2 \in V(S(G_1) - V(G_1))$ and $|E(S(G_1))| = 2|E(G_1)|$. So,

$$S_2 = 2e_1M_1(G_2) + 4e_2M_1(S(G_1)) - 16e_1e_2 + n_2EM_1(S(G_1)). \qquad (2)$$

Note that $M_1(S(G_1)) = M_1(G_1) + 4e_1$. So finally from (1) and (2), we have:

$$\begin{aligned}
EM_1(G_1 +_S G_2) &= 8e_2M_1(G_1) + 10e_1M_1(G_2) + n_1EM_1(G_2) \\
&\quad +n_2EM_1(S(G_1)) - 16e_1e_2. \qquad (3)
\end{aligned}$$

□

Let G be a graph, then $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Then for two connected graphs G_1 and G_2, $G_1 +_R G_2$

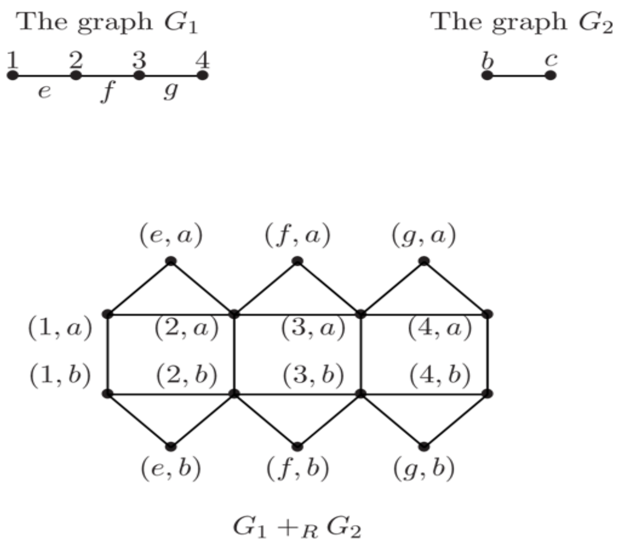

Figure 3: Graph G_1, G_2 and $G_1 +_R G_2$.

is defined with set of vertices $V(G_1 +_R G_2) = (V(G_1) \bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_R G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(R(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_R G_2$ is shown in Figure 3.

In the following theorem we compute $EM_1(G_1 +_R G_2)$.

Theorem 2.2. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1 +_R G_2) = 40e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + 4n_2EM_1(G_1) + 4n_2M_3(G_1) + 8n_2M_1(G_1) + 12n_2e_1 - 48e_1e_2$.

Proof. Let

$$d(u, v) = d_{G_1+_RG_2}(u, v) \text{ and } d((u_1, v_1)(u_2, v_2)) = d_{G_1+_RG_2}((u_1, v_1)(u_2, v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1+_RG_2$ respectively.

$$EM_1(G_1 +_R G_2) = \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_RG_2)} d^2((u_1, v_1)(u_2, v_2)) = S_1 + S_2,$$

where $S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u, v_1)(u, v_2))$ and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(R(G_1))} d^2((u_1, v)(u_2, v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2) - 2]^2 \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [4d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\ &= 16e_2 M_1(G_1) + n_1 EM_1(G_2) + 16e_1 M_1(G_2) - 32e_1 e_2. \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)), \\ u_1, u_2 \in V(G_1)}} d^2((u_1, v)(u_2, v)) \\ &+ \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)), \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} d^2((u_1, v)(u_2, v)) \\ &= S_1^{'} + S_2^{'} \end{aligned}$$

Note that $(i) u_1, u_2 \in V(G_1)$ and $u_1 u_2 \in E(R(G_1))$ if and only if $u_1 u_2 \in E(G_1)$, $(ii) d_{R(G_1)}(u_1) = 2d_{G_1}(u_1)$ for $u_1 \in V(G_1)$, and $d_{R(G_1)}(u_2) = 2$ for $u_2 \in V(R(G_1)) - V(G_1)$, we have

$$\begin{aligned} S_1^{'} &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)), \\ u_1, u_2 \in V(G_1)}} [d(u_1, v) + d(u_2, v) - 2]^2 \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)), \\ u_1, u_2 \in V(G_1)}} [2d_{G_2}(v) + d_{R(G_1)}(u_1) + d_{R(G_1)}(u_2) - 2]^2 \\ &= 4n_2 \sum_{\substack{u_1 u_2 \in E(R(G_1)), \\ u_1, u_2 \in V(G_1)}} [d_{G_1}(u_1) + d_{G_1}(u_2) - 1]^2 \\ &\quad + 4e_1 M_1(G_2) + 16e_2 M_1(G_1) - 16e_1 e_2 \\ &= 4n_2 \sum_{\substack{u_1 u_2 \in E(G_1), \\ u_1, u_2 \in V(G_1)}} [(d_{G_1}(u_1) + d_{G_1}(u_2) - 2)^2 + 4e_1 M_1(G_2) + 16e_2 M_1(G_1) \\ &\quad - 16e_1 e_2 + 2(d_{G_1}(u_1) + d_{G_1}(u_2) - 2) + 1] \\ &= 4e_1 M_1(G_2) + 16e_2 M_1(G_1) - 16e_1 e_2 + 4n_2 EM_1(G_1) \\ &\quad + 8n_2 M_1(G_1) - 12n_2 e_1. \end{aligned}$$

and

$$
\begin{aligned}
S_2' &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [d_{G_2}(v)+d_{R(G_1)}(u_1)+d_{R(G_1)}(u_2)-2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(R(G_1)),\\ u_1\in V(G_1),u_2\in V(R(G_1))-V(G_1)}} [2d_{G_1}(u_1)+d_{G_2}(v)]^2 \\
&= 4n_2M_3(G_1)+2e_1M_1(G_2)+8e_2M_1(G_1).
\end{aligned}
$$

Therefore, $EM_1(G_1 +_R G_2) = 40e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + 4n_2EM_1(G_1) + 4n_2M_3(G_1) + 8n_2M_1(G_1) + 12n_2e_1 - 48e_1e_2$. □

Let G be a graph, then $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on edges of G. For two connected graphs G_1 and G_2, $G_1 +_Q G_2$ is defined with set of vertices $V(G_1 +_Q G_2) = (V(G_1)\bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_Q G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(Q(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_Q G_2$ is shown in Figure 4

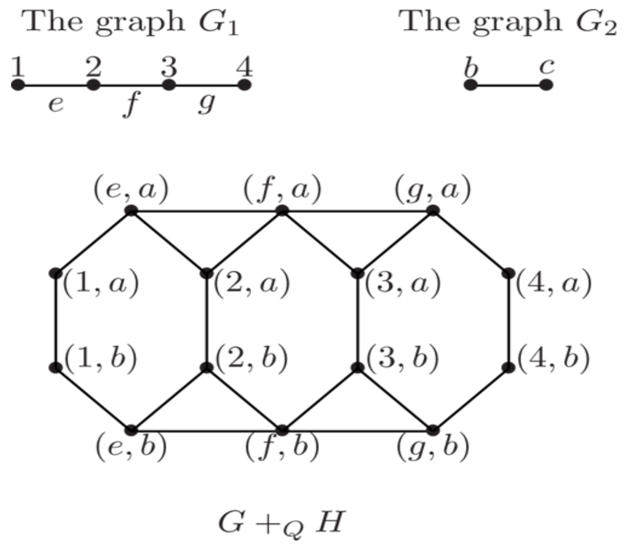

Figure 4: Graph G_1, G_2 and $G_1 +_Q G_2$.

In the following theorem we compute $EM_1(G_1 +_Q G_2)$.

Theorem 2.3. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1+_QG_2) = 16e_2M_1(G_1) + 10e_1M_1(G_2) - 32e_1e_2 + n_1EM_1(G_2) + n_2EM_1(Q(G_1)).$

Proof. Let

$$d(u,v) = d_{G_1+_Q G_2}(u,v) \text{ and } d((u_1,v_1)(u_2,v_2)) = d_{G_1+_Q G_2}((u_1,v_1)(u_2,v_2))$$

be the degrees of a vertex (u,v) and an edge $(u_1,v_1)(u_2,v_2)$ in the graph $G_1 +_Q G_2$, respectively

$$EM_1(G_1 +_Q G_2) = \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_Q G_2)} d^2((u_1,v_1)(u_2,v_2)) = S_1 + S_2,$$

where $S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u,v_1)(u,v_2))$ and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(Q(G_1))} d^2((u_1,v)(u_2,v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u,v_1) + d(u,v_2) - 2]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [2d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2) - 2]^2 \\ &= 4e_2M_1(G_1) + 8e_1M_1(G_2) - 16e_1e_2 + n_1EM_1(G_2). \end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &= S_1' + S_2'. \end{aligned}$$

Then,

$$\begin{aligned} S_1' &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} [d(u_1,v) + d(u_2,v) - 2]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1),u_2\in V(Q(G_1))-V(G_1)}} [d_{G_2}(v) + d_{Q(G_1)}(u_1) \\ &\qquad + d_{Q(G_1)}(u_2) - 2]^2. \end{aligned} \tag{5}$$

Note that $d_{T(G_1)}(u_2) = d_{G_1}(w_i) + d_{G_1}(w_j)$ for $u_2 \in V(T(G_1)) - V(G_1)$, where u_2 is the vertex inserted into the edge w_iw_j of G_1, we have

$$\begin{aligned} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d_{T(G_1)}(u_2) &= 2\sum_{w_iw_j\in E(G_1)} [d_{G_1}(w_i) + d_{G_1}(w_j)] \\ &= 2M_1(G_1). \end{aligned}$$

So,

$$\begin{aligned} S_1^{'} &= 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1\in V(G_1), u_2\in V(Q(G_1))-V(G_1)}} (d_{Q(G_1)}(u_1) + d_{Q(G_1)}(u_2) - 2)^2 \\ &= 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 + n_2EM_1(Q(G_1)) \\ &- \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1)),\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} [d(u_1,v) + d(u_2,v) - 2]^2. \end{aligned}$$

Hence,

$$(6) \qquad S_2 = 2e_1M_1(G_2) + 12e_2M_1(G_1) - 16e_1e_2 + n_2EM_1(Q(G_1))$$

So, finally from (4) and (6), we have:

$$\begin{aligned} EM_1(G_1 +_Q G_2) &= 16e_2M_1(G_1) + 10e_1M_1(G_2) - 32e_1e_2 \\ &+ n_1EM_1(G_2) + n_2EM_1(Q(G_1)). \end{aligned}$$

□

Let G be a graph, Then $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. For two connected graphs G_1 and G_2, $G_1 +_T G_2$ is defined with set of vertices $V(G_1 +_Q G_2) = (V(G_1) \bigcup E(G_2)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_T G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(T(G_1))]$. For example, if $G_1 = P_4$ and $G_2 = P_2$ then $G_1 +_T G_2$ is shown in Figure 5.

Theorem 2.4. *Let G_1 and G_2 be two connected graphs. Then* $EM_1(G_1 +_T G_2) = 48e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) + n_2EM_1(T(G_1)) - 64e_1e_2$.

Proof. Let

$$d(u,v) = d_{G_1+_TG_2}(u,v) \text{ and } d((u_1,v_1)(u_2,v_2)) = d_{G_1+_TG_2}((u_1,v_1)(u_2,v_2))$$

be the degrees of a vertex (u, v) and an edge $(u_1, v_1)(u_2, v_2)$ in the graph $G_1 +_T G_2$ respectively

$$\begin{aligned} EM_1(G_1 +_T G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_TG_2)} d^2((u_1,v_1)(u_2,v_2)) \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} d^2((u,v_1)(u,v_2))$$

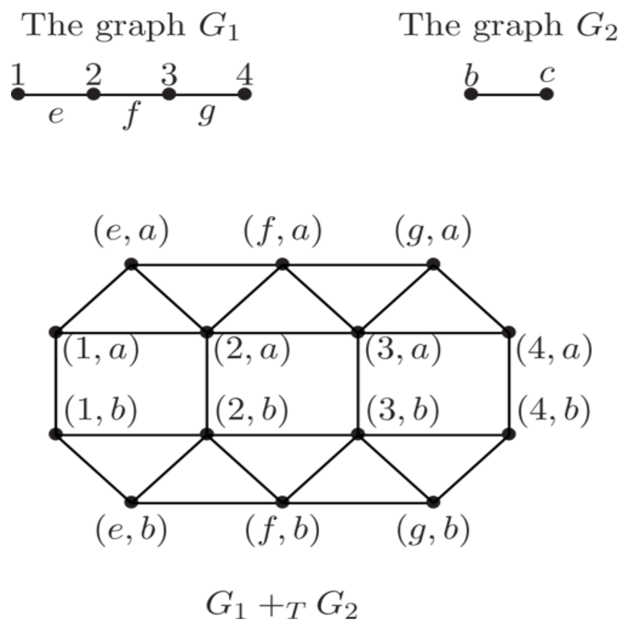

Figure 5: Graph G_1, G_2 and $G_1 +_T G_2$.

and

$$S_2 = \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(T(G_1))} d^2((u_1,v)(u_2,v)).$$

Now,

$$\begin{aligned} S_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d(u,v_1)+d(u,v_2)-2]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [4d_{G_1}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)-2]^2 \\ &= 16e_2M_1(G_1)+n_1EM_1(G_2)+16e_1M_1(G_2)-32e_1e_2. \qquad (7) \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(T(G_1))-V(G_1)}} d^2((u_1,v)(u_2,v)) \\ &= S_1^{'}+S_2^{'}+S_3^{'}. \end{aligned}$$

Note that $(i)u_1, u_2 \in V(G_1)$ and $u_1u_2 \in E(T(G_1))$ if and only if $u_1u_2 \in E(G_1)$, $(ii)d_{T(G_1)}(u_1) = 2d_{G_1}(u_1)$ for $u_1 \in V(G_1)$, we have

$$\begin{aligned}
S_1^{'} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(G_1),\\ u_1,u_2\in V(G_1)}} [2d_{G_2}(v)+d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2]^2 \\
&= 4e_1M_1(G_2)+16e_2M_1(G_1)-16e_1e_2 \\
&+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1,u_2\in V(G_1)}} (d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2.
\end{aligned}$$

Now,

$$\begin{aligned}
S_2^{'} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d(u_1,v)+d(u_2,v)-2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d_{G_2}(v)+d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2]^2 \\
&= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} [d_{G_2}^2(v)+2d_{G_2}(v)(2d_{G_1}(u_1) \\
&+d_{T(G_1)}(u_2)-2)+(d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2].
\end{aligned}$$

Note that $d_{T(G_1)}(u_2) = d_{G_1}(w_i) + d_{G_1}(w_j)$ for $u_2 \in V(T(G_1)) - V(G_1)$, where u_2 is the vertex inserted into the edge w_iw_j of G_1, we have

$$\begin{aligned}
\sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} d_{T(G_1)}(u_2) &= 2\sum_{w_iw_j\in E(G_1)} [d_{G_1}(w_i)+d_{G_1}(w_j)] \\
&= 2M_1(G_1).
\end{aligned}$$

So,

$$\begin{aligned}
S_2^{'} &= 16e_2M_1(G_1)+2e_1M_1(G_2)-16e_1e_2 \\
+\sum_{v\in V(G_2)} & \sum_{\substack{u_1u_2\in E(T(G_1)),\\ u_1\in V(G_1),u_2\in V(T(G_1))-V(G_1)}} (d_{T(G_1)}(u_1)+d_{T(G_1)}(u_2)-2)^2
\end{aligned}$$

finally,

$$S_2 = 32e_2M_1(G_1)+6e_1M_1(G_2)+n_2EM_1(T(G_1))-32e_1e_2. \tag{8}$$

Therefore, from (7) and (8), we have:

$$\begin{aligned} EM_1(G_1 +_T G_2) &= 48e_2M_1(G_1) + 22e_1M_1(G_2) + n_1EM_1(G_2) \\ &\quad + n_2EM_1(T(G_1)) - 64e_1e_2. \end{aligned}$$

□

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