The number of the generating matrices of the subspaces which represent an F_pW_n-submodule where $F_p = GF(p)$ and W_n is the Weyl group of type B_n

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Abstract. In this paper, we will find the number of the generating matrices of the subspaces, which represent any ℓ-dimensional F_pW_n-submodule N of an m-dimensional F_pW_n-module M and apply this result on three samples of submodules.

Keywords: finite field $F_p = GF(p)$, Weyl group W_n of type B_n, group ring F_pW_n, F_pW_n-module, F_pW_n-submodule, pair of partitions (λ, μ) of a positive integer n, Specht polynomial, Specht module, generating matrix of a subspace, statistics theorems

1. The number of bases of an F_pW_n-module M and the number of the generating matrices of the vector space representing M

Throughout this paper let F_p be the finite field of order p, which is the Galois field $GF(p)$, W_n be the Weyl group of type B_n.

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Theorem 1.1. *There are exactly*

$$\frac{(p^m - 1)\,(p^m - p)\ldots(p^m - p^{m-1})}{m!}$$

distinct F_p-bases of any m-dimensional F_pW_n-module M.

Proof. Let $B=\{b_1, b_2, \ldots, b_m\}$ be an F_p-basis of M and let $B'=\{b'_1, b'_2, \ldots, b'_m\}$ be any F_p-basis of M (B' and B are not necessarily distinct), then $b'_1, b'_2, \ldots, b'_m$ can be written as linear combinations of $b_1, b_2, \ldots, b_m$ as follows:

$$\begin{aligned} b'_1 &= c_{11}b_1 + c_{1\,2}b_2 + \ldots + c_{1\,m}b_m \\ b'_2 &= c_{21}b_1 + c_{2\,2}b_2 + \ldots + c_{2\,m}b_m \\ &\vdots \\ b'_m &= c_{m1}b_1 + c_{m\,2}b_2 + \ldots + c_{m\,m}b_m \end{aligned}$$

where $c_{ij} \in F_p$ for each $i = 1, 2, ..., m$, and for each $j = 1, 2, ..., m$ with the matrix $G = [c_{ij}]_{m\times m}$ is non-singular. b'_1 can be any nonzero element of M, which means that there are $(p^m - 1)$ ways to select b'_1 since $|M| = p^m$; b'_2 can be any element of M, which is not a multiple of b'_1 (i.e., $b'_2 \neq k_{21}b'_1$ with $k_{21} \in F_p$), which means that there are $(p^m - p)$ ways to select b'_2; b'_3 can be any element of M, which is not a linear combination of b'_1 and b'_2 (i.e., $b'_3 \neq k_{31}b'_1 + k_{32}b'_2$ with $k_{31}, k_{32} \in F_p$), which means that there are $(p^m - p^2)$ ways to select b'_3; ...; and b'_m can be any element of M, which is not a linear combination of $b'_1, b'_2, \ldots, b'_{m-1}$ (i.e., $b'_m \neq k_{m1}b'_1 + k_{m2}b'_2 + ... + k_{m\,m-1}b'_{m-1}$ with $k_{m1}, k_{m2}, ..., k_{m\,m-1} \in F_p$), which means that there are $(p^m - p^{m-1})$ ways to select b'_m.

Thus there are $(p^m - 1)(p^m - p)\ldots(p^m - p^{m-1})$ ordered bases of M.

Therefore the number of bases of M is

$$\frac{(p^m - 1)(p^m - p)\ldots(p^m - p^{m-1})}{m!},$$

since each basis can be written as $m!$ ordered bases. □

Corollary 1.1. *The number of generating matrices of F_p^m, which represent the F_pW_n-module M is $(p^m - 1)(p^m - p)\ldots(p^m - p^{m-1})$.*

Proof. Let B and B' be the bases given in theorem 1.1, and let G be the matrix representing the basis B' with respect to B, then the first row r_1 of G represents the first element b'_1 of the basis B', the second row r_2 of G represents the second element b'_2 of the basis B',..., the m^{th} row r_m of G represents the m^{th} element b'_m of the basis B', and hence the matrix G is a generating matrix of F_p^m.

Thus there are $(p^m - 1)$ different vectors, which can be the first row of G, and there are $(p^m - p)$ different vectors, which can be the second row of G,..., and there are $(p^m - p^{m-1})$ different vectors, which can be the m^{th} row of G. Therefore there are $(p^m - 1)(p^m - p)\ldots(p^m - p^{m-1})$ generating matrices G of F_p^m, which represent M. □

2. Three examples of F_pW_n-submodules

Example 2.1. Let $M_{F_5}((2,2),(1))$ be the F_5W_5-module corresponding to the pair of partitions $((2,2),(1))$ of 5, which is generated over F_5W_5 by the monomial $x_1x_2^3x_3x_4^3$, that is $M_{F_5}((2,2),(1)) = F_5W_5\, x_1x_2^3x_3x_4^3$, and let $S_{F_5}((2,2),(1))$ is the Specht module corresponding to the pair of partitions $((2,2),(1))$. $S_{F_5}((2,2),(1))$ is the F_5W_5-submodule of $M_{F_5}((2,2),(1))$, which is generated over F_5W_5 by the Specht polynomial $f_{(5)}(Z_1^{((2,2),(1))}) = x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3$, that is $S_{F_5}((2,2),(1)) = F_5W_5f_{(5)}\ (Z_1^{((2,2),(1))})$ $= F_5W_5(x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3)$. Then:

First: $\dim_{F_5} M_{F_5}((2,2),(1)) = \frac{n!}{\lambda_1!.\lambda_2!.\mu_1!} = \frac{5!}{2!.2!.1!} = 30$ (see [1] p.7 and [3] p.14).

Second: $M_{F_5}((2,2),(1))$ has a basis $B = \{b_1, b_2, \ldots, b_{30}\}$, where:
$b_1 = x_1^3x_2^3x_3x_4$, $b_2 = x_1^3x_2x_3^3x_4$, $b_3 = x_1x_2^3x_3^3x_4$, $b_4 = x_1^3x_2x_3x_4^3$, $b_5 = x_1x_2^3x_3x_4^3$, $b_6 = x_1x_2x_3^3x_4^3$, $b_7 = x_1^3x_2^3x_3x_5$, $b_8 = x_1^3x_2x_3^3x_5$, $b_9 = x_1x_2^3x_3^3x_5$, $b_{10} = x_1^3x_2x_3x_5^3$, $b_{11} = x_1x_2^3x_3x_5^3$, $b_{12} = x_1x_2x_3^3x_5^3$, $b_{13} = x_1^3x_2^3x_4x_5$, $b_{14} = x_1^3x_2x_4^3x_5$, $b_{15} = x_1x_2^3x_4^3x_5$, $b_{16} = x_1^3x_2x_4x_5^3$, $b_{17} = x_1x_2^3x_4x_5^3$, $b_{18} = x_1x_2x_4^3x_5^3$, $b_{19} = x_1^3x_3^3x_4x_5$, $b_{20} = x_1^3x_3x_4^3x_5$, $b_{21} = x_1x_3^3x_4^3x_5$, $b_{22} = x_1^3x_3x_4x_5^3$, $b_{23} = x_1x_3^3x_4x_5^3$, $b_{24} = x_1x_3x_4^3x_5^3$, $b_{25} = x_2^3x_3^3x_4x_5$, $b_{26} = x_2^3x_3x_4^3x_5$, $b_{27} = x_2x_3^3x_4^3x_5$, $b_{28} = x_2^3x_3x_4x_5^3$, $b_{29} = x_2x_3^3x_4x_5^3$, $b_{30} = x_2x_3x_4^3x_5^3$.

Third: $\dim_{F_5} S_{F_5}((2,2),(1)) = \frac{5!}{3.2.2.1.1} = 10$ (see [1], p.21, [2], p.305, and [3], p.17).

Fourth: $S_{F_5}((2,2),(1))$ has an F_5-basis $D = \{d_1, d_2, \ldots, d_{10}\}$, where $d_1, d_2, \ldots,$ d_{10} are the following Specht polynomials:

$$
\begin{aligned}
d_1 &= x_1^3x_2x_3^3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3,\\
d_2 &= x_1^3x_2^3x_3x_4 + 4x_1x_2^3x_3^3x_4 + 4x_1^3x_2x_3x_4^3 + x_1x_2x_3^3x_4^3,\\
d_3 &= x_1^3x_2x_3^3x_5 + 4x_1x_2^3x_3^3x_5 + 4x_1^3x_2x_3x_5^3 + x_1x_2^3x_3x_5^3,\\
d_4 &= x_1^3x_2^3x_3x_5 + 4x_1x_2^3x_3^3x_5 + 4x_1^3x_2x_3x_5^3 + x_1x_2x_3^3x_5^3,\\
d_5 &= x_1^3x_2x_4^3x_5 + 4x_1x_2^3x_4^3x_5 + 4x_1^3x_2x_4x_5^3 + x_1x_2^3x_4x_5^3,\\
d_6 &= x_1^3x_2^3x_4x_5 + 4x_1x_2^3x_4^3x_5 + 4x_1^3x_2x_4x_5^3 + x_1x_2x_4^3x_5^3,\\
d_7 &= x_1^3x_3x_4^3x_5 + 4x_1x_3^3x_4^3x_5 + 4x_1^3x_3x_4x_5^3 + x_1x_3^3x_4x_5^3,\\
d_8 &= x_1^3x_3^3x_4x_5 + 4x_1x_3^3x_4^3x_5 + 4x_1^3x_3x_4x_5^3 + x_1x_3x_4^3x_5^3,\\
d_9 &= x_2^3x_3x_4^3x_5 + 4x_2x_3^3x_4^3x_5 + 4x_2^3x_3x_4x_5^3 + x_2x_3^3x_4x_5^3,\\
d_{10} &= x_2^3x_3^3x_4x_5 + 4x_2x_3^3x_4^3x_5 + 4x_2^3x_3x_4x_5^3 + x_2x_3x_4^3x_5^3.
\end{aligned}
$$

The above polynomials $d_1,\ d_2, \ldots,\ d_{10}$ give the following generating matrix $\mu_{(5)}^{((2,2),(1))}$ for the subspace $V_{(5)}^{((2,2),(1))}$ (which represents the Specht module $S_{F_5}((2,2),(1)))$ of the vector space F_5^{30} (which represents the module $M_{F_5}((2,2),$

(1))):

$$\mu_{(5)}^{((2,2),(1))} = \left[\begin{matrix} 0&1&4&4&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&4&4&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&4&4&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&4&4&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&4&4&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&4&4&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 4&4&1&0&0&0&0&0&0&0\\ 4&4&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&4&4&1&0\\ 0&0&0&0&1&0&4&4&0&1 \end{matrix}\right]$$

There are fifteen pairs of equal columns in the matrix $\mu_{(5)}^{((2,2),(1))}$.

Example 2.2. Let $M_{F_3}((2,2),(1))$ be the F_3W_5-module corresponding to the pair of partitions $((2,2),1)$ of 5, which is generated over F_3W_5 by the monomial $x_1x_2^3x_3x_4^3$, that is $M_{F_3}((2,2),(1)) = F_3W_5x_1x_2^3x_3x_4^3$, and let $N_{F_3}((2,2),(1))$ be the F_3W_5-submodule of $M_{F_3}((2,2),(1))$, which is generated over F_3W_5 by the symmetrized Specht polynomial $f_{(3)}[Z_1^{((2,2),(1))}] = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3$, that is $N_{F_3}((2,2),(1)) = F_3W_5f_{(3)}[Z_1^{((2,2),(1))}] = F_3W_5(x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3)$. Then:

First: $\dim_{F_3} M_{F_3}((2,2),(1)) = \frac{n!}{\lambda_1!.\lambda_2!.\mu_1!} = \frac{5!}{2!.2!.1!} = 30$ (see [1], p.7 and [3], p.14).

Second: $M_{F_3}((2,2),(1))$ has a basis $B = \{b_1, b_2, \ldots, b_{30}\}$, given in second of Example 2.1.

Third: $N_{F_3}((2,2),(1))$ has an F_3-basis $D = \{d_1, d_2, \ldots, d_5\}$, where $d_1, d_2, \ldots, d_5$ are the following symmetrized Specht polynomials:

$$d_1 = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2x_3x_4^3 + x_1x_2^3x_3x_4^3 + x_1x_2x_3^3x_4^3,$$
$$d_2 = x_1^3x_2^3x_3x_5 + x_1^3x_2x_3^3x_5 + x_1x_2^3x_3^3x_5 + x_1^3x_2x_3x_5^3 + x_1x_2^3x_3x_5^3 + x_1x_2x_3^3x_5^3,$$
$$d_3 = x_1^3x_2^3x_4x_5 + x_1^3x_2x_4^3x_5 + x_1x_2^3x_4^3x_5 + x_1^3x_2x_4x_5^3 + x_1x_2^3x_4x_5^3 + x_1x_2x_4^3x_5^3,$$
$$d_4 = x_1^3x_3^3x_4x_5 + x_1^3x_3x_4^3x_5 + x_1x_3^3x_4^3x_5 + x_1^3x_3x_4x_5^3 + x_1x_3^3x_4x_5^3 + x_1x_3x_4^3x_5^3,$$
$$d_5 = x_2^3x_3^3x_4x_5 + x_2^3x_3x_4^3x_5 + x_2x_3^3x_4^3x_5 + x_2^3x_3x_4x_5^3 + x_2x_3^3x_4x_5^3 + x_2x_3x_4^3x_5^3.$$

Fourth: $\dim_{F_3} N_{F_3}((2,2),(1)) = 5$.
Fifth: The polynomials $d_1, d_2, \ldots, d_5$ given in Third above give the following generating matrix $\rho_{(3)}^{((2,2),(1))}$ for the subspace $U_{(3)}^{((2,2),(1))}$ (which represents the module $N_{F_3}((2,2),(1))$) of the vector space F_3^{30} (which represents the module $M_{F_3}((2,2),(1))$):

$$\rho_{(3)}^{((2,2),(1))} = \left[\begin{matrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1 \end{matrix}\right]$$

There are five of six equal columns in the matrix $\rho_{(3)}^{((2,2),(1))}$.

Example 2.3. Let $M_{F_5}((1,1,1),(2))$ be the F_5W_5-module corresponding to the pair of partitions $((1,1,1),(2))$ of 5, which is generated over F_5W_5 by the monomial $x_1x_2^3x_3^5$, that is $M_{F_5}((1,1,1),(2)) = F_5W_5x_1x_2^3x_3^5$, and let $N_{F_5}((1,1,1),(2))$ be the F_5W_5-submodule of $M_{F_5}((1,1,1),(2))$, which is generated over F_5W_5 by the symmetrized Specht polynomial $f_{(5)}[Z_1^{((1,1,1),(2))}] = 3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5$, that is $N_{F_5}((1,1,1),(2))$ $= F_5W_5f_{(5)}[Z_1^{((1,1,1),(2))}] = F_5W_5(3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5)$. Then:

First: $\dim_{F_5} M_{F_5}((1,1,1),(2)) = \frac{n!}{\lambda_1!.\lambda_2!.\lambda_3!.\mu_1!} = \frac{5!}{1!.1!.1!.2!} = 60$ (see [1], p.7 and [3], p.14).

Second: $M_{F_5}((1,1,1),(2))$ has a basis $B = \{b_1, b_2, \ldots, b_{60}\}$, where $b_1 = x_1^5x_2^3x_3$, $b_2 = x_1^3x_2^5x_3$, $b_3 = x_1^5x_2x_3^3$, $b_4 = x_1x_2^5x_3^3$, $b_5 = x_1^3x_2x_3^5$, $b_6 = x_1x_2^3x_3^5$, $b_7 = x_1^5x_2^3x_4$, $b_8 = x_1^3x_2^5x_4$, $b_9 = x_1^5x_3^3x_4$, $b_{10} = x_2^5x_3^3x_4$, $b_{11} = x_1^3x_3^5x_4$, $b_{12} = x_2^3x_3^5x_4$, $b_{13} = x_1^5x_2x_4^3$, $b_{14} = x_1x_2^5x_4^3$, $b_{15} = x_1^5x_3x_4^3$, $b_{16} = x_2^5x_3x_4^3$, $b_{17} = x_1x_3^5x_4^3$, $b_{18} = x_2x_3^5x_4^3$, $b_{19} = x_1^3x_2x_4^5$, $b_{20} = x_1x_2^3x_4^5$, $b_{21} = x_1^3x_3x_4^5$, $b_{22} = x_2^3x_3x_4^5$, $b_{23} = x_1x_3^3x_4^5$, $b_{24} = x_2x_3^3x_4^5$, $b_{25} = x_1^5x_2^3x_5$, $b_{26} = x_1^3x_2^5x_5$, $b_{27} = x_1^5x_3^3x_5$, $b_{28} = x_2^5x_3^3x_5$, $b_{29} = x_1^3x_3^5x_5$, $b_{30} = x_2^3x_3^5x_5$, $b_{31} = x_1^5x_4^3x_5$, $b_{32} = x_2^5x_4^3x_5$, $b_{33} = x_3^5x_4^3x_5$, $b_{34} = x_1^3x_4^5x_5$, $b_{35} = x_2^3x_4^5x_5$, $b_{36} = x_3^3x_4^5x_5$, $b_{37} = x_1^5x_2x_5^3$, $b_{38} = x_1x_2^5x_5^3$, $b_{39} = x_1^5x_3x_5^3$, $b_{40} = x_2^5x_3x_5^3$, $b_{41} = x_1x_3^5x_5^3$, $b_{42} = x_2x_3^5x_5^3$, $b_{43} = x_1^5x_4x_5^3$, $b_{44} = x_2^5x_4x_5^3$, $b_{45} = x_3^5x_4x_5^3$, $b_{46} = x_1x_4^5x_5^3$, $b_{47} = x_2x_4^5x_5^3$, $b_{48} = x_3x_4^5x_5^3$, $b_{49} = x_1^3x_2x_5^5$, $b_{50} = x_1x_2^3x_5^5$, $b_{51} = x_1^3x_3x_5^5$, $b_{52} = x_2^3x_3x_5^5$, $b_{53} = x_1x_3^3x_5^5$, $b_{54} = x_2x_3^3x_5^5$, $b_{55} = x_1^3x_4x_5^5$, $b_{56} = x_2^3x_4x_5^5$, $b_{57} = x_3^3x_4x_5^5$, $b_{58} = x_1x_4^3x_5^5$, $b_{59} = x_2x_4^3x_5^5$, $b_{60} = x_3x_4^3x_5^5$.

Third: $N_{F_5}((1,1,1),(2))$ has an F_5-basis $D = \{d_1, d_2, \dots, d_{10}\}$, where $d_1, d_2, \dots, d_{10}$ are the following symmetrized Specht polynomials:

$$\begin{aligned}
d_1 &= 3x_1^5x_2^3x_3 + 2x_1^3x_2^5x_3 + 2x_1^5x_2x_3^3 + 3x_1x_2^5x_3^3 + 3x_1^3x_2x_3^5 + 2x_1x_2^3x_3^5,\\
d_2 &= 3x_1^5x_2^3x_4 + 2x_1^3x_2^5x_4 + 2x_1^5x_2x_4^3 + 3x_1x_2^5x_4^3 + 3x_1^3x_2x_4^5 + 2x_1x_2^3x_4^5,\\
d_3 &= 3x_1^5x_2^3x_5 + 2x_1^3x_2^5x_5 + 2x_1^5x_2x_5^3 + 3x_1x_2^5x_5^3 + 3x_1^3x_2x_5^5 + 2x_1x_2^3x_5^5,\\
d_4 &= 3x_1^5x_3^3x_4 + 2x_1^3x_3^5x_4 + 2x_1^5x_3x_4^3 + 3x_1x_3^5x_4^3 + 3x_1^3x_3x_4^5 + 2x_1x_3^3x_4^5,\\
d_5 &= 3x_1^5x_3^3x_5 + 2x_1^3x_3^5x_5 + 2x_1^5x_3x_5^3 + 3x_1x_3^5x_5^3 + 3x_1^3x_3x_5^5 + 2x_1x_3^3x_5^5,\\
d_6 &= 3x_1^5x_4^3x_5 + 2x_1^3x_4^5x_5 + 2x_1^5x_4x_5^3 + 3x_1x_4^5x_5^3 + 3x_1^3x_4x_5^5 + 2x_1x_4^3x_5^5,\\
d_7 &= 3x_2^5x_3^3x_4 + 2x_2^3x_3^5x_4 + 2x_2^5x_3x_4^3 + 3x_2x_3^5x_4^3 + 3x_2^3x_3x_4^5 + 2x_2x_3^3x_4^5,\\
d_8 &= 3x_2^5x_3^3x_5 + 2x_2^3x_3^5x_5 + 2x_2^5x_3x_5^3 + 3x_2x_3^5x_5^3 + 3x_2^3x_3x_5^5 + 2x_2x_3^3x_5^5,\\
d_9 &= 3x_2^5x_4^3x_5 + 2x_2^3x_4^5x_5 + 2x_2^5x_4x_5^3 + 3x_2x_4^5x_5^3 + 3x_2^3x_4x_5^5 + 2x_2x_4^3x_5^5,\\
d_{10} &= 3x_3^5x_4^3x_5 + 2x_3^3x_4^5x_5 + 2x_3^5x_4x_5^3 + 3x_3x_4^5x_5^3 + 3x_3^3x_4x_5^5 + 2x_3x_4^3x_5^5.
\end{aligned}$$

Fourth: $\dim_{F_5} N_{F_5}((1,1,1),(2)) = 10$.

Fifth: The polynomials $d_1, d_2, \dots, d_{10}$ given in third above give the following generating matrix $\rho_{(5)}^{((1,1,1),(2))}$ for the subspace $U_{(5)}^{((1,1,1),(2))}$ (which represents the submodule $N_{F_5}((1,1,1),(2))$) of the vector space F_5^{60} (which represents the module $M_{F_5}((1,1,1),(2))$):

$$\rho_{(5)}^{((1,1,1),(2))} = \left[\begin{matrix}
3 & 2 & 2 & 3 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.$$

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0\\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0
\end{matrix}$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 2
\end{matrix}\right]$$

There are twenty of triple equal columns in the matrix $\rho_{(5)}^{((1,1,1),(2))}$.

3. The number of generating matrices representing a submodule

Theorem 3.1. *Let N be an ℓ-dimensional F_pW_n-submodule of an m-dimensional F_pW_n-module M, U is an ℓ-dimensional subspace of F_p^m, where U represents N, and the vector space F_p^m represents M, and G is a generating matrix of U, which has $g_1, g_2, \dots, g_s$ numbers of equal columns. Then the number of distinct generating matrices of all subspaces U that represent N is $\frac{m!}{g_1!.g_2!\dots g_s!.\ell!}(p^\ell - 1)(p^\ell - p)\dots(p^\ell - p^{\ell-1})$.*

Proof. Fix the order of the columns corresponding to the elements of a basis $B_M = \{b_1, b_2, \dots, b_m\}$ of M. Then by corollary 1.1 the number of generating matrices of an ℓ-dimensional subspace U representing N is $(p^\ell - 1)(p^\ell - p)\dots(p^\ell - p^{\ell-1})$.
Changing the order of the columns corresponding to the elements of B_M, we get that the number of the permutations of the columns of G is $\frac{m!}{g_1!.g_2!\dots g_s!.\ell!}$, since there are $g_1, g_2, ..., g_s$ numbers of equal columns and regarding that the changing of the order of the elements of a basis B_N of N (which implies the swapping of the columns representing any element of B_N with the columns representing another element of B_N) does not change the number of the generating matrices of U. Thus the number of the distincts generating matrices of all subspaces U that represent N is $\frac{m!}{g_1!.g_2!\dots g_s!.\ell!}(p^\ell - 1)(p^\ell - p)\dots(p^\ell - p^{\ell-1})$. □

Corollary 3.1. *The number of the distinct generating matrices of all the subspaces $V_{(5)}^{((2,2),(1))}$ of F_5^{30}, which represent the Specht module $S_{F_5}((2,2),(1))$ as an F_5W_5-submodule of the F_5W_5-module $M_{F_5}((2,2),(1))$ is $2^{302.7155187}$.*

Proof. The matrix $\mu_{(5)}^{((2,2),(1))}$ is a generating matrix of a subspace $V_{(5)}^{((2,2),(1))}$, which represents the Specht module $S_{F_5}((2,2),(1))$ for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \dots, b_{30}\}$ given in example 2.1.

Since the matrix $\mu_{(5)}^{((2,2),(1))}$ has fifteen pairs of equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $V_{(5)}^{((2,2),(1))}$ that represent $S_{F_5}((2,2),(1))$ is $\frac{30!}{(2!)^{15}.10!}(5^{10}-1)(5^{10}-5)...(5^{10}-5^9) = 2^{70.91800623}.2^{231.7975125} = 2^{302.7155187}$. □

Corollary 3.2. *The number of the distinct generating matrices of all the subspaces* $U_{(3)}^{((2,2),(1))}$ *of* F_3^{30}*, which represent the* F_3W_5*-submodule* $N_{F_3}((2,2),(1))$ *of the* F_3W_5*-module* $M_{F_3}((2,2),(1))$ *is* $2^{77.24959681}$.

Proof. The matrix $\rho_{(3)}^{((2,2),(1))}$ is a generating matrix of a subspace $U_{(3)}^{((2,2),(1))}$, which represents the F_3W_5-submodule $N_{F_3}((2,2),(1))$, for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \ldots, b_{30}\}$ given in example 2.1.

Since the matrix $\rho_{(3)}^{((2,2),(1))}$ has five of six equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $U_{(3)}^{((2,2),(1))}$ that represent $N_{F_3}((2,2),(1))$ is
$\frac{30!}{(6!)^5.10!}.(3^5-1)(3^5-3)...(3^5-3^4) = 2^{38.45874075}.2^{38.79085606} = 2^{77.24959681}$. □

Corollary 3.3. *The number of the distinct generating matrices of all the subspaces* $U_{(5)}^{((1,1,1),(2))}$ *of* F_5^{60}*, which represent the* F_5W_5*-submodule* $N_{F_5}((1,1,1),(2))$ *of the* F_5W_5*-module* $M_{F_5}((1,1,1),(2))$ *is* $2^{430.4401317}$.

Proof. The matrix $\rho_{(5)}^{((1,1,1),(2))}$ is a generating matrix of a subspace $U_{(5)}^{((1,1,1),(2))}$, which represents the F_5W_5-submodule $N_{F_5}((1,1,1),(2))$ for a specific order of the columns representing the elements of the basis $B = \{b_1, b_2, \ldots, b_{60}\}$ given in Example 2.3.

Since the matrix $\rho_{(5)}^{((1,1,1),(2))}$ has twenty of triple equal columns. Then by theorem 3.1, the number of the distinct generating matrices of all subspaces $U_{(5)}^{((1,1,1),(2))}$ that represent $N_{F_5}((1,1,1),(2))$ is $\frac{60!}{(3!)^{20}.10!}(5^{10}-1)(5^{10}-5)...(5^{10}-5^9) = 2^{198.6426192}.2^{231.7975125} = 2^{430.4401317}$. □

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Accepted: 16.10.2019