Bochner-Martinelli type formula over the quaternionic Heisenberg group and the octonionic Heisenberg group

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Abstract. The tangential k-Cauchy-Fueter operator on the quaternionic Heisenberg group is counterpart of the tangential Cauchy-Riemann operator on the Heisenberg group in the theory of several complex variables. In this paper, we establish the Bochner-Martinelli type formula for tangential k-Cauchy-Fueter type operators over the quaternionic Heisenberg group and the octonionic Heisenberg group.
Keywords: the tangential k-Cauchy-Fueter operator, the Bochner-Martinelli formula, the quaternionic Heisenberg group, the octonionic Heisenberg group.

1. Introduction

The k-Cauchy-Fueter complex has been extensively studied (cf. [2,5,6,20,22] and references therein), which are used to investigate properties of k-regular functions (cf. [7, 20] and references therein), and 1-Cauchy-Fueter operator is exactly the usual Cauchy-Fueter operator in several quaternionic variables.

Quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ is the simplest model of quaternionic contact manifolds (cf. [3, 8, 9] and references therein for more information about quaternionic contact manifolds). A natural generalization of k-Cauchy-Fueter operator on the quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ is the tangential k-Cauchy-Fueter operator, which is quaternionic counterpart of the tangential Cauchy-Riemann operator on the Heisenberg group in the theory of several complex variables. More generally, we can also define this kinds of operators on the octonionic Heisenberg group.

The Bochner-Martinelli integral representation formula in several complex variables was appeared in the works of Martinelli and Bochner at the begin-

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ning of the 1940's. It have a lot of applications in several complex variables. See [11] for more details of the Bochner-Martinelli integral representation formula in several complex variables. The Bochner-Martinelli type formula for k-Cauchy-Fueter operator was obtained in [20]. H.-Y. Wang and G.-B. Ren found the explicit Bochner-Martinelli kernel [17]. Moreover they have established the Bochner-Martinelli formula for a regular type operator in octonionic space [16]. It was generalized to a massless field operator on $\mathbb{R}^6$ by Q.-Q. Kang and W. Wang [10]. Recently we have generalised Bochner-Martinelli formula to k-Cauchy-Fueter operator over $(4n+1)$-dimensional Heisenberg group [12].

In this paper, we will generalise the Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operators over the quaternionic Heisenberg group and the octonionic Heisneberg group.

In Section 2, we introduce the basic knowledge of quaternions, quaternionic Heisenberg group and octonions, octonionic Heisneberg group. We also give the definition of the tangential k-Cauchy-Fueter operator on quaternionic Heisenberg group and octionionic Heisenberg group, and their basic properties. In Section 3, we give the Bochner-Martinelli formula for tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group. In Section 4, we give the Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group.

2. Primaries

2.1 The quaternionic Heisenberg group and the tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group

The *quaternionic Heisenberg group* $\mathscr{H}_{\mathbb{H}}$ is $\mathbb{H}^n \oplus \text{Im}\ \mathbb{H}$ equipped with the multiplication given by

$$(y,s)\cdot(y',s') = \left(y+y', s+s'+2\text{Im}\left(y\bar{y'}\right)\right),$$

where $y, y' \in \mathbb{H}^n$ and $s, s' \in \text{Im}\ \mathbb{H}$. The norm of quaternionic Heisenberg group is defined by

$$\|(y,s)\| = \left(|y|^4 + |s|^2\right)^{\frac{1}{4}},$$

where $(y,s) = (y_1, \cdots, y_{4n}, s_1, s_2, s_3)$.

The multiplication of the quaternionic Heisenberg group $\mathscr{H}_{\mathbb{H}}$ can be written in terms of real variables (cf. [19, (2.13)]) as

$$(y,s)\cdot(y',s') = \left(y+y', s_\beta + s'_\beta + 2\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B^\beta_{kj} y_{4l+k} y'_{4l+j}\right),$$

for $y, y' \in \mathbb{R}^{4n}$, $s, s' \in \mathbb{R}^3$, $\beta = 1, 2, 3$, where B^β_{kj} is the (k, j)-th entry of the following matrices

$$B^1 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^2 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$B^3 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

satisfying the commutating relation of quaternions $(B^1)^2 = (B^2)^2 = (B^3)^2 = -\mathrm{id}$, $B^1B^2 = B^3$. Then

$$Y_{4l+j} := \partial_{y_{4l+j}} + 2\sum_{\beta=1}^{3}\sum_{k=1}^{4} B^\beta_{kj} y_{4l+k}\partial_{s_\beta},$$

are the left invariant vector fields of $\mathscr{H}_{\mathbb{H}}$, whose brackets are

$$[Y_{4l+k}, Y_{4l+j}] = 4\sum_{\beta=1}^{3} B^\beta_{kj}\partial_{s_\beta}, \qquad \text{and} \qquad [Y_{4l+k}, Y_{4l'+j}] = 0 \quad \text{for } l \neq l',$$

where $l, l' = 0, 1, \cdots, n-1$, $j, k = 1, \cdots, 4$. We denote *SubLaplacian* $\Delta_{\mathbb{H}}$ *on the quaternionic Heisenberg group* by

$$\Delta_{\mathbb{H}} := -\sum_{k=1}^{4n} Y_k^2. \tag{2.1}$$

The standard $\mathbb{R}^3$-valued contact form of $\mathscr{H}_{\mathbb{H}}$ is

$$\Theta_{\mathbb{H}} := \mathrm{d}s - y\cdot \mathrm{d}\bar{y} + \mathrm{d}y\cdot\bar{y}.$$

If we write $\Theta_{\mathbb{H}} = (\theta_{\mathbb{H};1}, \theta_{\mathbb{H};2}, \theta_{\mathbb{H};3})$, then we have

$$\theta_{\mathbb{H};\beta} = \mathrm{d}s_\beta - 2\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B^\beta_{kj} y_{4l+k}\mathrm{d}y_{4l+j}, \quad \beta = 1, 2, 3,$$

(cf. [19, (2.20)]). Denote

$$(Z_{AA'}) := \begin{pmatrix} Y_1 + \mathbf{i}Y_2 & -Y_3 - \mathbf{i}Y_4 \\ Y_3 - \mathbf{i}Y_4 & Y_1 - \mathbf{i}Y_2 \\ \vdots & \vdots \\ Y_{4l+1} + \mathbf{i}Y_{4l+2} & -Y_{4l+3} - \mathbf{i}Y_{4l+4} \\ Y_{4l+3} - \mathbf{i}Y_{4l+4} & Y_{4l+1} - \mathbf{i}Y_{4l+2} \\ \vdots & \vdots \end{pmatrix}$$

they are complex left invariant vector fields, where $A = 0, 1, \cdots, 2n-1$, $A' = 0', 1'$. It is motivated by the embedding τ of quaternionic algebra $\mathbb{H}$ into $\mathfrak{gl}(2, \mathbb{C})$:

$$\tau(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}.$$

We will use matrices

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower primed indices, e.g. $Z_A^{A'} = \sum_{B'=0',1'} Z_{AB'}\varepsilon^{B'A'}$. Here $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. Then we get

$$\left(Z_A^{A'}\right) := \begin{pmatrix} -Y_3 - \mathbf{i}Y_4 & -Y_1 - \mathbf{i}Y_2 \\ Y_1 - \mathbf{i}Y_2 & -Y_3 + \mathbf{i}Y_4 \\ \vdots & \vdots \\ -Y_{4l+3} - \mathbf{i}Y_{4l+4} & -Y_{4l+1} - \mathbf{i}Y_{4l+2} \\ Y_{4l+1} - \mathbf{i}Y_{4l+2} & -Y_{4l+3} + \mathbf{i}Y_{4l+4} \\ \vdots & \vdots \end{pmatrix}. \tag{2.2}$$

We denote $\odot^p\mathbb{C}^2$ by the p-th symmetric power of $\mathbb{C}^2$. Its element is denoted by an 2^p-tuple $(f_{A'_1\cdots A'_p})$, $A'_1, \cdots, A'_p = 0', 1'$, which are invariant under permutations of subscripts. The *tangential k-Cauchy-Fueter operator $\mathscr{D}_0^{(k)}$ over $\mathscr{H}_{\mathbb{H}}$* is given by

$$\left(\mathscr{D}_0^{(k)} f\right)_{AA'_2\cdots A'_k} := \sum_{A'_1=0',1'} Z_A^{A'_1} f_{A'_1A'_2\cdots A'_k},$$

for $f \in C^1(\Omega, \odot^k\mathbb{C}^2)$ and $k = 1, 2, \cdots$ (cf. [21, (2.19)]). We have isomorphisms

$$\odot^k\mathbb{C}^2 \cong \mathbb{C}^{k+1}, \qquad \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n} \cong \mathbb{C}^{2nk},$$

by identifying $f \in \odot^k\mathbb{C}^2$ and $F \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}$ with

$$f := \begin{pmatrix} f_{0'0'0'\cdots 0'} \\ f_{1'0'0'\cdots 0'} \\ f_{1'1'0'\cdots 0'} \\ \vdots \\ f_{1'1'1'\cdots 1'} \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} F_{0'0'\cdots 0'0} \\ F_{0'0'\cdots 0'1} \\ \vdots \\ F_{0'0'\cdots 0'(2n-1)} \\ F_{1'0'\cdots 0'0} \\ \vdots \\ F_{1'1'\cdots 1'0} \\ \vdots \\ F_{1'1'\cdots 1'(2n-1)} \end{pmatrix},$$

respectively. In the following of this paper we write $\mathscr{D}_0$ instead of $\mathscr{D}_0^{(k)}$ for simplicity. On a domain $\Omega \subset \mathscr{H}_{\mathbb{H}}$, denote the inner product

$$(u, v)_{\mathbb{H}} := \int_{\Omega} u \cdot \overline{v} \mathrm{d}V_{\mathbb{H}},$$

for $u, v \in L^2(\Omega, \mathbb{C})$, where

$$\mathrm{d}V_{\mathbb{H}} = \theta_{\mathbb{H};1} \wedge \theta_{\mathbb{H};2} \wedge \theta_{\mathbb{H};3} \wedge (\mathrm{d}\theta_{\mathbb{H};\beta})^{2n}, \quad \beta = 1, 2, 3$$

is the volume form on $\mathscr{H}_{\mathbb{H}}$. The inner product of $L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n})$ is defined as

$$\langle f, h \rangle_{\mathbb{H}} := \sum_{A=0}^{2n-1} \sum_{A'_2, \cdots, A'_k = 0', 1'} \left(f_{AA'_2 \cdots A'_k}, h_{AA'_2 \cdots A'_k} \right)_{\mathbb{H}}$$

for $f, h \in L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n})$. Now we list some properties of $Z_A^{A'}$'s and $\mathscr{D}_0$'s given in [14].

Proposition 2.1 ([14, Proposition 2.2])**.** *The formal adjoint operator of* $Z_A^{A'}$ *is*

$$\left(Z_A^{A'} \right)^* = \delta_{A'}^A, \quad \text{where} \quad \delta_{A'}^A := -\overline{Z_A^{A'}}.$$

Lemma 2.1 ([14, Lemma 2.3])**.** *For* $f \in C_0^1(\mathscr{H}_{\mathbb{H}}, \odot^{k-1}\mathbb{O}^2 \otimes \mathbb{C}^{2n})$, *we have*

$$(\mathscr{D}_0^* f)_{A'_1 \cdots A'_k} = \sum_{A=0}^{2n-1} \delta_{(A'_1}^A f_{A'_2 \cdots A'_k)A},$$

where $\mathscr{D}_0^*$ *is the formal adjoint operator of* $\mathscr{D}_0$.

Lemma 2.2 ([14, Lemma 2.4])**.** *For* $A', B' = 0', 1'$, *we have*

$$\sum_{A=0}^{2n-1} Z_A^{A'} \overline{Z_A^{B'}} = -\delta_{A'B'} \Delta_{\mathbb{H}},$$

where $\Delta_{\mathbb{H}}$ *is the SubLaplacian defined in* (2.1).

$\mathscr{D}_0^* \mathscr{D}_0$ is diagonal by the following proposition.

Proposition 2.2 ([14, Proposition 2.3])**.** *For* $f \in C^2(\Omega, \odot^k \mathbb{C}^2)$, *we have*

$$\mathscr{D}_0^* \mathscr{D}_0 f = \Delta_{\mathbb{H}} f.$$

2.2 Octonions, the octonionic Heisenberg group and the tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group

The octonionic algebra has a basis $\{e_j\}_{j=0}^7$ satisfying the relation $e_ie_0 = e_0e_i, i = 0, 1, \cdots, 7$, and

$$e_ie_j = -\delta_{ij} + \epsilon_{ijk}e_k, \tag{2.3}$$

$i, j = 1, \cdots, 7$. The constants ϵ_{ijk} in (2.3) are completely anti-symmetric in i, j, k, and equal the value $+1$ for $(i, j, k) \in \Omega$ (cf. e.g. [1, 18]), where

$$\Omega = \{(1, 2, 3), (2, 4, 6), (4, 3, 5), (3, 6, 7), (6, 5, 1), (5, 7, 2), (7, 1, 4)\}. \tag{2.4}$$

ϵ_{ijk} is nonzero only when (i, j, k) is a permutation of triples in Ω. So, $\epsilon_{ijk}e_k$ in (2.3) can be regarded as $\sum_{k=1}^7 \epsilon_{ijk}e_k$, where there exists only one k such that $\epsilon_{ijk} \neq 0$. The octonionic algebra $\mathbb{O}$ is a non-associative algebra and octonionic multiplication is neither commutative nor associative, i.e.

$$(e_ie_j)e_k \neq e_i(e_je_k)$$

for some choice of i, j, k.

The octonionic Heisenberg group $\mathscr{H}_{\mathbb{O}}$ is $\mathbb{O} \oplus \mathrm{Im}\mathbb{O}$ equipped with multiplication given by

$$(x, t) \cdot (x', t') = \left(x' + x', t + t' + 2\mathrm{Im}\left(x\bar{x'}\right)\right), \tag{2.5}$$

where $x, x' \in \mathbb{O}, t, t' \in \mathrm{Im}\ \mathbb{O}$. The multiplication of the octonionic Heisenberg group in terms of real variables can be written as (cf. eg. [15])

$$(x, t) \cdot (x', t') = \left(x' + x, t_\beta + t'_\beta + 2\sum_{k,j=1}^{8} E^\beta_{kj}x_kx'_j\right),$$

where $x = (x_1, \cdots, x_8) \in \mathbb{R}^8$, $t = (t_1, \cdots, t_7) \in \mathbb{R}^7$, and E^β are given by

$$E^\beta = \begin{pmatrix} 0 & -\nu_\beta \\ \nu^t_\beta & \varepsilon^\beta \end{pmatrix}, \quad \beta = 1, \cdots, 7.$$

Here ε^β are (7×7)-matrices with

$$\varepsilon^\beta_{kj} = \epsilon_{jk\beta},$$

and

$$\nu_\beta = (\nu_{\beta;1}, \cdots, \nu_{\beta;7}) \in \mathbb{R}^7, \quad \nu_{\beta;j} = \delta_{\beta j}.$$

Then E^β are anti-symmetric matrices. The norm of the octonionic Heisenberg group $\mathscr{H}_{\mathbb{O}}$ is defined by

$$\|(x, t)\| := (|x|^4 + |t|^2)^{\frac{1}{4}},$$

for $(x,t) \in \mathscr{H}_{\mathbb{O}}$. By definition,

$$X_j = \partial_{x_j} + 2\sum_{\beta=1}^{7}\sum_{k=1}^{8} E_{kj}^{\beta} x_k \partial_{t_\beta}, \quad j = 1, \cdots, 8,$$

are the left invariant vector fields on $\mathscr{H}_{\mathbb{O}}$, whose brackets are

$$[X_k, X_j] = 4\sum_{\beta=1}^{7} E_{kj}^{\beta} \partial_{t_\beta}, \tag{2.6}$$

for $k, j = 1, \cdots, 8$. We denote *SubLaplacian* $\Delta_{\mathbb{O}}$ *on the octonionic Heisenberg group* by

$$\Delta_{\mathbb{O}} := -\sum_{k=1}^{8} X_k^2. \tag{2.7}$$

The standard $\mathbb{R}^7$-valued contact form of $\mathscr{H}_{\mathbb{O}}$ is

$$\Theta_{\mathbb{O}} := \mathrm{d}t - x \cdot \mathrm{d}\bar{x} + \mathrm{d}x \cdot \bar{x}.$$

If we write $\Theta_{\mathbb{O}} = (\theta_{\mathbb{O};1}, \cdots, \theta_{\mathbb{O};7})$, then we have

$$\theta_{\mathbb{O};\beta} = \mathrm{d}t_\beta - 2\sum_{j,k=1}^{8} E_{kj}^{\beta} x_k \mathrm{d}x_j,$$

by direct calculation. On a domain $\Omega \subset \mathscr{H}_{\mathbb{O}}$, denote the inner product

$$(u, v)_{\mathbb{O}} := \int_{\Omega} u \cdot \overline{v} \mathrm{d}V_{\mathbb{O}},$$

for $u, v \in L^2(\Omega, \mathbb{C})$, where

$$\mathrm{d}V_{\mathbb{O}} = \theta_{\mathbb{O};1} \wedge \cdots \wedge \theta_{\mathbb{H};7} \wedge (\mathrm{d}\theta_{\mathbb{H};\beta})^4, \quad \beta = 1, \cdots, 7$$

is the volume form on $\mathscr{H}_{\mathbb{O}}$. The inner product of $L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^4)$ is defined as

$$\langle f, h\rangle_{\mathbb{O}} := \sum_{A=0}^{3} \sum_{A_2', \cdots, A_k' = 0', 1'} \left(f_{AA_2'\cdots A_k'}, h_{AA_2'\cdots A_k'}\right)_{\mathbb{O}}$$

for $f, h \in L^2(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^4)$.

Define

$$\left(\nabla_A^{A'}\right) := \begin{pmatrix} -X_3 - \mathbf{i}X_4 & -X_1 - \mathbf{i}X_2 \\ X_1 - \mathbf{i}X_2 & -X_3 + \mathbf{i}X_4 \\ -X_7 - \mathbf{i}X_8 & X_5 - \mathbf{i}X_6 \\ -X_5 - \mathbf{i}X_6 & -X_7 + \mathbf{i}X_8 \end{pmatrix}, \tag{2.8}$$

they are complex left invariant vector fields, for $A = 0, \cdots, 3, A' = 0', 1'$. It is a bit different from the traditional operator $Z_A^{A'}$ in (2.2) by take X_5 into $-X_5$, this comes from the technical difficult to establish the following Lemma 2.4 in the remaining case. Define

$$\left(\mathcal{D}_0^{(k)} f\right)_{AA'_2\cdots A'_k} := \sum_{A'_1=0',1'} \nabla_A^{A'_1} f_{A'_1 A'_2 \cdots A'_k},$$

for $f \in C^1(\Omega, \odot^k \mathbb{C}^2)$, we will still call it tangential k-Cauchy-Fueter type operator for simplicity. Although this difference, it is still interesting to study the function theory of this operator. Similar with the quaternionic case we will write $\mathcal{D}_0$ instead of $\mathcal{D}_0^{(k)}$ for simplicity.

Proposition 2.3. *The formal adjoint operator of* $\nabla_A^{A'}$ *is*

$$\left(\nabla_A^{A'}\right)^* = \delta'^A_{A'}, \quad \text{where} \quad \delta'^A_{A'} := -\overline{\nabla_A^{A'}}.$$

Lemma 2.3. *For* $f \in C_0^1(\mathscr{H}_{\mathbb{O}}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^4)$, *we have*

$$(\mathcal{D}_0^* f)_{A'_1\cdots A'_k} = \sum_{A=0}^{2n-1} \delta'^A_{(A'_1} f_{A'_2\cdots A'_k)A}, \tag{2.9}$$

where $\mathcal{D}_0^*$ *is the formal adjoint operator of* $\mathcal{D}_0$.

The proof of Proposition 2.3 and Lemma 2.3 are similar with Proposition 2.1 and Lemma 2.1 in the quaternionic case, respectively. We omit the details.

Lemma 2.4. *For* $A', B' = 0', 1'$, *we have*

$$\sum_{A=0}^{3} \overline{\nabla_A^{A'}} \nabla_A^{B'} = \sum_{A=0}^{3} \nabla_A^{B'} \overline{\nabla_A^{A'}} = -\delta_{A'B'} \Delta_{\mathbb{O}}, \tag{2.10}$$

where $\Delta_{\mathbb{O}}$ *is the SubLaplacian defined in* (2.7).

Proof. Note that for $A' = B' = 0'$, we have

$$\begin{aligned}
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{0'}}\nabla_{2l}^{0'} + \overline{\nabla_{2l+1}^{0'}}\nabla_{2l+1}^{0'}\right) \\
=&(X_3 - \mathbf{i}X_4)(-X_3 - \mathbf{i}X_4) + (-X_1 - \mathbf{i}X_2)(X_1 - \mathbf{i}X_2) \\
&+ (X_7 - \mathbf{i}X_8)(-X_7 - \mathbf{i}X_8) + (X_5 - \mathbf{i}X_6)(-X_5 - \mathbf{i}X_6) \\
=&-\sum_{k=1}^{8} X_k^2 - \mathbf{i}[X_3, X_4] + \mathbf{i}[X_1, X_2] - \mathbf{i}[X_7, X_8] - \mathbf{i}[X_5, X_6] = \Delta_{\mathbb{O}},
\end{aligned}$$

by (2.6). For $A' = 0', B' = 1'$, we have

$$\begin{aligned}
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{0'}}\nabla_{2l}^{1'} + \overline{\nabla_{2l+1}^{0'}}\nabla_{2l+1}^{1'}\right)\\
=&(X_3 - \mathbf{i}X_4)(-X_1 - \mathbf{i}X_2) + (-X_1 - \mathbf{i}X_2)(-X_3 + \mathbf{i}X_4)\\
&+ (X_7 - \mathbf{i}X_8)(X_5 - \mathbf{i}X_6) + (X_5 - \mathbf{i}X_6)(-X_7 + \mathbf{i}X_8)\\
=&[X_1, X_3] + [X_2, X_4] - [X_5, X_7] + [X_6, X_8]\\
&- \mathbf{i}[X_1, X_4] + \mathbf{i}[X_2, X_3] + \mathbf{i}[X_5, X_8] + \mathbf{i}[X_6, X_7] = 0,
\end{aligned}$$

by (2.6). Similarly, (2.10) holds for $A' = 1, B' = 0'$ and $A' = B' = 1'$ by

$$\begin{aligned}
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{1'}}\nabla_{2l}^{0'} + \overline{\nabla_{2l+1}^{1'}}\nabla_{2l+1}^{0'}\right)\\
=&-[X_1, X_3] - [X_2, X_4] + [X_5, X_7] - [X_6, X_8]\\
&- \mathbf{i}[X_1, X_4] + \mathbf{i}[X_2, X_3] + \mathbf{i}[X_5, X_8] + \mathbf{i}[X_6, X_7] = 0,\\
&-\sum_{l=0}^{1}\left(\overline{\nabla_{2l}^{1'}}\nabla_{2l}^{1'} + \overline{\nabla_{2l+1}^{1'}}\nabla_{2l+1}^{1'}\right)\\
=&\Delta_{\mathbb{O}} - \mathbf{i}[X_1, X_2] + \mathbf{i}[X_3, X_4] + \mathbf{i}[X_5, X_6] + \mathbf{i}[X_7, X_8] = \Delta_{\mathbb{O}}.
\end{aligned}$$

Similarly we have $\sum_{A=0}^{3} \nabla_A^{B'}\overline{\nabla_A^{A'}} = -\delta_{A'B'}\Delta_{\mathbb{O}}$. Then (2.10) follows. □

Similarly with the quaternionic case, $\mathcal{D}_0^*\mathcal{D}_0$ is also diagonal by the following proposition.

Proposition 2.4. *For $f \in C^2(\Omega, \odot^k\mathbb{C}^2)$, we have*

$$\mathcal{D}_0^*\mathcal{D}_0 f = \Delta_{\mathbb{O}} f.$$

Proof. Recall that for a $\otimes^k\mathbb{C}^2$-valued function $F_{A'_1\cdots A'_k}$ symmetric in $A'_2\cdots A'_k$, we have

$$F_{(A'_1\cdots A'_k)} = \frac{1}{k}\left(F_{A'_1A'_2\cdots A'_k} + \cdots + F_{A'_sA'_1\cdots\widehat{A'_s}\cdots A'_k} + \cdots + F_{A'_kA'_1\cdots\widehat{A'_k}}\right),$$

by the definition of *symmetrisation* of primed indices

$$f_{\cdots(A'_1\cdots A'_p)\cdots} := \frac{1}{p!}\sum_{\sigma\in S_p} f_{\cdots A'_{\sigma(1)}\cdots A'_{\sigma(p)}\cdots}.$$

As usual, a hatnote means omittance of the corresponding index. Then for fixed $A'_1, \cdots, A'_k = 0', 1'$,

$$
\begin{aligned}
(\mathcal{D}_0^* \mathcal{D}_0 f)_{A'_1 \cdots A'_k} &= \sum_A \delta'^{A}_{(A'_1} (\mathcal{D}_0 f)_{A'_2 \cdots A'_k) A} = \frac{1}{k} \sum_{s=1}^{k} \delta'^{A}_{A'_s} (\mathcal{D}_0 f)_{\cdots \widehat{A'_s} \cdots A'_k A} \\
&= -\frac{1}{k} \sum_{s=1}^{k} \sum_{A,A'} \overline{\nabla^{A'_s}_{A}} \nabla^{A'}_{A} f_{A' \cdots \widehat{A'_s} \cdots A'_k} \\
&= \frac{1}{k} \sum_{s=1}^{k} \sum_{A'} \Delta_{\mathbb{O}} f_{A' \cdots \widehat{A'_s} \cdots A'_k} \delta_{A'_s A'} = \Delta_{\mathbb{O}} f_{A'_1 \cdots A'_k},
\end{aligned}
$$

by using Lemma 2.4 and f symmetric in the primed indices, where $\mathcal{D}_0^*$ is given by (2.9). The proposition is proved. □

3. The Bochner-Martinelli formula for tangential k-Cauchy-Fueter operator over quaternionic Heisenberg group

Let 1-forms $\{\theta^1_{\mathbb{H}}, \cdots, \theta^{4n}_{\mathbb{H}}, \theta_{\mathbb{H};1}, \theta_{\mathbb{H};2}, \theta_{\mathbb{H};3}\}$ be the basis dual to $\{Y_1, \cdots, Y_{4n}, \partial_{s_1}, \partial_{s_2}, \partial_{s_3}\}$. It is known that

$$
K(\eta) = \frac{C_Q}{\|\eta\|^{Q-2}}, \quad \eta = (y, s) \in \mathscr{H}_{\mathbb{H}},
$$

is the fundamental solution to the SubLaplacian $\Delta_{\mathbb{H}}$ on the quaternionic Heisenberg group (cf. e.g. [13, Appendix]), where C_Q is a positive constant and $Q = 4n + 6$ is the homogeneous dimension of $\mathscr{H}_{\mathbb{H}}$. Denote $(4n+2)$-forms

$$
\mathrm{d}\sigma^{A'}_{A} := i_{Z^{A'}_{A}} \mathrm{d}V_{\mathbb{H}}, \tag{3.1}
$$

for $A = 0, 1, \cdots, 2n-1$, $A' = 0', 1'$, where i is the interior product. Define the *quaternionic Bochner-Martinelli kernel* to be a $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$-valued function

$$
\mathcal{E}^{A}_{A'}(\eta) := \overline{Z^{A'}_{A}} K(\eta) \quad \text{on} \quad \mathscr{H}_{\mathbb{H}} \setminus (0, 0). \tag{3.2}
$$

See [12, (1.16)] for Bochner-Martinelli kernel on $(4n+1)$-dim Heisenberg group. The following procedure of proving our main Theorem 3.1 is similar with which in the Heisenberg case [12].

Lemma 3.1. *Let Ω be a domain in $\mathscr{H}_{\mathbb{H}}$ and $w \in C^1(\Omega, \mathbb{C})$. Then we have*

$$
\int_{\partial\Omega} w \mathrm{d}\sigma^{A'}_{A}(\eta) = \int_{\Omega} Z^{A'}_{A} w \mathrm{d}V_{\mathbb{H}}(\eta), \tag{3.3}
$$

for $\eta \in \mathscr{H}_{\mathbb{H}}$.

Proof. Note that, for fixed $a,b\in\{1,2,\cdots,4n\}$,

$$\left(\sum_{j=1}^{4n} Y_j w\cdot\theta_{\mathbb{H}}^j\right)\wedge i_{c_1Y_a+c_2Y_b}\mathrm{d}V_{\mathbb{H}}(\eta)=(c_1Y_a+c_2Y_b)\,w\cdot\mathrm{d}V_{\mathbb{H}}(\eta).$$

Thus we have

$$\begin{aligned}\mathrm{d}\left(w\cdot\mathrm{d}\sigma_A^{A'}\right)=&\left(\sum_{j=1}^{4n}Y_jw\cdot\theta_{\mathbb{H}}^j+\sum_{\beta=1}^{3}\partial_{s_\beta}w\cdot\theta_{\mathbb{H};\beta}\right)\wedge i_{Z_A^{A'}}\mathrm{d}V_{\mathbb{H}}(\eta)\\ =&Z_A^{A'}w\cdot\mathrm{d}V_{\mathbb{H}}(\eta).\end{aligned}$$

Then (3.3) follows by Stokes formula. □

Lemma 3.2. *Let Ω be a domain in $\mathscr{H}_{\mathbb{H}}$ with smooth boundary. For $\Phi\in C\left(\bar\Omega,\odot^k\mathbb{C}^2\right)\cap C^1\left(\Omega,\odot^k\mathbb{C}^2\right)$ and a fixed point $\xi\in\mathscr{H}_{\mathbb{H}}\backslash\bar\Omega$, for any fixed $A_2',\cdots,A_k'$, we have*

$$\begin{aligned}&\sum_{A,A'}\int_{\partial\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\Phi_{A'A_2'\cdots A_k'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\\ =&\sum_{A}\int_{\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\Phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{H}}(\eta).\end{aligned}$$

Proof. Applying Lemma 3.1 to $w(\eta)=\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\Phi_{A'A_2'\cdots A_k'}(\eta)$, we have

$$\begin{aligned}&\sum_{A,A'}\int_{\partial\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\Phi_{A'A_2'\cdots A_k'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\\ =&\sum_{A,A'}\int_{\Omega}\left[Z_A^{A'}\left(\overline{Z_A^{A_1'}}K(\xi^{-1}\eta)\right)\Phi_{A'A_2'\cdots A_k'}(\eta)+\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\cdot Z_A^{A'}\Phi_{A'A_2'\cdots A_k'}(\eta)\right]\mathrm{d}V_{\mathbb{H}}(\eta)\\ =&\sum_{A}\int_{\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\Phi(\eta)\right)_{AA_2'\cdots A_k'}\mathrm{d}V_{\mathbb{H}}(\eta),\end{aligned}$$

by (3.2) and

$$\sum_{A}Z_A^{A'}\left(\overline{Z_A^{A_1'}}K(\xi^{-1}\eta)\right)=\sum_{A}\left(Z_A^{A'}\overline{Z_A^{A_1'}}K\right)(\xi^{-1}\eta)=-\delta_{A'A_1'}\Delta_{\mathbb{H}}K(\xi^{-1}\eta)=0.$$

It holds by left invariance and Lemma2.2. □

Lemma 3.3. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{H}}$ with C^1 boundary and $\phi\in C\left(\bar\Omega,\odot^k\mathbb{C}^2\right)\cap C^1\left(\Omega,\odot^k\mathbb{C}^2\right)$. Then we have*

$$\sum_{A,A'}\int_{\partial\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma_A^{A'}(\eta)=\begin{cases}N\cdot\phi_{A_1'A_2'\cdots A_k'}(\xi) & \text{if}\quad \xi\in\Omega,\\ 0 & \text{if}\quad \xi\notin\bar\Omega,\end{cases}$$

where

$$N=-C_Q(Q-2)(Q+2)\int_{\|(y,s)\|\le 1}|y|^2\mathrm{d}V_{\mathbb{O}}(\eta). \tag{3.4}$$

Proof. For a fixed point $\xi \notin \bar{\Omega}$, apply Lemma 3.2 to $\Phi(\eta) \equiv \phi(\xi)$ we have

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi) \mathrm{d}\sigma^{A'}_{A}(\eta) = 0,$$

by $\mathscr{D}_0\Phi(\eta) \equiv 0$. When $\xi \in \Omega$, let $B_{\mathbb{H}}(\xi,\epsilon) := \left\{\eta \in \mathscr{H}_{\mathbb{H}} : \|\xi^{-1}\eta\| \leq \epsilon\right\} \subset \Omega$ be the quaternionic Heisenberg ball with ϵ sufficient small. Applying Lemma 3.2 to $\Phi(\eta) = \phi(\xi)$ and $\Omega \setminus B_{\mathbb{H}}(\xi,\epsilon)$ we get

$$\sum_{A,A'} \int_{\partial(\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon))} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi) \mathrm{d}\sigma^{A'}_{A}(\eta) = 0.$$

Therefore we have

$$\begin{aligned} &\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi) \mathrm{d}\sigma^{A'}_{A}(\eta) \\ =&\sum_{A,A'} \int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^{A}_{A'_l}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_k A'}(\xi) \mathrm{d}\sigma^{A'}_{A}(\eta). \end{aligned} \tag{3.5}$$

Note that

$$\mathcal{E}^{A}_{A'}(\eta) = \overline{Z^{A'}_{A}} \frac{1}{\|\eta\|^{Q-2}} = -\frac{Q-2}{4} \frac{\overline{Z^{A'}_{A}}\|\eta\|^4}{\|\eta\|^{Q+2}}, \tag{3.6}$$

we have

$$\begin{aligned} &\sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta) \mathrm{d}\sigma^{A'}_{A}(\eta) \\ =& -\frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} Z^{A'}_{A} \overline{Z^{A'_1}_{A}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ =& -\frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \sum_{A} Z^{A'}_{A} \overline{Z^{A'_1}_{A}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ =& \frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \sum_{A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \delta_{A'A'_1} \Delta_{\mathbb{H}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta) \\ =& \frac{(Q-2)C_Q}{4\epsilon^{Q+2}} \phi_{A'_1 A'_2\cdots A'_k}(\xi) \int_{B_{\mathbb{H}}(\xi,\epsilon)} \Delta_{\mathbb{H}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{H}}(\eta), \end{aligned} \tag{3.7}$$

by using (3.6), Lemma 2.2, Lemma3.1 and left invariance. Note that

$$Y_{4l+j}\|\eta\|^4 = 4|y|^2 y_{4l+j} + 4\sum_{\beta=1}^{3}\sum_{k=1}^{4} B^{\beta}_{kj} y_{4l+k} s_{\beta}, \tag{3.8}$$

we have

$$
\begin{aligned}
\Delta_{\mathbb{H}}\|\eta\|^4 =& -\sum_{l=0}^{n-1}\sum_{j=1}^{4}\left(8y_{4l+j}^2+4|y|^2+8\sum_{\beta=1}^{3}\sum_{k,k'=1}^{4}B_{k'j}^{\beta}B_{kj}^{\beta}y_{4l+k'}y_{4l+k}\right)\\
=& -4(Q+2)|y|^2,
\end{aligned}
\tag{3.9}
$$

by using (3.8) and $\sum_{j=1}^4 B_{kj}^{\beta}B_{jk'}^{\beta'}=(B^{\beta}B^{\beta'})_{kk'}$, antisymmetry for $B^{\beta}B^{\beta'}$ of $\beta\neq\beta'$ and $\left(B^{\beta}\right)^2=-\mathrm{id}$. As the Jacobi determinant of dilation $(y,s)\to(\epsilon y,\epsilon^2 s)$ is ϵ^Q, we have

$$
\begin{aligned}
&\frac{(Q-2)C_Q}{4\epsilon^{Q+2}}\int_{B_{\mathbb{H}}(\xi,\epsilon)}\Delta_{\mathbb{H}}\|\xi^{-1}\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\
=&\frac{(Q-2)C_Q}{4\epsilon^{Q+2}}\int_{B_{\mathbb{H}}(0,\epsilon)}\Delta_{\mathbb{H}}\|\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\
=&\frac{(Q-2)C_Q}{4}\int_{B_{\mathbb{H}}(0,1)}\Delta_{\mathbb{H}}\|\eta\|^4\mathrm{d}V_{\mathbb{H}}(\eta)\\
=&-C_Q(Q-2)(Q+2)\int_{\|(y,s)\|\leq 1}|y|^2\mathrm{d}V_{\mathbb{H}}(\eta)=N,
\end{aligned}
$$

by (3.9). The lemma is proved. □

Now we can prove the Bochner-Martinelli type formula for tangential k-Cauchy-Fueter operator over the quaternionic Heisenberg group.

Theorem 3.1. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{H}}$ with C^1 boundary. For $\phi\in\left(\bar{\Omega},\odot^k\mathbb{C}^2\right)\cap\ C^1\left(\Omega,\odot^k\mathbb{C}^2\right)$, $\xi\in\Omega$ and $k=1,2,\cdots$, we have*

$$
\begin{aligned}
\phi_{A_1'A_2'\cdots A_k'}(\xi)=\frac{1}{N}\sum_{\substack{A=0,\cdots,2n-1\\ A'=0',1'}}&\left\{\int_{\partial\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\right.\\
&\left.-\int_{\Omega}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{H}}(\eta)\right\},
\end{aligned}
$$

where N is given by (3.4).

Proof. For any fixed $\xi\in\Omega$, by Lemma 3.2 and the domain $\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon)$, we get

$$
\begin{aligned}
&\sum_{A,A'}\left\{\int_{\partial\Omega}-\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)}\right\}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta)\\
=&\sum_{A}\int_{\Omega\setminus B_{\mathbb{H}}(\xi,\epsilon)}\mathcal{E}_{A_1'}^{A}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A_2'\cdots A_k'A}\mathrm{d}V_{\mathbb{H}}(\eta).
\end{aligned}
$$

By (3.6) and (3.8), we have $\mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta) = O\left(\frac{1}{\|\xi^{-1}\eta\|^{Q-1}}\right)$. Since $Z_A^{A'_1}\phi(\eta)_{A'_1A'_2\cdots A'_k}$ is locally bounded, we find that

$$\lim_{\epsilon\to 0}\sum_{A}\int_{B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\left(\mathscr{D}_0\phi(\eta)\right)_{A'_2\cdots A'_kA}\,\mathrm{d}V_{\mathbb{H}}(\eta) = 0.$$

On the other hand

$$\lim_{\epsilon\to 0}\sum_{A,A'}\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\left(\phi_{A'_2\cdots A'_kA'}(\eta) - \phi_{A'_2\cdots A'_kA'}(\xi)\right)\mathrm{d}\sigma_A^{A'}(\eta) = 0,$$

by using $\left|\phi_{A'_2\cdots A'_kA'}(\eta) - \phi_{A'_2\cdots A'_kA'}(\xi)\right| \le \left\|\phi_{A'_2\cdots A'_kA'}\right\|_{C^1}|\xi-\eta|$. Then by Lemma 3.3, we get

$$\lim_{\epsilon\to 0}\sum_{A,A'}\int_{\partial B_{\mathbb{H}}(\xi,\epsilon)} \mathcal{E}^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_kA'}(\eta)\mathrm{d}\sigma_A^{A'}(\eta) = N\cdot\phi_{A'_1A'_2\cdots A'_k}(\xi).$$

The theorem is proved. □

4. The Bochner-Martinelli formula for tangential k-Cauchy-Fueter type operator over octonionic Heisenberg group

It is known that

$$K'(\eta) = \frac{C_{Q'}}{\|\eta\|^{Q'-2}},\quad \eta = (x,t)\in\mathscr{H}_{\mathbb{O}}, \tag{4.1}$$

is the fundamental solution to the SubLaplacian $\Delta_{\mathbb{O}}$ on the octonionic Heisenberg group (cf. e.g. [4]), where $C_{Q'}$ is a positive constant and $Q' = 22$ is the homogeneous dimension of $\mathscr{H}_{\mathbb{O}}$. Denote 14-forms

$$\mathrm{d}\sigma'^{A'}_{A} := i_{\nabla_A^{A'}}\mathrm{d}V_{\mathbb{O}}, \tag{4.2}$$

for $A = 0,\cdots,3$, $A' = 0',1'$, where i is the interior product. Define the *octonionic Bochner-Martinelli kernel* to be a $\mathbb{C}^2\otimes\mathbb{C}^4$-valued function

$$\mathcal{E}'^{A}_{A'}(\eta) := \overline{\nabla_A^{A'}}K'(\eta)\quad\text{on}\quad \mathscr{H}_{\mathbb{O}}\setminus(0,0). \tag{4.3}$$

Lemma 4.1. *For a domain Ω in $\mathscr{H}_{\mathbb{O}}$ and $w\in C^1(\Omega,\mathbb{C})$, we have*

$$\int_{\partial\Omega} w\mathrm{d}\sigma'^{A'}_{A}(\eta) = \int_{\Omega}\nabla_A^{A'}w\mathrm{d}V_{\mathbb{O}}(\eta), \tag{4.4}$$

for $\eta\in\mathscr{H}_{\mathbb{O}}$.

Lemma 4.2. *Let Ω be a domain in $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary. For $\Phi \in C\left(\bar{\Omega}, \odot^k\mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k\mathbb{C}^2\right)$ and a fixed point $\xi \in \mathscr{H}_{\mathbb{O}} \backslash \bar{\Omega}$, for any fixed $A_1', A_2', \cdots, A_k'$, we have*

$$\begin{aligned}&\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\Phi_{A'A_2'\cdots A_k'}(\eta)\mathrm{d}\sigma'^{A'}_{A}(\eta)\\ =&\sum_{A} \int_{\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\left(\mathcal{D}_0\Phi(\eta)\right)_{A_2'\cdots A_k'A}\,\mathrm{d}V_{\mathbb{O}}(\eta).\end{aligned}$$

The proof of Lemma 4.1 and 4.2 is just like Lemma 3.1 and 3.2, respectively. We omit the details.

Lemma 4.3. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary and $\phi \in C\left(\bar{\Omega}, \odot^k\mathbb{C}^2\right) \cap C^1\left(\Omega, \odot^k\mathbb{C}^2\right)$. Then we have*

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = \begin{cases} N'\cdot\phi_{A_1'A_2'\cdots A_k'}(\xi) & \text{if } \ \xi\in\Omega,\\ 0 & \text{if } \ \xi\notin\bar{\Omega},\end{cases}$$

where

$$N' = -480C_{Q'}\int_{\|(x,t)\|\le 1} |x|^2\mathrm{d}V_{\mathbb{O}}(\eta). \tag{4.5}$$

Proof. For a fixed point $\xi \notin \bar{\Omega}$, applying Lemma 4.2 with $\Phi(\eta) \equiv \phi(\xi)$, we have

$$\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = 0,$$

by $\mathcal{D}_0\Phi(\eta) \equiv 0$. When $\xi \in \Omega$, let $B_{\mathbb{O}}(\xi,\epsilon) := \left\{\eta \in \mathscr{H}_{\mathbb{O}} : \|\xi^{-1}\eta\| \le \epsilon\right\} \subset \Omega$ be the octionionic Heisenberg ball with ϵ sufficient small. Applying Lemma 4.2 to $\Phi(\eta) = \phi(\xi)$ and $\Omega \setminus B_{\mathbb{O}}(\xi,\epsilon)$ we get

$$\sum_{A,A'} \int_{\partial(\Omega\setminus B_{\mathbb{O}}(\xi,\epsilon))} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta) = 0,$$

i.e.

$$\begin{aligned}&\sum_{A,A'} \int_{\partial\Omega} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta)\\ =&\sum_{A,A'} \int_{\partial B_{\mathbb{O}}(\xi,\epsilon)} \mathcal{E}'^{A}_{A_1'}(\xi^{-1}\eta)\phi_{A_2'\cdots A_k'A'}(\xi)\mathrm{d}\sigma'^{A'}_{A}(\eta).\end{aligned} \tag{4.6}$$

Note that

$$\mathcal{E}'^{A}_{A'}(\eta) = \overline{\nabla^{A'}_{A}}\frac{C_{Q'}}{\|\eta\|^{Q'-2}} = -C_{Q'}\frac{Q'-2}{4}\frac{\overline{\nabla^{A'}_{A}}\|\eta\|^4}{\|\eta\|^{Q'+2}}, \tag{4.7}$$

we have

$$\sum_{A,A'} \phi_{A'_2\cdots A'_k A'}(\xi) \int_{\partial B_{\mathbb{O}}(\xi,\epsilon)} \mathcal{E}'^{A}_{A'_1}(\xi^{-1}\eta)\mathrm{d}\sigma'^{A'}_{A}(\eta)$$
$$= \frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}} \phi_{A'_1A'_2\cdots A'_k}(\xi) \int_{B_{\mathbb{O}}(\xi,\epsilon)} \Delta_{\mathbb{O}} \|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta),$$

similar with (3.7). Note that

$$X_j\|\eta\|^4 = 4|x|^2 x_j + 4\sum_{\beta=1}^{7}\sum_{k=1}^{8} E^{\beta}_{kj} x_k t_\beta, \tag{4.8}$$

we have

$$\Delta_{\mathbb{O}}\|\eta\|^4 = -\sum_{j=1}^{8}\left(8x_j^2 + 4|x|^2 + 8\sum_{\beta=1}^{7}\sum_{k,k'=1}^{8} E^{\beta}_{k'j}E^{\beta}_{kj} x_{k'} x_k\right)$$
$$= -4(Q'+2)|x|^2, \tag{4.9}$$

by using (4.8) and $\sum_{j=1}^{8} E^{\beta}_{kj}E^{\beta'}_{jk'} = (E^{\beta}E^{\beta'})_{kk'}$, antisymmetry for $E^{\beta}E^{\beta'}$ of $\beta \neq \beta'$ and $\left(E^{\beta}\right)^2 = -\mathrm{id}$ (cf. [15]). As the Jacobi determinant of dilation $(x,t) \to (\epsilon x, \epsilon^2 t)$ is $\epsilon^{Q'}$ and $\Delta_{\mathbb{O}}$ is left invariant, we have

$$\begin{aligned}\frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}}\int_{B_{\mathbb{O}}(\xi,\epsilon)} \Delta_{\mathbb{O}}\|\xi^{-1}\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta) &= \frac{(Q'-2)C_{Q'}}{4\epsilon^{Q'+2}}\int_{B_{\mathbb{O}}(0,\epsilon)} \Delta_{\mathbb{O}}\|\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta)\\ &= \frac{(Q'-2)C_{Q'}}{4}\int_{B_{\mathbb{O}}(0,1)} \Delta_{\mathbb{O}}\|\eta\|^4 \mathrm{d}V_{\mathbb{O}}(\eta)\\ &= -480C_{Q'}\int_{\|(x,t)\|\le 1} |x|^2 \mathrm{d}V_{\mathbb{O}}(\eta) = N',\end{aligned}$$

by (4.9). The lemma is proved. □

Now we have the Bochner-Martinelli type formula for the tangential k-Cauchy-Fueter type operator over the octonionic Heisenberg group.

Theorem 4.1. *Let Ω be an open bounded set of $\mathscr{H}_{\mathbb{O}}$ with C^1 boundary. For $\phi \in \left(\bar{\Omega}, \odot^k\mathbb{C}^2\right) \cap\ C^1\left(\Omega, \odot^k\mathbb{C}^2\right)$, $\xi \in \Omega$ and $k = 1, 2, \cdots$, we have*

$$\phi_{A'_1A'_2\cdots A'_k}(\xi) = \frac{1}{N'}\sum_{\substack{A=0,\cdots,3\\A'=0',1'}}\left\{\int_{\partial\Omega} \mathcal{E}'^{A}_{A'_1}(\xi^{-1}\eta)\phi_{A'_2\cdots A'_kA'}(\eta)\mathrm{d}\sigma'^{A'}_{A}(\eta)\right.$$
$$\left. - \int_{\Omega} \mathcal{E}'^{A}_{A'_1}(\xi^{-1}\eta)\left(\mathcal{D}_0\phi(\eta)\right)_{A'_2\cdots A'_kA} \mathrm{d}V_{\mathbb{O}}(\eta)\right\},$$

where N' is given by (4.5)*.*

The proof is similar with the quaternionic case, we omit the detail.

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