Graphs whose completely regular endomorphisms form a monoid

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Abstract. In this paper, we show that if the completely regular endomorphisms of $X+Y$ form a monoid, then the completely regular endomorphisms of X and Y form a monoid respectively. We give several approaches to construct new graphs whose completely regular endomorphisms form a monoid. In particular, we determine trees and the joins of trees whose completely regular endomorphisms form a monoid.
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1. Introduction and preliminary concepts

As is known, endomorphism monoids of graphs are generalizations of automorphism groups of graphs. Endomorphism monoids of graphs have been studied for a long time and many interesting results concerning graphs and their endomorphism monoids have been obtained (cf. [3], [7-8], [12], [14-16]). The endomorphism monoids of graphs have valuable applications (cf. [10]) and are related to automata theory (cf. [11], [13]). Let X be a graph. Denote by $End(X)$ the set of all endomorphisms of X. It is known that $End(X)$ forms a monoid with respect to composition of mappings. We call $End(X)$ the endomorphism monoid of graph X. An element a of a semigroup S is said to be *completely regular* if $a = axa$ and $xa = ax$ hold for some $x \in S$. Let $f \in End(X)$. Then f is called a *completely regular endomorphism* of X if it is a completely regular element in $End(X)$. Denote by $cEnd(X)$ the set of all completely regular endomorphisms of graph X. For a monoid S, the composition of two completely regular elements of S is not completely regular in general. So it is natural to ask: Under what conditions does the set $cEnd(X)$ form a monoid for a graph X?

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However, it seems difficult to obtain a general answer to this question. Therefore a natural strategy for dealing with this question is to find various kinds of conditions for various kinds of graphs. In [9], completely regular endomorphisms of split graphs were characterized and the conditions under which the completely regular endomorphisms of split graphs form a monoid were given. In this paper, we show that if $cEnd(X+Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form monoids. We give several approaches to construct new graphs whose completely regular endomorphisms form monoids. In particular, we determine trees and the joins of trees whose completely regular endomorphisms form monoids.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in graph X, the edge joining x_1 and x_2 is denoted by $\{x_1, x_2\}$. Let $v \in V(X)$. Set $N(v) = \{x \in V(X) | \{x, v\} \in E(X)\}$. The cardinality of $N(v)$ is called the *degree* of v in X and is denoted by $d(v)$. A connected graph X is called a *tree* if and only if it has no cycle. A *path* with n vertices is written as P_n. If X has an u, v-path, then the *distance* from u to v, denoted by $d(u, v)$, is the least length of the u, v-path. The *diameter* of X is the greatest distance between any two vertices in X, denoted by $diam(X)$. Let X and Y be two graphs. The *join* of X and Y, denoted by $X+Y$, is a graph such that $V(X+Y) = V(X) \cup V(Y)$ and $E(X+Y) = E(X) \cup E(Y) \cup \{\{a, b\} | a \in V(X), b \in V(Y)\}$.

Let X and Y be graphs. An adjacency preserving mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* from X to Y, i.e. for any $x_1, x_2 \in V(X)$, if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp. isomorphism) f from X to itself is called an *endomorphism* (resp. *automorphism*) of X. The sets of all endomorphisms and automorphisms of X are written as $End(X)$ and $Aut(X)$, respectively. Always, we denote an endomorphism f of a graph X with $V(X) = \{v_1, v_2, \cdots, v_n\}$ in the obvious sense as

$$f = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_{i_1} & v_{i_2} & \cdots & v_{i_n} \end{pmatrix}.$$

i.e. a mapping f from $V(X)$ to itself such that $f(v_1) = v_{i_1}$, $f(v_2) = v_{i_2}$, $\cdots$, $f(v_n) = v_{i_n}$.

A *proper coloring* of a graph X is a map from $V(X)$ into some finite set of colors such that no two adjacent vertices are assigned the same colors. If X can be properly colored with a set of k colors, then we say that X can be properly k-colored. The least value of k for which X can be properly k-colored is the *chromatic number* of X, and is denoted by $\chi(X)$. We know that if there is a homomorphism from X to Y, then $\chi(X) \leq \chi(Y)$. A *retraction* of a graph X is a homomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ of f to $V(Y)$ is the identity mapping on $V(Y)$. It is known that the idempotents of $End(X)$ are retractions of X. The set of all idempotents of $End(X)$ is denoted by $Idpt(X)$. Let $f \in End(X)$. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e. for any $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. The equivalence class of a under ρ_f is denoted by $[a]_{\rho_f}$.

We use the standard terminology and notation of semigroup theory as in [4,11] and of graph theory as in [2,5]. We list some known results which are used within this paper.

Lemma 1.1 ([16])**.** *Let X be a graph and let $f \in End(X)$. Then f is completely regular if and only if $f|_{I_f} \in Aut(I_f)$.*

Lemma 1.2 ([1])**.** *Let X be a bipartite graph. Then X is End-completely-regular if and only if X is one of K_1, K_2, P_3, $2K_1$, $2K_2$ and $K_1 \bigcup K_2$.*

Lemma 1.3 ([6])**.** *Let X and Y be two bipartite graphs. Then $X + Y$ is End-completely-regular if and only if one of them is End-completely-regular and the other is K_1 or K_2.*

Lemma 1.4 ([8])**.** *Let X and Y be two K_3-free graphs. If both of them are non-bipartite, then for any endomorphism f of $X + Y$, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(X) \subseteq Y$ and $f(Y) \subseteq X$.*

2. Trees whose completely regular endomorphisms form a monoid

In this section, we investigate the completely regular endomorphisms of a tree and give the conditions under which all completely regular endomorphisms of the tree form a monoid.

Lemma 2.1. *Let X be a graph and $f \in End(X)$. Then f is completely regular if and only if $f(a) \neq f(b)$ for any $a, b \in V(I_f)$ with $a \neq b$.*

Proof. This follows directly from Lemma 1.1. □

Lemma 2.2. *Let X be a graph and Y be a retract of X. If $cEnd(X)$ forms a monoid, then $cEnd(Y)$ forms a monoid.*

Proof. Let $g_1, g_2 \in cEnd(Y)$. We only need to prove $g_1g_2 \in cEnd(Y)$. As Y is a retract of X, there exists a retraction h from X to Y such that the restriction $h|_Y$ of h to $V(Y)$ is the identity mapping on $V(Y)$. Then $g_1h, g_2h \in End(X)$. It is easy to check that $I_{g_1h} = I_{g_1}$ and $I_{g_2h} = I_{g_2}$. Let $a, b \in V(I_{g_1})$ be such that $a \neq b$. Then $g_1h(a) = g_1(a)$ and $g_1h(b) = g_1(b)$. Since g_1 is completely regular, by Lemma 2.1 $g_1(a) \neq g_1(b)$. Thus $g_1h(a) \neq g_1h(b)$. Hence $g_1h \in cEnd(X)$. A similar argument will show that $g_2h \in cEnd(X)$.

Denote $f = g_1hg_2h = g_1g_2h$. Then $f \in End(X)$. It is easy to check that $I_f = I_{g_1g_2}$. Since $cEnd(X)$ forms a monoid, f is completely regular. Let $c, d \in V(I_{g_1g_2})$ be such that $c \neq d$. Then $c, d \in V(I_f)$. Note that f is completely regular. By Lemma 2.1, $f(c) \neq f(d)$. This means that $g_1g_2h(c) \neq g_1g_2h(d)$. Now $g_1g_2(c) = g_1g_2h(c)$ and $g_1g_2(d) = g_1g_2h(d)$. Thus $g_1g_2(c) \neq g_1g_2(d)$. By Lemma 2.1, $g_1g_2 \in cEnd(Y)$. □

Lemma 2.3. *Let X be a graph and Y be a retract of X. If $cEnd(Y)$ does not form a monoid, then $cEnd(X)$ does not form a monoid.*

Proof. This follows directly from Lemma 2.2. □

Now we start to seek the conditions for a tree T under which $cEnd(T)$ forms a monoid

Lemma 2.4. *Let T be a tree. If $diam(T) \geq 3$, then $cEnd(T)$ does not form a monoid.*

Proof. Let T be a tree. If $diam(T) \geq 3$, then P_4 (see Fig.1) is a retract of T. By Lemma 2.3, we only need to show that $cEnd(P_4)$ does not form a monoid. Let

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

Then $f \in Idpt(P_4)$, $g \in Aut(P_4)$. So $f, g \in cEnd(P4)$. Now

$$fg = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_2 & x_1 \end{pmatrix}.$$

It is not hard to see that $x_1, x_3 \in V(I_{fg})$. But $fg(x_1) = fg(x_3) = x_2$. By Lemma 2.1, fg is not completely regular. Hence $cEnd(P_4)$ does not form a monoid. □

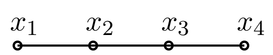

Fig.1 Graph P_4

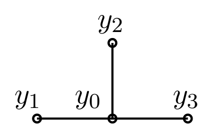

Fig.2 Graph S_3

Lemma 2.5. *Let T be a tree. If there exists $x \in V(T)$ such that $d(x) \geq 3$, then $cEnd(T)$ does not form a monoid.*

Proof. Let T be a tree. If there exists $x \in V(T)$ such that $d(x) \geq 3$, then the star S_3 (see Fig.2) is a retract of T. We only need to show that $cEnd(S_3)$ does not form a monoid. Let

$$f = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_1 & y_3 \end{pmatrix}.$$

and

$$g = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_1 & y_3 & y_3 \end{pmatrix}.$$

Then $f \in Aut(S_3)$, $g \in Idpt(S_3)$. So $f, g \in cEnd(S_3)$. Now

$$fg = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_3 & y_3 \end{pmatrix}.$$

It is easy to see that $y_2, y_3 \in V(I_{fg})$. But $fg(y_2) = fg(y_3) = y_3$. By Lemma 2.1, fg is not completely regular. Hence $cEnd(S_3)$ does not form a monoid. □

Theorem 2.1. *Let T be a tree. Then $cEnd(T)$ forms a monoid if and only if T is one of K_1, K_2 and P_3.*

Proof. Necessity. This follows from Lemmas 2.4 and 2.5.

Sufficiency. By Lemma 1.2, K_1, K_2 and P_3 are End-completely-regular. This means that $End(T) = cEnd(T)$ for $T \in \{K_1, K_2, P_3\}$. Therefore, $cEnd(T)$ forms a monoid. □

3. Graphs whose completely regular endomorphisms form a monoid

In this section, we show that if $cEnd(X+Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form a monoid. We give several approaches to construct new graphs whose completely regular endomorphisms form a monoid.

Lemma 3.1. *Let X and Y be two graphs. If $cEnd(X + Y)$ forms a monoid, then both $cEnd(X)$ and $cEnd(Y)$ form a monoid.*

Proof. Let $f_1, f_2 \in cEnd(X)$. Then $f_1 f_2 \in End(X)$. To prove that $cEnd(X)$ forms a monoid, we only need to prove that $f_1 f_2 \in cEnd(X)$. By Lemma 2.1, we only need to show that $f_1 f_2(x_1) \neq f_1 f_2(x_2)$ for any $x_1, x_2 \in I_{f_1 f_2}$ with $x_1 \neq x_2$. Define two mappings from $X + Y$ to itself by

$$F_1(x) = \begin{cases} f_1(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

and

$$F_2(x) = \begin{cases} f_2(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

Then $F_1, F_2 \in End(X+Y)$. It is not difficult to see that $I_{F_1} = I_{f_1}+Y$. Note that $F_1(x) = f_1(x) \in V(X)$ for any $x \in V(X)$ and $f_1 \in cEnd(X)$. Then $F_1|_{I_{f_1}} = f_1|_{I_{f_1}} \in Aut(I_{f_1})$. On the other hand, $F_1(x) = x$ for any $x \in V(Y)$. This means $F_1|_Y$ is an identity mapping. Then $F_1|_Y \in Aut(Y)$. Thus $F_1|_{I_{F_1}} \in Aut(I_{F_1})$. Hence $F_1 \in cEnd(X+Y)$. A similar argument will show that $F_2 \in cEnd(X+Y)$.

Since $cEnd(X + Y)$ forms a monoid, $F_1F_2 \in cEnd(X + Y)$. Furthermore,

$$F_1F_2(x) = \begin{cases} f_1f_2(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

Let $x_1, x_2 \in I_{f_1f_2}$ be such that $x_1 \neq x_2$. Then $x_1, x_2 \in I_{F_1F_2}$. By Lemma 2.1, $F_1F_2(x_1) \neq F_1F_2(x_2)$. Note that $F_1F_2(x_1) = f_1f_2(x_1)$ and $F_1F_2(x_2) = f_1f_2(x_2)$. Then $f_1f_2(x_1) \neq f_1f_2(x_2)$. By Lemma 2.1, $f_1f_2 \in cEnd(X)$. Hence $cEnd(X)$ forms a monoid. A similar argument will show that $cEnd(Y)$ forms a monoid. □

However, both $cEnd(X)$ and $cEnd(Y)$ form a monoid can not yield that $cEnd(X + Y)$ forms a monoid. In the following, we give an example in the range of trees.

Example 3.1. *Let $T_1 = T_2 = P_3$. By Theorem 2.1, both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid.*

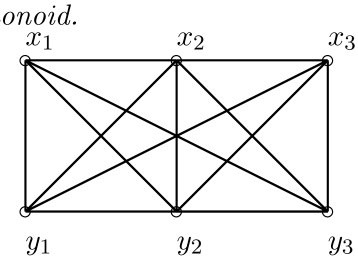

Fig.3 Graph $P_3 + P_3$

Let

$$f_1 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ x_3 & x_2 & x_3 & y_1 & y_2 & y_3 \end{pmatrix}.$$

Then $f_1 \in Aut(P_3+P_3)$ and $f_2 \in Idpt(P_3+P_3)$. Thus $f_1, f_2 \in cEnd(P_3+P_3)$. Now

$$f_1f_2 = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ y_3 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}.$$

It is easy to check that $x_1, x_3 \in V(I_{f_1f_2})$. But $f_1f_2(x_1) = f_1f_2(x_3) = y_3$. By Lemma 2.1, f_1f_2 is not completely regular. Thus $cEnd(P_3 + P_3)$ does not form a monoid.

Theorem 3.1. *Let T_1 and T_2 be two trees. Then $cEnd(T_1+T_2)$ forms a monoid if and only if*

(1) *Both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid, and*

(2) *$T_1 + T_2 \neq P_3 + P_3$.*

Proof. Necessity. This follows from Lemma 3.1 and Example 3.1.

Sufficiency. If both $cEnd(T_1)$ and $cEnd(T_2)$ form a monoid and $T_1 + T_2 \neq P_3 + P_3$, then $T_1 + T_2 \in \{K_1 + K_1, K_1 + K_2, K_1 + P_3, K_2 + K_2, K_2 + P_3\}$. By Lemma 1.3, they are End-completely-regular. This means that $End(T_1 + T_2) = cEnd(T_1 + T_2)$. Therefore, $cEnd(T_1 + T_2)$ forms a monoid. □

In the following, we give some sufficient conditions under which $cEnd(X+Y)$ forms a monoid.

Lemma 3.2. *Let X and Y be two graphs such that both $cEnd(X)$ and $cEnd(Y)$ form monoids. If for any $f \in End(X + Y)$, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, then $cEnd(X + Y)$ forms a monoid.*

Proof. Let $f, g \in cEnd(X + Y)$. Then $fg \in End(X + Y)$. Denote $f_1 = f|_X$, $f_2 = f|_Y$, $g_1 = g|_X$ and $g_2 = g|_Y$. Note that $f(X) \subseteq X$ and $f(Y) \subseteq Y$ for any $f \in End(X + Y)$. Then $f_1, g_1 \in End(X)$, $f_2, g_2 \in End(Y)$ and $I_{fg} = I_{f_1g_1} + I_{f_2g_2}$. Since f is completely regular, $f_1(x_1) = f(x_1) \neq f(x_2) = f_1(x_2)$ for any $x_1, x_2 \in I_{f_1}$ and $f_2(y_1) = f(y_1) \neq f(y_2) = f_2(y_2)$ for any $y_1, y_2 \in I_{f_2}$. Hence $f_1 \in cEnd(X)$ and $f_2 \in cEnd(Y)$ by Lemma 2.1. Similarly, we have $g_1 \in cEnd(X)$ and $g_2 \in cEnd(Y)$. Note that $cEnd(X)$ and $cEnd(Y)$ form monoids. Then $f_1g_1 \in cEnd(X)$ and $f_2g_2 \in cEnd(Y)$.

Let $x, y \in V(I_{fg})$ with $x \neq y$. If $x, y \in V(I_{f_1g_1})$, then $fg(x) = f_1g_1(x) \neq f_1g_1(y) = fg(y)$. Similarly, if $x, y \in I_{f_2g_2}$, then $fg(x) \neq fg(y)$. If $x \in I_{f_1g_1}$ and $y \in I_{f_2g_2}$, then $fg(x) \in V(X)$ and $fg(y) \in V(Y)$. Clearly, $fg(x) \neq fg(y)$. By Lemma 2.1, $fg \in cEnd(X + Y)$. Therefore, $cEnd(X + Y)$ forms a monoid. □

Theorem 3.2. *Let X and Y be two graphs such that both $cEnd(X)$ and $cEnd(Y)$ form monoids. Then:*

(1) *If X and Y are two K_3-free non-bipartite graphs and the cores of X and Y are not isomorphic, then $cEnd(X + Y)$ forms a monoid.*

(2) *If X is a bipartite graph and Y is a K_3-free non-bipartite graph, then $cEnd(X + Y)$ forms a monoid.*

(3) *If X is a K_3-free non-bipartite graph and Y has at least one triangle with $\chi(Y) < \chi(X) + 1$, then $cEnd(X + Y)$ forms a monoid.*

Proof. (1) Let X and Y be two K_3-free non-bipartite graphs and $f \in End(X + Y)$. By Lemma 1.4, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(Y) \subseteq X$ and $f(X) \subseteq Y$. In the second case, the restriction $f|_X$ of f to $V(X)$ is a homomorphism from X to Y and the restriction $f|_Y$ of f to $V(Y)$ is a homomorphism from Y to X. Thus X and Y are homomorphically equivalent. Hence the cores of X and Y

are isomorphic. This is a contradiction. Therefore, $f(X) \subseteq X$ and $f(Y) \subseteq Y$. By Lemma 3.2, $cEnd(X+Y)$ forms a monoid.

(2) Let $f \in End(X+Y)$. We hope to prove that $f(X) \subseteq X$ and $f(Y) \subseteq Y$. There are two cases.

Case 1. Assume that $E(X) = \phi$. First, we claim that $f(X) \subseteq X$. Otherwise, there exists $x \in V(X)$ such that $f(x) \in Y$. As Y is non-bipartite, Y has an odd cycle. Note that every endomorphism image of an odd cycle must contain an odd cycle. Then $f(Y)$ contains an odd cycle. Note that $E(X) = \phi$. Then $f(Y)$ has an edge in Y, say $\{f(y_1), f(y_2)\}$. Thus $f(y_1), f(y_2), f(x)$ form a triangle in Y. This contradicts that Y is K_3-free.

Next, we prove $f(Y) \subseteq Y$. Otherwise, there exists $y \in V(Y)$ such that $f(y) \in V(X)$. Let $x \in V(X)$. Then $\{y, x\} \in E(X+Y)$. It follows that $\{f(y), f(x)\} \in E(X+Y)$. As $f(y), f(x) \in V(X)$, $\{f(y), f(x)\} \in E(X)$. This is impossible.

Case 2. Assume that $E(X) \neq \phi$. First, we prove that $f(X) \subseteq V(X)$. Suppose for contradiction that $f(X) \not\subseteq V(X)$. Then either $f(X) \subseteq V(Y)$, or there exist $x_1, x_2 \in V(X)$ such that $f(x_1) \in V(X)$ and $f(x_2) \in V(Y)$. In the first case, since X contains at least one edge, $f(X)$ contains at least one edge, say $\{a, b\}$. Now we claim that $f(Y) \subseteq V(X)$. Otherwise, there exists $y_0 \in V(Y)$ such that $f(y_0) \in V(Y)$. Then $a, b, f(y_0)$ form a triangle in Y. This contradicts that Y is K_3-free. Hence $f|_Y$ is a homomorphism from Y to X, and so $\chi(Y) \leq \chi(X)$. Note that $\chi(X) = 2$ and $\chi(Y) \geq 3$. This is a contradiction. In the second case, since Y has an odd cycle, $f(Y)$ also has an odd cycle. Thus $f(Y)$ either has an edge in X or has an edge in Y. If there exist $y_1, y_2 \in V(Y)$ such that $\{f(y_1), f(y_2)\} \in E(Y)$, then $f(x_2), f(y_1), f(y_2)$ form a triangle in Y. This contradicts that Y is K_3-free. If there exist $y_1, y_2 \in V(Y)$ such that $\{f(y_1), f(y_2)\} \in E(X)$, then $f(x_1), f(y_1), f(y_2)$ form a triangle in X. This is impossible since X is bipartite. Hence $f(X) \subseteq X$.

Next, we prove that $f(Y) \subseteq Y$. Otherwise, there exists $y \in V(Y)$ such that $f(y) \in V(X)$ and $f(y)$ is adjacent to every vertex of $f(X)$. Note that $f(X)$ contains at least one edge in X, say $\{f(x_1), f(x_2)\}$. Thus $f(x_1), f(x_2), f(y)$ form a triangle in X. This is a contradiction. Now the assertion follows directly from Lemma 3.2.

(3) We show that $f(Y) \not\subseteq X$. Otherwise, $f|_Y$ is a homomorphism from Y to X. Note that any homomorphism f maps a triangle to a triangle and Y contains at least one triangle. Then X must contain a triangle. This contradicts that X is K_3-free. Hence either $f(Y) \subseteq Y$, or there exist $y_1, y_2 \in V(Y)$ such that $f(y_1) \in V(Y)$ and $f(y_2) \in V(X)$.

In the second case, if $f(X) \subseteq V(X)$, then $f|_X$ is a homomorphism from X to itself. Hence $\chi(X) = \chi(I_{f|_X})$. On the other hand, $f(y_2)$ is adjacent to every vertex of $I_{f|_X}$. Hence $\chi(X) \geq \chi(I_{f|_X}) + 1$. This is a contradiction. If $f(X) \subseteq V(Y)$, then $f|_X$ is a homomorphism from X to Y. Note that $f(y_1)$ is adjacent to every vertex of $I_{f|_X}$. Then $\chi(Y) \geq \chi(I_{f|_X}) + 1 \geq \chi(X) + 1$. A contradiction. If there exist $x_1, x_2 \in V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$, then both $f(X)$ and $f(Y)$ have no edge in X, otherwise, there exists a triangle in X. Denote $A = f(X) \cap V(X)$ and $B = f(Y) \cap V(X)$. Then the subgraph H of X induced by $A \cup B$ is bipartite. It is easy to see that I_f is a subgraph of $H + Y$. It follows that $\chi(I_f) \leq \chi(H + Y) = \chi(Y) + 2$. On the other hand, $\chi(X + Y) = \chi(X) + \chi(Y) \leq \chi(I_f)$. Hence $\chi(X) + \chi(Y) \leq \chi(Y) + 2$. This means that $\chi(X) \leq 2$. This is a contradiction since X is non-bipartite. Hence $f(Y) \subseteq Y$.

If $f(X) \nsubseteq X$, then there exists $x \in V(X)$ such that $f(x) \in V(Y)$. Clearly, $f(x)$ is adjacent to every vertex in $V(I_{f|_Y})$. Thus we have $\chi(Y) \geq \chi(I_{f|_Y}) + 1 \geq \chi(Y) + 1$. This is a contradiction. Hence $f(X) \subseteq X$. By Lemma 3.2, $cEnd(X + Y)$ forms a monoid. □

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