On locally F-semiregular and locally δ-semiregular modules

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Abstract. In this paper, we introduce locally $F-$ semiregular, locally $\delta-$ semiregular and locally finitely $\delta-$supplemented modules and investigate some properties of these new classes of modules. We prove that, if M is a self-projective module with the finite exchange property, then M is $\delta(M)-$semipotent. Finally, we show that the properties of $F-$semiregular, $\delta-$semiregular and $\delta-$supplemented modules which can be extended to their localizations.

Keywords: δ- small submodule, locally F-semiregular, locally δ-supplemented module, locally δ-semiregular.

1. Introduction

Unless stated otherwise, all rings are associative and have identity, all modules are unital left modules. Semiregular, supplemented, δ-semiregular, δ-supplemented module are constitute classes of modules that establishes nice homological, non-homological and algebraic properties. The concept of semiregular modules has been generalized to δ-semirgular module by Zhou [11].

In this paper, we construct the notions of locally δ-semiregular and locally δ-suppl-emented modules as generalizations of δ-semiregular and δ-supplemented modules, respectively. In section two, we investigate some properties and prove some new results on δ-semiregular modules. An $R-$module M is called $F-$semiregular module, if for every $x \in M$, there exists a decomposition $M = A \oplus B$ such that $A \subseteq Rx$ is projective and $B \cap Rx \subseteq F$. In third section, we define an element $x \in M$ is locally $F-$semiregular element with respect to a multiplicative closed system S in R if there exists $s \in S$ such that $\frac{x}{s}$ is F_S-semirgular in M_S . Therefore, a module M is said to be locally $F-$semiregular, if every element

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of M is locally $F-$semiregular. If R is a commutative ring with identity and P is a prime ideal of R, then the localization of R at P is local ring. We generalized this properties for some special types of modules among them we provide that, if M is an $F-$semiregular and $End\left(\left(\frac{M}{F}\right)_P\right)$ is commutative, then $\left(\frac{M}{F}\right)_P$ is local as an R_P-module, for any prime ideal P of R. Moreover, a module M is locally $F-$semiregular with respect to a multiplicative system S in R if and only if $\left(\frac{M}{F}\right)_S$ is a regular module and direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.

In section four, we investigate some properties of δ-semiregular modules that can be extended to their localizations. A submodule N of M is locally δ-small if and only if $X_P+N_P = M_P$, then there exists a projective semisimple submodule Y_P of N_P such that $X_P \oplus Y_P = M_P$ for every maximal ideal P of R. We prove that an $R-$module M is locally δ-semiregular module if and only if $\frac{M}{N}$ has a locally projective δ-cover, for every finitely generated submodule N of M. By finitely locally δ-supplemented module we mean, a module M such that every finitely generated submodule N_P of M_P has a supplemente submodule in M_P. Furthermore, every locally δ-semiregular module is locally finitely δ-supplemented module.

2. Results on δ-semiregular modules

In this section, we establish some results on endomorphism of a δ- semiregular module and providing some conditions under which the endomorphism of a δ-semiregular module would be δ- semiregular. Also, we investigate some new results on $U-$semipotent modules, where a module $M$is called $U-$ semipotent[6]if for every submodule $A$of Msuch that $A \nsubseteq U$, there exists a direct summand $B$of Msuch that $B \subseteq A$and $B \nsubseteq U$, for a submodule U of M.

Lemma 2.1. *Let M be an $R-$module and X be a submodule of M. If every submodule $N \ll_\delta M$ with $M = N + X$, then M is $X-$semipotent.*

Proof. Let $N \ll_\delta M$ with $M = N + X$, then by [11, lemma 2.1], there exists $Y \subseteq N$ such that $M = X \oplus Y$ and Y is projective semisimple $Y \nsubseteq X$ and hence M is an $X-$semipotent module. □

It is obvious that, every non-zero $N \ll_\delta M$ with $M = X + N$ is a $0-$potent.

Proposition 2.1. *If M is a self-projective module with the finite exchange property, then M is $\delta(M)-$semipotent.*

Proof. Let $N \leq M$ such that $N \nsubseteq \delta(M)$. Then there exist $x \in (N - \delta(M))$, this implies that $x \notin RadM$, since $RadM \subseteq \delta(M)$. So, there exists a maximal submodule K of M such that $M = Rx + K$. Since M has a finite exchange property, then exists a decomposition $M = P \oplus Q$ such that $P \leq Rx$ and $Q \leq K$. If $P \leq \delta(M)$, then $\frac{M}{P}$ is simple. Therefore, P is a maximal submodule and hence $P = K$. Also, $K = M$ which is a contradiction. Thus $P \nsubseteq \delta(M)$, so M is $\delta(M)-$semipotent. □

Proposition 2.2. *Let M be an $R-$module and $m \in M$. Then $Rm \ll_\delta M$ if and only if $m \in \delta(M)$.*

Proof. Suppose $Rm \ll_\delta M$, then Rm is a submodule of $\delta(M)$ and hence $m \in \delta(M)$.

Conversely, let $m \in \delta(M)$ and suppose that Rm is no δ-small, then there exists a maximal submodule N of M with $\frac{M}{N}$ is singular such that $m \notin N$ and hence $m \notin \delta(M)$. □

Proposition 2.3. *$\delta(M) = M$ if and only if all finitely generated submodules are δ-small.*

Proof. Suppose that $\delta(M) = M$ and N is a finitely generated submodule of M with $N + L = M$, where $\frac{M}{L}$ is singular. Since $N = Rx_1 + Rx_2 + \ldots. + Rx_n$, where $x_i \in M = \delta(M)$, which implies that $Rx_i \ll_\delta M$ for each $i = 1, \ldots, n$. Therefore, $L = M$.

Conversely, if $m \in M$, then Rm is a finitely generated submodule of M and hence $Rm \ll_\delta M$. Thus $m \in \delta(M)$. □

Proposition 2.4. *Let $\delta(M) \ll_\delta M$. Then $\frac{M}{\delta(M)}$ has no nonzero δ-small submodule.*

Proof. Let $\frac{L}{\delta(M)}$ be a δ-small submodule of $\frac{M}{\delta(M)}$. Therefore, $L \ll_\delta M$ [9, Lemma 2.1], then $L \leq \delta(M)$ and hence $L = \delta(M)$. Whence $\frac{L}{\delta(M)}$ is a zero submodule of $\frac{M}{\delta(M)}$. □

Let A be a submodule of M. Then A is called a coclosed submodule of M if A/B is not small in M/B for any proper submodule B of A [7].

Proposition 2.5. *Let N be a coclosed submodule of an $R-$module M. Then $\delta(N) = N \cap \delta(M)$.*

Proof. Clearly, $\delta(N) \leq N \cap \delta(M)$. Let $x \in (N \cap \delta(M))$. Then $x \in N$ implies that $Rx \leq N$ and $x \in \delta(M)$. Hence $Rx \ll_\delta M$. Since N is coclosed submodule, then $Rx \ll_\delta N$, therefore $x \in \delta(N)$. □

An R-module M is called hollow if every submodule is small [2]. In the following Lemma, we give a result on hollow module.

Lemma 2.2. *Let M be a hollow module, which has a maximal submodule K with $\frac{M}{K}$ is singular. Then $\delta(M) = K$.*

Proposition 2.6. *Let M be a hollow module. Then $\delta(M) \neq M$ if and only if M is cyclic.*

Proof. Let M be a hollow module and $\delta(M) \neq M$. Then $\delta(M) \ll_\delta M$. Therefore, $\delta(M)$ is unique maximal submodule with $\frac{M}{\delta(M)}$ singular and simple. So, $\frac{M}{\delta(M)}$ is cyclic. Suppose $\frac{M}{\delta(M)} = Rm + \delta(M)$ for $m \in M$. We claim that $M = Rm$. Let $w \in M$. Then $(w + \delta(M)) \in \frac{M}{\delta(M)}$, therefore there is $r \in R$ such that $w + \delta(M) = r(m + \delta(M)) = rm + \delta(M)$, which implies that $w - rm = y$, for some $y \in \delta(M)$. Thus, $w = rm + y \in (Rm + \delta(M))$. Hence $M = Rm + \delta(M)$, then $M = Rm$.

Conversely, since M is cyclic, then every submodules of M is essential, so M contains a maximal essential submodule. Hence $\delta(M) \neq M$. □

Lemma 2.3. *Let M be a local module. Then $\delta(M) = RadM$.*

In the following proposition we give a characterization of local modules.

Proposition 2.7. *M is local if and only if $\delta(M)$ is a maximal and δ-small in M.*

Proof. Let $\delta(M)$ be a δ-small and maximal submodule in M. To show M is local, first we must show that $\delta(M)$ is unique maximal submodule of M. Suppose that L is another maximal submodule in M. Then $M = L + \delta(M)$, but $\delta(M) \ll_\delta M$, $\frac{M}{L}$ singular simple module, so $L = M$, which is a contradiction. Thus $\delta(M)$ is a unique maximal submodule in M. We claim that every proper submodule of M contained in $\delta(M)$. Let N be a proper submodule of M. Since if not $N + \delta(M) = M$ and $\frac{M}{N}$ is singular simple module, so we get $N = M$, which is a contradiction. Therefore, M is a local module. Conversely, is obvious. □

Theorem 2.1. *Let P be a projective module. Then the following are equivalent:*

1. *P has a δ-small and maximal submodule .*
2. *$\delta(P)$ is a δ-small and maximal submodule .*
3. *P is a local module.*
4. *$End(P)$ is a local ring.*
5. *Every maximal submodule in P is a δ-small submodule of P.*
6. *P is a projective δ-cover, for a simple module .*

Proof. $(1 \Rightarrow 2)$Let N be δ-small and maximal submodule of P. Then $N = \delta(P)$. Moreover,

$N \ll_\delta P$, this implies that $\delta(P) \ll_\delta N$. Hence $N = \delta(P)$.

$(2 \Rightarrow 3)$ P is a local module.

$(3 \Rightarrow 4)$ Since P is a local projective module, then $End(P)$ is a local ring.

($4 \Rightarrow 5$) Let N be a maximal submodule in P. We must show that $N \ll_\delta P$. Now since P is a projective module, then $End(P)$ is a local ring. Therefore P is a local module and hence P is hollow. Thus $N \ll P$ and hence $N \ll_\delta P$.

($5 \Rightarrow 6$) Since P is a projective module and $\delta(P) \neq P$, then P has a maximal submodule, say N. Now $\frac{P}{N}$ is a simple. Let $\pi{:}P \to \frac{P}{N}$ is a natural epimorphism. We have $ker\pi \ = N$ and $N {\ll_\delta} P$, then (P, π) is a projective δ-cover for $\frac{P}{N}$.

($6 \Rightarrow 1$) Let P be a projective δ-cover for a simple module , say M. So, there exists an epimorphism $g : P \to M$ such that $kerg \ll_\delta P$. We only have to show $kerg$ is a maximal submodule in P by First Isomorphism Theorem, $\frac{P}{kerg} \cong M$ and M is a simple module, then $\frac{P}{kerg}$ is also simple and $kerg$ is a maximal submodule in P. □

Proposition 2.8. *Let M be $\delta(M)-$semipotent. Then every indecomposable direct summand N of M with $N \nsubseteq \delta(M)$ is local.*

Proof. Suppose that N is indecomposable direct summand of M with $N \nsubseteq \delta(M)$. We claim that for every proper submodule K of N, $\frac{N}{K}$ is simple module and hence $K \leq RadN$. Assume not, K is not submodule of $RadN$, since $RadN = N \ \cap \ RadM$. We have $N \nsubseteq \delta(M)$, this implies that $N \nsubseteq RadM$ and hence $N \nsubseteq RadN$. Since M is $\delta(M)-$ semipotent, then there exists a summand X of M such that $X \nsubseteq \delta(M)$. Thus $X \nsubseteq RadM$ and hence X is a summand of N, but N is indecomposable, where $X = N = K$, which is a contradiction. Hence N local. □

Theorem 2.2. *Let U be a proper submodule of a module M. If M is indecomposable and $\delta(M) \ll_\delta M$, then the following are equivalent:*

1. *U respect every finitely generated submodule of M.*
2. *M is $U-$semipotent.*
3. *M is local.*

Proof. (**1 ⇒ 2**) Let A be a submodule of M with $A \nsubseteq U$**.** Suppose that $a \ \in \ (A - U)$, then $M = X \ \oplus \ Y$, where $X \leq Ra$ and $Y \ \cap \ Ra \leq U$.This implies that $Ra = X \ \oplus \ (Y \ \cap \ Ra)$, so $X \nsubseteq U$. Hence, M is $U-$semipotent.

(**2 ⇒ 3**) By (Proposition 2.7), M is local . Since $RadM$ is a maximal and $RadM \ \subseteq \ \delta(M)$, then $RadM = \delta(M)$. Hence $U \leq RadM = \delta(M)$.

Now, let $x \ \in \ \delta(M) - U$. Then there exists a direct summand $B \leq Rx$ and $B \nsubseteq U$. Since M local, then it is hollow and every hollow indecomposable, $B = 0$, which is a contradiction. Hence $U = \delta(M)$.

(**3 ⇒ 1**) Let N be a finitely generated submodule of M. If $N = M$ there is nothing to prove. Assume $N \neq M$, this implies $N \leq RadM$, since M is local, hence $N \leq \delta(M)$. Then $M = \{0\} \oplus M$, therefore $M \cap N = N \leq \delta(M)$. Hence $U = \delta(M)$ respects every finitely generated submodule of M. □

A submodule N of M is said to be PSD in M, if there exists a projective summand S of M such that $S \leq N$ and $M = S \oplus X$, whenever $N + X = M$, for any submodule X of M [10].

Lemma 2.4. *If M is δ-semiregular, then every finitely generated submodule N of M contains a PSD submodule of M.*

Proof. Let N be a finitely generated submodule of M. Then there exists a decomposition $M = P \oplus Q$ such that $P \subseteq N$ is projective and $N \cap Q$ is δ-small in Q. Now, suppose that $M = N \cap Q + X$, for a submodule X of M, then there exists a submodule S of $N \cap Q$ such that S is projective and $M = S \oplus X$. □

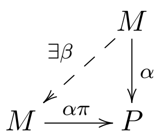
Proposition 2.9. *Let P be a projective module and N be a submodule of P. If $\frac{P}{N}$ has a projective δ-cover, then N contains a PSD submodule.*

Proof. Let $q : Q \to \frac{P}{N}$ be a projective δ-cover and $t : P \to \frac{P}{N}$ be the canonical epimorphism. Then there exists a decomposition $P = X \oplus Y$ such that $t|x : X \to \frac{P}{N}$ is projective δ-cover and $Y \subseteq kert = N$. Thus $X \cap N = kert|x \ll_\delta X$, since X is a direct summand of P, $X \cap N \ll_\delta P$, then we have $S = Y$ is a projective summand of P. □

Theorem 2.3. *Suppose that M is a module, where $E = End\ M$ and $A = \{\alpha \in E; \alpha(M) \ll_\delta M$ and $\frac{M}{(1-\alpha)M}$ is singular$\}$. Then $A \subseteq \delta(E)$, where M is direct projective. Moreover, if A is an essential, then E is δ-semiregular and $A = \delta(E)$ if and only if there exists a decomposition $M = P \oplus Q$ such that $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$, for every $\alpha \in E$.*

Proof. First to show $A \subseteq \delta(E)$, suppose $\alpha \in A$, then we have $\alpha(M) + (1-\alpha)(M) = M$, so $M = (1-\alpha)(M)$. Since M is a direct projective, then $1-\alpha$ has an inverse in E. So, we get $\alpha \in J(E) \subseteq \delta(E)$. Suppose that E is δ-semiregular and $A = \delta(E)$. Let $\alpha \in E$. Then there exists $\pi^2 = \pi \in E\alpha$ such that $\alpha - \alpha\pi \in \delta(E) = A$, so $(\alpha - \alpha\pi)(M)$ is δ-small. Now, $M = \pi(M) \oplus (1-\pi)(M)$, where $\pi(M) \subseteq \alpha(M)$ and $\alpha(M) \cap (1-\pi)(M) = (\alpha - \alpha\pi)(M)$ is δ-small.

Conversely, suppose that there exists a decomposition $M = P \oplus Q$ such that $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$, for $\alpha \in E$. Let $\pi : M \to P$ be the projection map. Then $\pi(M) = P$ and $ker\pi = Q$. Since $\alpha : M \to P$, then $\alpha\pi : M \to P$ is an epimorphism and M is a direct projective, then the following diagram is commutative:



Then there exists $\beta \in E$ such that $\beta\alpha\pi = \pi$. Define $\tau = \pi\beta\alpha$. Then $\tau^2 = \tau \in \alpha(E)$, and $\tau(M) \subseteq \alpha(M)$, $ker\tau = ker\pi = Q$. Therefore,

$(\alpha - \alpha\tau)(M) = \alpha(1-\alpha)(M) \subseteq (\alpha(M) \cap Q)$, which is δ-small in M. So, $(\alpha - \alpha\tau)(M)$ is a δ-small in M and $\frac{M}{(1-\alpha+\alpha\tau)(M)}$ is singular. Therefore, $(\alpha - \alpha\tau) \in A \subseteq \delta(E)$, so α is δ-semiregular, this implies that E is semiregular. Since $\frac{E}{A}$ is regular, then A is essential maximal ideal, so $\delta(E) \subseteq A$. Hence $A = \delta(E)$. □

Theorem 2.4. *Let M be a δ-semiregular module. If every epimorphism in $EndM$ has a left inverse, $A = \{\gamma \in EndM; \gamma(M) \ll_\delta M\}$ and $G = \{\alpha \in EndM; \alpha(M)$ is Noetherian$\}$ are essential ideal in $EndM$, then G is δ-semiregul-ar.*

Proof. Let $E = EndM$ and $\alpha \in G$. Since $\alpha(M)$ is finitely generated, then there exists a decomposition $M = P \oplus Q$, where $P \subseteq \alpha(M)$ is projective and $\alpha(M) \cap Q \ll_\delta M$. Let π be the projection mapping with $\pi(M) = P$ and $ker\pi = Q$. Since P is projective, then there exists $\beta : P \to M$ such that $\alpha\beta = \pi$. If we set, $\beta(Q) = 0$, then $\beta(M) \subseteq P \subseteq \alpha(M)$ is Noetherian, so $b \in G$. Moreover, $(\alpha - \alpha\pi)(M) \subseteq (\alpha(M) \cap ker\pi) = \alpha(M) \cap Q$, then $(\alpha - \alpha\pi) \in A \cap G$. We show that $\delta(G) = A \cap G$, since $\frac{G}{A \cap G}$ is regular, so $\delta(E) \subseteq (A \cap G)$. On the other hand, since each epimorphism in E has a left inverse, then $A \subseteq J(E) \subseteq \delta(E)$. So, $(A \cap G) \subseteq (\delta(E) \cap G) = \delta(G)$. Hence, $\delta(G) = A \cap G$ and this gives that G is δ-semiregular. □

Proposition 2.10. *Suppose that R is a δ-semiregular ring, if $ann\left(\frac{M}{N}\right)$ is finitely generated ideal, for every finitely generated submodules N of M. Then every faithful multiplication $R-$module is δ-semiregular module.*

Proof. Let N be a finitely generated submodule of M. Then $N = ann\left(\frac{M}{N}\right)M$, where $ann\left(\frac{M}{N}\right)$ is finitely generated ideal in R. Since R is δ-semiregular, so there exists a decomposition $R = A \oplus B$, such that $A \subseteq ann\left(\frac{M}{N}\right)$ is projective and $(ann\left(\frac{M}{N}\right) \cap B) \ll_\delta B$. Therefore, $M = AM \oplus BM$ and $AM \subseteq ann\left(\frac{M}{N}\right)M = N$, and $(BM \cap N) \ll_\delta BM$. □

Corollary 2.1. *Over a Noetherian δ-semiregular ring, every faithful multiplication module is δ-semiregular.*

3. locally $F-$semiregular modules

In this section, we construct a localization for $F-$semiregular module and investigate whether the properties of $F-$semiregular modules are extends under the localization. We start this section with the following definition.

Definition 3.1. *Suppose that M is an $R-$module, F is a submodule of M and S is a multiplicative closed system in R, then an element $x \in M$ is said to be locally $F-$semiregular with respect to S if $\frac{x}{s}$ is F_S-semiregular in M_S for some $s \in S$. If every element of M is locally $F-$semiregular, then M is called locally $F-$semiregular module with respect to S.*

The following is an example of locally $F-$semiregular in Z_{12}.

Example 3.1. *Consider Z_{12} as a $Z_{12}-$module and the closed set $S = \{1\ ,\ 11\}$, then $(Z_{12})_S = \left\{\frac{0}{1},\ \frac{1}{1}\ ,\ \frac{2}{1},\ldots,\frac{11}{1}\right\}$. It can be seen that $3\ \in\ Z_{12}$ is locally $F-$semiregular element in Z_{12}, where $F = \{0\ ,\ 3\ ,\ 6\ ,\ 9\}$, since $(Z_{12})_S = I_S\ \oplus\ J_S$, where $I = \{0\ ,\ 6\}$ and $J = \{0\ ,\ 4\ ,\ 8\}$, then $I_S = \left\{\frac{0}{1},\frac{6}{1}\right\}$, $J_S = \left\{\frac{0}{1},\ \frac{4}{1},\ \frac{8}{1}\right\}$, $I_S \subseteq \left\{\frac{0}{1},\frac{3}{1},\ \frac{6}{1},\ \frac{9}{1}\right\}$ is projective and $\left\{\frac{0}{1},\ \frac{3}{1},\ \frac{6}{1},\ \frac{9}{1}\right\} \cap J_S = \left\{\frac{0}{1}\right\} \subseteq F_S$, so $\frac{3}{1}$ is F_S-semiregular.*

If $\alpha\ \in\ M^*$, then $\alpha : M \to R$ is a homomorphism that induces a homomorphism $\alpha_S : M_S \to R_S$ which is defined by $\alpha_S\left(\frac{m}{s}\right) = \frac{\alpha(m)}{s}$, for every $m\ \in\ M$ and $s\ \in\ S$. This notation leads us to study the relations between $F-$semiregular, which is defined in [1] and locally $F-$semiregular.

Proposition 3.1. *Every $F-$semiregular element in M is locally $F-$semiregular.*

Proof. Suppose that x is $F-$semiregular, then there exists an $\alpha\ \in\ M^*$ such that $(\alpha(x))^2 = \alpha(x)$ and $x - \alpha(x)x\ \in\ F$, therefore α_S, which is mentioned in above, exists. Suppose that s is an idempotent in S, then $\alpha_S\left(\frac{x}{s}\right) = \frac{\alpha(x)}{s} = \frac{(\alpha(x))^2}{ss} = \frac{\alpha(x)}{s}\frac{\alpha(x)}{s} = \alpha_S\left(\frac{x}{s}\right)\alpha_S\left(\frac{x}{s}\right)$, then $\left(\alpha_S\left(\frac{x}{s}\right)\right)^2 = \alpha_S\left(\frac{x}{s}\right)$ and $\frac{x}{s} - \alpha_S\left(\frac{x}{s}\right).\frac{x}{s} = \frac{x}{s} - \frac{\alpha(x)}{s}.\frac{x}{s} = \frac{x}{s} - \frac{\alpha(x)x}{s^2} = \frac{s(x-\alpha(x)x)}{s}\ \in\ F_S$, this implies that $\frac{x}{s}$ is an F_S-semiregular. Hence x is locally $F-$semiregular. □

Theorem 3.1. *Suppose that there exists $\lambda\ \in\ M^*$ such that $\lambda(m) = e = e^2$ and $(1-e)m$ is a locally $F-$semiregular with respect to $s = s^2\ \in\ S^2$, then m is a locally $F-$semiregular.*

Proof. Since $(1-e)m$ is a locally $F-$semiregular, then there exists $s\ \in\ S$, such that $\frac{(1-e)m}{s}$ is F_S-semiregular in M_S . Therefore, there exists $\beta\ \in\ M_S^*$ such that $f = \beta\left(\frac{(1-e)m}{s}\right)$ is an idempotent and $\left(\frac{s}{s} - f\right)\frac{(1-e)m}{s}\ \in\ F_S$ and $\frac{e}{s}f = \frac{e}{s}\beta\left(\frac{(1-e)m}{s}\right) = \frac{e}{s}.\frac{1-e}{s}\beta\left(\frac{ms}{s}\right) = \frac{e-e^2}{s^2}\beta\left(\frac{ms}{s}\right) = \frac{0}{s}$. So, $\frac{e}{s}f = \frac{0}{s}$. Now, $g^2 = \left(\frac{e}{s} + f - f\frac{e}{s}\right)^2 = \frac{e}{s} + f - f\frac{e}{s} = g$ and $\left(\frac{s}{s} - \frac{e}{s}\right)\frac{m}{s} = \frac{(1-e)m}{s}$, then $\left(\frac{s}{s} - g\right)\frac{m}{s} = \left(\frac{s}{s} - \frac{e}{s} - f + f\frac{e}{s}\right)\frac{m}{s} = \left(\frac{s}{s} - f\right)\left(\frac{(1-e)m}{s}\right)\ \in\ F_S$. So, $\left(\frac{s}{s} - g\right)\frac{m}{s}\ \in\ F_S$. Let $\alpha = \lambda_S + \beta\left(\frac{m}{s}\right)\left(\frac{s}{s} - \frac{e}{s}\right)(\beta - \lambda_S)$, where $\lambda_S\left(\frac{m}{s}\right) = \frac{\lambda(m)}{s}$ for all $m\ \in\ M$. Then $\alpha\left(\frac{m}{s}\right) = g$. Thus, $\alpha\ \in\ M_S^*$ and $\left(\alpha\left(\frac{m}{s}\right)\right)^2 = \alpha\left(\frac{m}{s}\right)$ and $\left(\frac{s}{s} - \alpha\left(\frac{m}{s}\right)\right)\frac{m}{s}\ \in\ F_S$. Hence $\frac{m}{s}$ is an F_S-semiregular. □

Proposition 3.2. *Suppose that $M = N\ \oplus\ K$ is an $R-$module, where N , K and F are submodules of M and S is a multiplicative system in R. If $m = n + k$, where $n\ \in\ F$ and k is a locally $F\ \cap\ K-$semiregular element in K, then m is a locally $F-$semiregular element in M.*

Proof. Let k be a locally $F\ \cap\ K-$semiregular in K. Then there exists $s\ \in\ S$ such that $\frac{k}{s}$ is $F_S\ \cap\ K_S-$semiregular. Therefore, there exists $\lambda : K_S \to R_S$

such that $\lambda\left(\frac{k}{s}\right)=e=e^2$ and $\left(\frac{s}{s}-e\right)\frac{k}{s} \in F_S \cap K_S$. Define $\alpha : M_S \to R_S$ by $\alpha\left(N_S\right)=0$, then $\frac{m}{s}=\lambda\left(\frac{k}{s}\right)=e$ and $\left(\left(\frac{s}{s}-e\right)\frac{m}{s}=\left(\frac{s}{s}-e\right)\frac{n}{s}+\left(\frac{s}{s}-e\right)\frac{k}{s}\right) \in F_S$. So, $\left(\frac{s}{s}-e\right)\frac{m}{s} \in F_S$. Hence m is a locally $F-$semiregular element. □

Theorem 3.2. *If $M = N \oplus K$ is an $R-$module, where N, K and F are submodules of M and S is a multiplicative system in R, then n is a locally $F \cap N-$semiregular in N if and only if n is a locally $F-$semiregular element in M.*

Proof. Suppose that n is a locally $F \cap N-$semiregular element in N, then there exists $s \in S$ such that $\frac{n}{s}$ is an $F_S \cap N_S-$semiregular. We chose $\lambda : N_S \to R_S$, such that $\lambda\left(\frac{n}{s}\right)=e=e^2$ and $\left(\frac{s}{s}-e\right)\frac{n}{s} \in F_S \cap N_S$. Define $\alpha : M_S \to R_S$ by $\alpha\left(\frac{n}{s}+\frac{k}{t}\right)=\lambda\left(\frac{n}{s}\right)$, then $\alpha\left(\frac{n}{s}\right)=\lambda\left(\frac{n}{s}\right)=e$ and $\left(\frac{s}{s}-e\right)\frac{n}{s} \in F_S$, so $\frac{n}{s}$ is an F_S-semiregular element in M_S.

Conversely, let $n \in N$ be locally $F-$semiregular element in M. Then there exists $s \in S$ such that $\frac{n}{s}$ is an F_S-semiregular element in M_S. We chose $\gamma : M_S \to R_S$ such that $\gamma\left(\frac{n}{s}\right)=e=e^2$ and $\left(\frac{s}{s}-e\right)\frac{n}{s} \in F_S$. Define $\lambda=\gamma|_{N_S} : N_S \to R_S$, then $\lambda\left(\frac{n}{s}\right)=\gamma\left(\frac{n}{s}\right)=e$ and since $\frac{n}{s} \in N_S$, therefore $\left(\frac{s}{s}-e\right)\frac{n}{s} \in \left(F_S \cap N_S\right)$. So, $\frac{n}{s}$ is $F_S \cap N_S-$semiregular in N_S. Hence n is a locally $F \cap N-$semiregular element in N. □

Proposition 3.3. *The following are equivalent, for $x \in M$:*

1. *x is a locally $F-$semiregular with respect to S.*
2. *there exists $\alpha \in M_S^*$ and $s \in S$ such that $\left(\alpha\left(\frac{x}{s}\right)\right)^2=\alpha\left(\frac{x}{s}\right)$ and $\left(\frac{x}{s}-\left(\alpha\left(\frac{x}{s}\right)\right)\frac{x}{s}\right) \in F_S$.*
3. *there exists a homomorphism $\gamma : M_S \to (Rx)_S$ such that $\gamma^2=\gamma$; $\gamma\left(M_S\right)$ is projective and $\left(\frac{x}{s}-\gamma\left(\frac{x}{s}\right)\right) \in F_S$.*

Proof. Straight forward to [1, Proposition 2.2]. □

Theorem 3.3. *An element $x \in M$ is a locally $F-$semiregular with respect to S if and only if there exists a regular element $y \in (Rx)_S$ and $s \in S$ such that $\left(\frac{x}{s}-y\right) \in F_S$ and $(Rx)_S=R_S y \oplus R_S\left(\frac{x}{s}-y\right)$.*

Proof. It is similar to [1, Proposition 2.2]. □

Theorem 3.4. *Let F be a submodule of M such that F_S is a fully invariant in M_S, then the following are equivalent:*

1. *M is a locally $F-$semiregular with respect to S.*
2. *For every finitely generated submodule N_S of M_S there exists a homomorphism $\gamma : M_S \to N_S$ such that $\gamma^2=\gamma$, $\gamma\left(M_S\right)$ is a projective $\left(I_S-\gamma\right)\left(N_S\right) \subseteq F_S$.*

3. *For any finitely generated submodule N_S of M_S, there exists a decomposition $M_S = A_S \oplus B_S$ such that A_S is a projective submodule of N_S and $(N \cap B)_S \subseteq F_S$.*

4. *For any finitely generated submodule N_S of M_S, then N_S can be written as $N_S = A_S \oplus K_S$, where A_S is a projective direct summand of M_S and $K_S \subseteq F_S$.*

Proof. It is similar to [1, Proposition 2.3]. □

Theorem 3.5. *If M is a locally $F-$semiregular module with respect to S, then every finitely generated submodule of $\left(\frac{M}{F}\right)_S$ is a direct summand.*

Proof. Let $\left(\frac{A}{F}\right)_S \subseteq \left(\frac{M}{F}\right)_S$ be a finitely generated submodule. Then there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$. Therefore, there exists a decomposition $M_S = C_S \oplus D_S$ where C_S and D_S are submodules of M_S such that $C_S \subseteq N_S$ is projective and $(N \cap D)_S \subseteq F_S$. Now, $N_S = C_S + (N \cap D)_S$ and $\left(\frac{A}{F}\right)_S = \left(\frac{N+F}{F}\right)_S = \left(\frac{C+F}{F}\right)_S$. Since $F_S = (F \cap D)_S \oplus (F \cap C)_S$ and $(D+F)_S \cap (C+F)_S = (D+(F \cap C))_S \cap (C+(F \cap D))_S = F_S$. Then $\left(\frac{C+F}{F}\right)_S \oplus \left(\frac{D+F}{F}\right)_S = \left(\frac{M}{F}\right)_S$. Hence $\left(\frac{A}{F}\right)_S$ is a direct summand of $\left(\frac{M}{F}\right)_S$. □

Corollary 3.1. *If M is a locally $F-$semiregular module with respect to S, then $\frac{M}{F}$ is a locally regular module with respect to S.*

Theorem 3.6. *If M is a locally $F-$semiregular with respect to S, then direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.*

Proof. Let $\left(\frac{M}{F}\right)_S = \left(\frac{A}{F}\right)_S \oplus \left(\frac{B}{F}\right)_S$, where A, B are submodules of M and $\left(\frac{A}{F}\right)_S$ is a finitely generated submodule of $\left(\frac{M}{F}\right)_S$. Then there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$. Therefore, there exists a decomposition $M_S = C_S \oplus D_S$ such that $C_S \subseteq N_S$ is projective and $(N \cap D)_S \subseteq F_S$. Then $M_S = C_S + B_S$, so by [5, Lemma 1.16], we can write $M_S = C_S \oplus Q_S$, where $Q_S \subseteq B_S$, then $\left(\frac{C+F}{F}\right)_S = \left(\frac{A}{F}\right)_S$ and $\left(\frac{Q+F}{F}\right)_S = \left(\frac{B}{F}\right)_S$. □

Theorem 3.7. *Let M be a projective $R-$module an S a multiplicative closed set in R. If F is a submodule of M such that F_S is fully invariant, then M is locally $F-$semiregular with respect to S if and only if*

1. *$\frac{M}{F}$ is a locally regular with respect to S.*

2. *Direct summands of $\left(\frac{M}{F}\right)_S$ can be lifted to M_S.*

Proof. Let M be a locally $F-$semiregular with respect to S. Then by Corollary 3.1 and Theorem 3.6, (1) and (2) are hold respectively.

Conversely, suppose that (1) and (2) are hold and N_S is a finitely generated submodule of M_S, then $\left(\frac{N+F}{F}\right)_S$ is finitely generated submodule of $\left(\frac{M}{F}\right)_S$, so there exists a submodule B_S of M_S such that $F_S \subseteq B_S$ and $\left(\frac{N}{F}\right)_S = \left(\frac{N+F}{F}\right)_S \oplus \left(\frac{B}{F}\right)_S$, therefore by (2), we get $M_S = P_S \oplus Q_S$ with $P_S \subseteq N_S$, $\left(\frac{P+F}{F}\right)_S = \left(\frac{N+F}{F}\right)_S$ and $\left(\frac{B}{F}\right)_S = \left(\frac{Q+F}{F}\right)_S$. It is easy to see that $N_S = P_S + (N \cap F)_S$ and $F_S = (F \cap P)_S \oplus (F \cap Q)_S$, then $(Q \cap N)_S \subseteq F_S$. Hence M is a locally $F-$semiregular with respect to S. $\square$

Proposition 3.4. *If M is a locally $F-$semiregular module with respect to S, then M_S is an F_S semipotent module.*

Proof. Let A_S be a submodule of M_S with $A_S \nsubseteq F_S$ and $\frac{a}{s} \in A_S \backslash F_S$, then a is a locally $F-$semiregular in M, so $M_S = X_S \oplus Y_S$, where $X_S \subseteq (Ra)_S$ is projective and $Y_S \cap (Ra)_S \subseteq F_S$. Now, $(Ra)_S = X_S \oplus (Y \cap Ra)_S$. Therefore, $X_S \nsubseteq F_S$. Hence M_S is F_S-semipotent. $\square$

Theorem 3.8. *If M_S is a finitely generated projective and locally $F-$semiregular module with respect to S, then M_S is a F_S-potent module.*

Proof. By Proposition 3.4, M_S is F_S-semipotent, it remains to show F_S is a strongly lifting submodule of M_S. Since M_S is a finitely generated and projective, then M_S is Noetherian. Therefore, $\left(\frac{M}{F}\right)_S$ is Noetherian, this implues that every submodule of $\left(\frac{M}{F}\right)_S$ is finitely generated. If $M_S = \left(\frac{A+F}{F}\right)_S \oplus \left(\frac{B+F}{F}\right)_S$, then $\left(\frac{A+F}{F}\right)_S$ is finitely generated and there exists a finitely generated submodule N_S of M_S such that $\left(\frac{N}{F}\right)_S = \left(\frac{A+F}{F}\right)_S$, therefore $M_S = C_S \oplus D_S$, where $C_S \subseteq N_S$ and $(D \cap N)_S \subseteq F_S$. Since $N_S = C_S \oplus (D \cap N)_S$, $M_S = (A+F)_S + B_S$, C_S is a summand of M_S and M_S is projective, then there exists a direct summand Q_S of M_S such that $M_S = C_S \oplus Q_S$ and $Q_S \subseteq F_S + B_S$. It is easy to see that $C_S \subseteq A_S$. So, $\left(\frac{C+F}{F}\right)_S = \left(\frac{A+F}{F}\right)_S$ and $\left(\frac{Q+F}{F}\right)_S = \left(\frac{B+F}{F}\right)_S$. Therefore, F_S is strongly lifting. $\square$

Proposition 3.5. *If M is an $F-$semiregular module, then M is a locally $F-$semir-egular module with respect to any multiplicative closed sets in R.*

Proof. Let S be a multiplicative closed set in R and $x' \in M_S$. Then there exists $x \in M$ and $s \in S$ such that $x' = \frac{x}{s}$, $M = P \oplus Q$, $P \subseteq Rx$ is projective and $Rx \cap Q \subseteq F$. Therefore, $M_S = P_S \oplus Q_S$, $P_S \subseteq (Rx)_S = R_S x'$ is projective and $R_S x' \cap Q_S = (Rx \cap Q)_S \subseteq F_S$. Thus x' is F_S-semiregular. Hence M is a locally $F-$semiregular with respect to S. $\square$

The converse of the above proposition is not true in general. For example, consider Z_{12} as $Z_{12}-$module and $F = 0$, then it is easy to see that Z_{12} is a locally $F-$semiregular, but it is not $F-$semiregular. Now, we investigate some conditions for which the converse will be true.

Theorem 3.9. *Suppose that P is a maximal ideal in R, then a projective $R-$module M is $F-$semiregular if and only if it is locally $F-$semiregular with respect to $R\backslash P$.*

Proof. If M is $F-$semiregular, then M is a locally $F-$semiregular with respect to $R\backslash P$ by Proposition 3.5.

Conversely, suppose that M is locally $F-$semiregular with respect to $R\backslash P$ and N is a finitely generated submodule of M, then N_P is a finitely generated submodule of M_P and $N_P = A_P \oplus S_P$, where A_P is a projective direct summand of M_P and $S_P \subseteq F_P$. So, A is a projective direct summand of M and $S \subseteq F$. Hence M is $F-$semiregular. □

Theorem 3.10. *If every elements of S are unit, then M is an $F-$semiregular if and only if it is a locally $F-$semiregular with respect to S.*

Proof. By (Propositions 3.5 and 3.9). □

If R is a commutative ring with identity and P is a prime ideal of R, then the localization of R at P is local ring. In the following propositions we generalize this idea for some special classes of modules, before this we need the following theorem, which is mentioned in [3].

Theorem 3.11. *If M is a multiplication $R-$module and S is a multiplicative closed set in R, then M_S is a multiplication R_S-module.*

Proof. It is obvious. □

Theorem 3.12. *Suppose that P is a prime ideal in R and M is a multiplication $R-$module, then M_P is local as an R_P-module.*

Proof. Let M be a multiplication $R-$module. Then by Proposition 3.5, M_P is a multiplication R_P-module and every proper submodules of M_P contained in a maximal submodule. It remains to show that M_P has unique maximal submodule, if N_P and L_P are two maximal submodules of M_P, then $N_P = I_P M_P$ and $L_P = J_P M_P$, where I_P and J_P are maximal ideals in R_P, this implies that $I_P = J_P$, therefore $N_P = L_P$. Hence M_P is local. □

Corollary 3.2. *If M is a regular $R-$module and $End(M)$ is a commutative, then M_P is a local R_P-module, for every prime ideal P of R.*

Proposition 3.6. *If M is an $F-$semiregular and $End\left(\left(\frac{M}{F}\right)_P\right)$ is commutative, then $\left(\frac{M}{F}\right)_P$ is a local as an R_P-module, for any prime ideal P of R.*

Proof. Follows from corollary 3.2 and [4, Proposition 2.2]. □

Proposition 3.7. *Every semisimple projective R_P-module M is a locally F-semi-regular.*

Proof. Let $(Rx)_P$ be a cyclic submodule of M_P. Since M_P is semisimple, then $M_P = (Rx)_P \oplus B_P$ for some submodule $B_P$of M_P. Now, M_P is a projective module, therefore $(Rx)_P$ is also projective and $B_P \cap (Rx)_P = \{0\} \subseteq F_P$. Hence M_P is an F_P-semiregular module. □

Proposition 3.8. *Let M be a locally $F-$semiregular module and K_P a submodule of M_P such that $F_P \subseteq K_P$. Then K is a locally F_P-semiregular module.*

Proof. Let $(Rx)_P$ be a cyclic submodule of K_P, then $(Rx)_P$ is a finitely generated submodule of M_P, since M_P is semiregular, so there exists a decomposition $M_P = A_P \oplus B_P$, where A_P is a projective submodule of $(Rx)_P$ and $(Rx \cap B)_P \subseteq F_P$. We have $K_P = A_P \oplus (B \cap K)_P$ and $(Rx \cap B \cap K)_P = (Rx \cap B)_P \subseteq F_P$, therefore K_P is an F_P-semiregular. Hence K is a locally $F-$semiregular. □

According to the above proposition, if M is a locally $F-$semiregular module and F, K are submodules of M with $F \subseteq F_1$, then M is a locally F_1-semiregular and K is locally $(F \cap K)-$semiregular module.

Proposition 3.9. *Let $M = \oplus_{i \in I} M_i$ be a directsum of submodules of M_i. If M is a locally $F-$semiregular, then each M_i is a locally F_i-semiregular, where F is a submodule of M and $F_i = F \cap M_i$ for all $i \in I$.*

Proof. Let $x \in M_i$ for some $i \in I$. Since M is $F-$semiregular, then there exists a projective submodule A_P of $(Rx)_P$ such that $M_P = A_P \oplus B_P$ and $(Rx \cap)_P \subseteq F_P$ for some submodule B of M. Now, $(M_i)_P = A_P \oplus (B \cap M_i)_P$ and $(Rx)_P \cap (B \cap M_i)_P \subseteq (Rx)_P \cap B_P \subseteq (F \cap M_i)_P = (F_i)_P$. Therefore, $(M_i)_P$ is $(F_i)_P-$semiregular. Hence, M_i is a locally F_i-semiregular module. □

Proposition 3.10. *Let M_1 and M_2 be two $R-$modules such that $Ann\,(M_1) + Ann\,(M_2) = R$ and $M = M_1 \oplus M_2$. If M_1 is a locally F_1-semiregular and M_2 is a locally F_2-semirgular, then M is a locally $F_1 \oplus F_2-$semiregular.*

Proof. Let N be a finitely generated submodule of M. Since $Ann\,(M_1) + Ann\,(M_2) = R$, then $N = N_1 \oplus N_2$, where $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$. Since N is a finitely generated submodule, N_1 and N_2 are direct summands of N, then N_1 and N_2 are finitely generated, so there exists a decomposition $(M_i)_P = (A_i)_P \oplus (B_i)_P$, such that $(A_i)_P \subseteq N_i$ is projective and $(B_i \cap N_i)_P \subseteq (F_i)_P$. Now, $M_P = (A_1)_P \oplus (A_2)_P \oplus (B_1)_P \oplus (B_2)_P$, where $(A_1)_P \oplus (A_2)_P \subseteq (F_1)_P \oplus (F_2)_P$ is projective and $((B_1 \oplus B_2) \cap N)_P = (B_1 \cap N_1)_P \oplus (B_2 \cap N_2)_P \subseteq ((F_1)_P \oplus (F_2)_P)$. Hence M is a locally $F_1 \oplus F_2-$semiregular module. □

Theorem 3.13. *Let F_P be a fully invariant submodule of M_P. If M is a locally $F-$semiregular module, then*

1. *For all* $\frac{x}{p} \in M_P$*, there exists a regular element* $\frac{y}{t} \in M_P$ *such that* $(\frac{x}{p} - \frac{y}{t}) \in F_P$.
2. *Every submodule of* M_P *that is not contained in* F_P *contains a regular element not in* F_P.
3. $Rad\,(M_P) \subseteq F_P$ *and* $Z\,(M_P) \subseteq F_P$.

Proof.

1. Let $\frac{x}{p} \in M_P$. Since M is $F-$semiregular, then there exists $\alpha_P \in M_P^*$ such that $\left(\alpha_P\left(\frac{x}{p}\right)\right)^2 = \alpha_P\left(\frac{x}{p}\right)$ and $\left(\frac{x}{p} - \alpha_P\left(\frac{x}{p}\right)\frac{x}{p}\right) \in F_P$. If $\frac{y}{t} = \alpha_P\left(\frac{x}{p}\right)\frac{x}{p}$ then it is easy to show that $\frac{y}{t}$ is regular element and $(\left(\frac{x}{p} - \frac{y}{t}\right)) \in F_P$.
2. Assume that N_P is a submodule of M_P not contained in F_P, $\frac{x}{p} \in N_P$ and $\frac{x}{p} \notin F_P$. Since M_P is F_P-semiregular, then there exists a regular element $\frac{y}{t} \in (Rx)_P$ such that $(\frac{x}{p} - \frac{y}{t} \in F_P$. If $\frac{y}{t} \in F_P$, then $\frac{x}{p} \in F_P$ which is a contradiction. Thus $\frac{y}{p} \notin F_P$.
3. Let $\frac{x}{p} \in Rad\,(M_P)$. Since M_P is F_P-semiregular, then $(Rx)_P = A_P \oplus S_P$, where A_P is a projective direct summand of M_P and $S_P \subseteq F_P$. Since $(Rx)_P$ is a small in M_P and A_P is a direct summand of M_P, therefore $A_P = 0$. So, $(Rx)_P = S_P \subseteq F_P$, this implies that $Rad\,(M_P) \subseteq F_P$. The second part, can be proved in similar manner.

□

4. Locally $\delta-$semiregular modules

In this section we localized the concept of δ-small and investigate some properties of locally δ-small submodule. Also, we introduce the notion of locally δ-semiregular and locally δ-supplemented modules as the localizations for δ-semiregular δ-supplemented.

Definition 4.1. *A submodule* N *of* M *is said to be locally* δ*-small in* M *if* N_P *is* δ*-small submodule in* M_P *for every maximal ideal* P *of* R.

Every δ-small submodule of M is locally δ-small submodule. If N is any δ-small submodule of M and P is a maximal ideal of R. Suppose that $N_P + L_P = M_P$, then $N + L = M$, since L is δ-submodule of M, then there exists a projective semisimple submodule $Y \subseteq N$ such that $Y \oplus L = M$, $Y_P \subseteq M_P$ is projective semisimple submodule and $Y_P \oplus L_P = M_P$, so N is locally δ-small submodule. But the converse is not true in general. For example, consider that Z_6 as a $Z-$module, $N = \{0,\ 3\}$ and $K = \{0,\ 2,\ 4\}$, then $M = N + K$ and $\frac{M}{K}$ is singular. So, N is not δ-small. If P is a maximal ideal in Z, then

$P = \langle p\rangle$, where p is a prime number, then $p = 2$ or $p = 2n + 1$, $n \in N$. If $P = \langle 2\rangle = \{\dots, -2, -1, 0, 1, 2, \dots\}$, then $Z - P = \{2n+1 : n \in Z\}$. Therefore, $\frac{3}{2n+1} = \frac{3}{2m+1}$, since $1.(6n + 3 - 6m - 3) = 0$ and $\frac{3}{1} = \frac{3}{2n+1}$. Thus, $N_P = \{\frac{0}{1}, \frac{3}{1}\}$. Also, $3 \in (Z - P)$, then $\frac{0}{1} = \frac{2}{1} = \frac{2}{m} = \frac{4}{m} = \frac{4}{1}$, for all $m \in Z$, so $K_P = \{0\}$. If $P = \langle p\rangle$, where p is a prime, then $2 \in (Z - P)$, $\frac{3}{1} = \frac{0}{1} = \frac{0}{m} = \frac{3}{m}$, therefore $N_P = \{0\}$. In each case we conclude that N_P is δ-small.

Theorem 4.1. *Let N be a submodule of M. Then the following are equivalent:*

1. *N is a locally δ-small.*
2. *If $X_P + N_P = M_P$, then there exists a projective semisimple submodule Y_P of N_P such that $X_P \oplus Y_P = M_P$ for every maximal ideal P of R.*
3. *$X_P + N_P = M_P$ with $\left(\frac{M}{X}\right)_P$ is Goldie torsion, then $X_P = M_P$ for every maximal ideal P of R.*

Proof. Similar to [11, Lemma 2.1]. □

Lemma 4.1. *If A, B and K are submodules of M such that $A \subseteq B$, $M = A+K$ and $B \cap K$ is a locally δ-small in M, then $\frac{B}{A}$ is a locally δ-small in $\frac{M}{A}$.*

Proof. Follows from Theorem 4.1. □

Theorem 4.2. *Let M be an $R-$module. Then*

1. *For submodules N, K and L with $K \subseteq N$, then we have:*
2. *N is a locally δ-small in M if and only if K is a locally δ-small in M and $\frac{N}{K}$ is a locally δ-small in $\frac{M}{K}$.*
3. *$N + L$ is a locally δ-small in M if and only if N and L are locally δ-small in M.*
4. *If K is a locally δ-small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is a locally δ-small in N. In particular, if K is locally δ-mall in M, then K is a locally δ-small in N.*
5. *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2$ is a locally δ-small in $M_1 \oplus M_2$ if and only if K_1 is a locally δ-small in M_1 and K_2 is a locally δ-small in M_2.*

Proof. 1. (a) Assume that N is a locally $\delta-$small, this implies that $N_P \ll_\delta M_P$ for every maximal ideal P of R. We have $K_P \subseteq M_P$ and $N_P \ll_\delta M_P$. Hence $K_P \ll_\delta M_P$ by [11, Lemma 2.1]. Since $N_P \ll_\delta M_P$, $\frac{N_P}{K_P} \ll_\delta \frac{M_P}{K_P}$, therefore $\frac{N}{K}$ is a locally $\delta-$small.

Conversely, suppose that K is a locally $\delta-$small in M and $\frac{N}{K}$ is a locally $\delta-$small in $\frac{M}{K}$, since K is a locally $\delta-$small in M, this implies that

$K_P \ll_\delta M_P$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$, then $\frac{N_P}{K_P} \ll_\delta \frac{M_P}{K_P}$, hence $N_P \ll_\delta K_P$.

(b) Let $N+L$ be a locally $\delta-$small. Then $(N+L)_P \ll_\delta M_P$ and we have $(N+L)_P = N_P + L_P$, this implies that $N_P \ll_\delta M_P$ and $L_P \ll_\delta M_P$. For the converse direction the proof is similar.

2. Let K be a locally δ-small in M. Then $K_P \ll_\delta M_P$. Since $f : M \to N$ is a homomorphism, then $f_P : M_P \to N_P$ is a homomorphism, so $f_P(K_P) \ll_\delta N_P$.

3. Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is locally δ-small in $M_1 \oplus M_2$, this implies that $(K_1 \oplus K_2)_P \ll_\delta (M_1 \oplus M_2)_P$. Therefore, $K_{1P} \ll_\delta M_{1P}$ and $K_{2P} \ll_\delta M_{2P}$. Hence K_1 and K_2 are locally δ-small in M. The converse part is obvious.

□

Recall that if ϕ is that class of all singular simple modules. For a module M let $\delta(M) = \cap \left\{N \subseteq M;\ \frac{M}{N} \in \phi\right\} = \sum\{L \subseteq M; L \ll_\delta M\}$.

Proposition 4.1. *Let M be an $R-$module. Then $\delta(M_P) = (\delta(M))_P$ for every maximal ideal of R.*

Proof. It is obvious. □

Proposition 4.2. *Let A, B and C be submodules of an $R-$module M. If $M = A+B$, $B \subseteq C$ and $\frac{C}{B}$ is a locally δ-small in $\frac{M}{B}$, then $\frac{A \cap C}{A \cap B}$ is a locally δ-small in $\frac{M}{A \cap B}$.*

Proof. It is follows from **Theorem 4.2**. □

Theorem 4.3. *Let M be a module and $m \in M$. Then the following are equivalent:*

1. *mR is not locally δ-small in M.*
2. *There is a maximal submodule N_P of M_P such that $\frac{m}{s} \notin N_P$ and $\frac{M_P}{N_P}$ is singular, for every maximal ideal P in R.*

Proof. $(1 \Rightarrow 2)$ Suppose that mR is not locally δ-small in M, then it is not δ-small. Therefore, by [11, Lemma 2.2], there exists a maximal submodule N of M such that $m \notin N$, $\frac{M}{N}$ is a simple module, this implies that $Z\left(\frac{M}{N}\right) = \frac{M}{N}$. Then $\frac{M}{N}$ is singular. So, N_P is a maximal submodule of M_P such that $\frac{m}{s} \notin N_P$ and $\frac{M_P}{N_P}$ is singular for every maximal submodule P of R.

$(2 \Rightarrow 1)$ Suppose that there exists a maximal submodule N_P of M_P such that $\frac{m}{s} \notin N_P$ with $\frac{M_P}{N_P}$ is singular, then $\frac{m}{s}R_P + N_P = M_P$. Hence $\frac{m}{s}R_P$ is not δ-small in M_P, that is, mR is not locally δ-small in M. □

Theorem 4.4. *Let M be a module, K is a direct summand of M and $N \subseteq K$. Then N is locally δ-small in M if and only if N is a locally δ-small in K.*

Proof. Assume that N is a locally δ-small in K, then $N_P \ll_\delta K_P$ for every maximal ideal P in R, then $N_P \ll_\delta M_P$.

Conversely, suppose that $N_P \ll_\delta M_P$ and $X_P + N_P = K_P$, then $M_P = K_P \oplus Y_P = (X_P + N_P) \oplus Y_P = N_P + (X_P + Y_P)$. Therefore, there exists a projective semisimple submodule Z_P of N_P such that $M_P = Z_P \oplus (X_P + Y_P)$, then $K_P = (K \cap Z)_P \oplus X_P$ and $(K \cap Z)_P \subseteq N_P$ is projective semisimple, so N is a locally δ-small in K. □

Proposition 4.3. *Suppose that M is a module K, L and H are submodules of M. If L is a locally δ-small, then L is a locally δ-small in $K + H$.*

Proof. Obvious by Theorem 4.3. □

Definition 4.2. *M is called locally δ-semiregular if for every finitely generated submodule N of M, there is a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $N \cap B$ is locally $\delta-$small in B.*

Every regular, semiregular and δ-semiregular modules are locally δ-semiregular. But the converse is not true, in general (See example 3.2). If M is a projective module, then the converse is also true.

Theorem 4.5. *Let M be a module. Then the following are equivalent:*

1. *M is a locally δ-semiregular.*
2. *For every cyclic submodule N of M there exists a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $N \cap B$ is a locally δ-small in B.*
3. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ is projective and $(1-e)(N)$ is a locally δ-small in M.*

Proof. $(1 \Rightarrow 2)$ is obvious.
$(2 \Rightarrow 1)$ Let N be a finitely generated submodule of M with generators $x_1, \ldots, x_n$. If $n = 1$, then the result is hold, suppose that the result is hold, for every $(n-1)-$generated submodule of M. Since N is $n-$generated submodule of M. Then there exists an $(n-1)-$generated submodule A and a cyclic submodule B of M such that $N = A + B$. By our hypothesis there exists a decomposition $M = P_1 \oplus Q_1$ such that $P_1 \subseteq A$ is projective and $A \cap Q_1$ is locally δ-small in M. Let $h : M \to Q_1$ be the projection, since $P_1 \subseteq A \subseteq N$, then $N = P_1 \oplus h(N) = P_1 \oplus (h(A) + h(B))$ and $A = P_1 \oplus h(A)$, where $h(N) = N \cap Q_1$ and $h(A) = A \cap Q_1$. Since $h(B)$ is cyclic submodule of M, then there exists a decomposition $M = P_2 \oplus Q_2$ such that

$P_2 \subseteq h(B) \subseteq h(N) \subseteq Q$ and $h(B) \cap Q_2$ is a locally δ-small in M. So, $Q_1 = P_2 \oplus Q$, where $Q \subseteq Q_2$ and $h(B) \cap Q$ is a locally δ-small in M. Suppose that $P = P_1 \oplus P_2$, then $M = P \oplus Q$ and $P \subseteq N$ is projective. Now, to show $N \cap Q$ is a locally δ-small in M, let $\pi : Q_1 \to Q$ be the projection. Since $h(A)$ is a locally δ-small in M, then $\pi(h(A))$ is a locally δ-small in M therefore, $\pi(h(A)) + (h(B) \cap Q)$ is a locally δ-small in M. Since $P_2 \subseteq h(B)$, then $h(A)+h(B) = \pi(h(A))+P_2+(h(B) \cap Q)$. Now, $N \cap Q = h(N) \cap Q = (h(A) + h(B)) \cap Q = \pi(h(A))+P_2 \cap Q+h(B) \cap Q = \pi(h(A))+(h(B) \cap Q)$, which is a locally δ-small in M.

$(1 \Rightarrow 3)$ and $(3 \Rightarrow 1)$ are Obvious. □

According to Theorem 4.3, if M is an $R-$module, then an element $\frac{m}{s}$ of M_P is belonging to $\delta(M_P)$ if and only if $\frac{m}{s} R_P$ is a δ-small submodule of M, that is, $\delta(M_P) = \left\{\frac{m}{s} \in M_P; R_P\frac{m}{s} \ll_\delta M_P\right\}$. It can be seen that [8, Corollary 2.4] stated that a module M with $\delta(M) \ll_\delta M$ is a δ-semiregular module if and only if for every finitely generated submodule N of M, there exists an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ and $(1-e)(N) \subseteq \delta(M)$.

In the following proposition we provide the result for a module and its localization without using the condition $\delta(M) \ll_\delta M$ and $\delta(M)$ is a locally δ-small submodule in M.

Theorem 4.6. *For an $R-$module M, the following are equivalent:*

1. *M is a δ-semiregular module.*
2. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M)$ such that $e(M) \subseteq N$ is projective and $(1-e)(N) \subseteq \delta(M)$.*

Proof. It is obvious by Theorem 4.5. □

Theorem 4.7. *For an $R-$module M, the following are equivalent:*

1. *M is a Locally δ-semiregular module.*
2. *For every finitely generated submodule N of M, there is an idempotent $e \in End(M_P)$ such that $e(M_P) \subseteq N_P$ is projective and $(1-e)(N_P) \subseteq \delta(M_P)$, for every maximal ideal P in R.*

Proof. It is obvious by Theorem 4.5. □

Theorem 4.8. *If M is a locally δ-semiregular module, then M_P is δ-simiregular, for every maximal ideal P of R.*

Proof. Let N' be a cyclic submodule of M_P, then there exists a cyclic submodule N of M such that $N' = N_P$. Therefore, there exists a decomposition $M = A \oplus B$ such that $A \subseteq N$ is projective and $(N \cap B)_P \ll_\delta M_P$, then $M_P = A_P \oplus B_P$ and $A_P \subseteq N'$ is projective and $N' \cap B_P \ll_\delta M_P$, so M_P is δ-semiregular. □

Corollary 4.1. *If M is a locally δ-semiregular module with $End\left(\frac{M}{\delta(M)}\right)$ is commutative, then $\frac{M_P}{\delta(M_P)}$ is local module over R_P.*

For a projective module the converse of the above theorem is true.

Theorem 4.9. *Suppose that M is a projective module with $\delta(M)$ is locally δ-small submodule in M. Then M is locally δ-semiregular if and only if $\frac{M_P}{\delta(M_P)}$ is regular and direct summands of $\frac{M_P}{\delta(M_P)}$ can be lifted to M_P.*

Proof. Clear by [5, Proposition 1.7] □

Definition 4.3. *A submodule N is called locally δ-supplemented in M if there exists a submodule L in M such that $M = N + L$ and $N \cap L$ is locally δ-small in L. A module M is called locally δ-supplemented if every submodule of M has a locally δ-supplement. Also M is said to be locally finitely δ-supplemented if every finitely generated submodule has a locally δ-supplemented.*

Lemma 4.2. *Every locally δ-semiregular module is a locally finitely δ-supplemented module.*

Proof. Let M be a locally δ-semiregular and U be a finitely generated submodule. Then there exists a decomposition $M = A \oplus B$, $A \subseteq U$ is projective and $U \cap B$ is a locally δ-small in B, therefore, $M = U + B$ and $U \cap B$ is a locally δ-small in B, so M is a locally finitely δ-supplemented. □

Definition 4.4. *A pair (T, π) is called a locally projective δ-cover of a module M, if T is projective and $\pi : T \to M$ is an epimorphism with $ker\pi$ is locally δ-small in P.*

Proposition 4.4. *Suppose that $M = M_1 \oplus M_2 \ldots \oplus M_n$ such that all $\pi_i : T_i \to M_i$, for $i = 1, \ldots, n$ are locally projective δ-covers. If $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n$, then $\pi = \oplus_{i=1}^{n} \pi_i : T \to M$ is a locally projective δ-cover.*

Proof. Since T_i is projective module, for $i = 1, \ldots, n$, then $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ is also projective module and every π_i is an epimorphism, so $\pi = \oplus_{i=1}^{n} \pi_i$ is an epimorphism. Now, $ker\pi = \oplus_{i=1}^{n} ker\pi_i$ and $ker\pi_i$ is a locally δ-small, for $i = 1, \ldots, n$, then $ker\pi$ is δ-small. Thus, (T, π) is a locally projective δ-cover of M. □

Theorem 4.10. *Let $\pi : T \to M$ be a locally projective δ-cover. If Q is projective and $q : Q \to M$ is an epimorphism, then there exists decompositions $T = A \oplus B$ and $Q = X \oplus Y$ such that:*

1. *$A \cong X$.*
2. *$\pi|_A : A \to M$ is a locally projective δ-cover.*
3. *$q|_X : X \to M$ is a locally projective δ-cover.*

4. B is a projective semisimple module, with $B \subseteq ker\pi$ and $Y \subseteq kerq$.

Proof. Since Q is projective, then there exists $h : Q \to T$ such that $q = \pi \circ h$. Thus, $\pi(h(Q)) = M = \pi(T)$ and so $T = h(Q) + ker\pi$. We have, $ker\pi$ is a locally δ-small in T. Then by Theorem 4.1 $T_P = A_P \oplus B_P$, where $A_P = h_P(Q_P)$ and B_P is projective semisimple with $B_P \subseteq (ker\pi)_P$. So, $T = A \oplus B$, B is projective semisimple with $B \subseteq ker\pi$ and $\pi|_A : A \to M$ is a projective δ-cover. Since $ker\pi|_A \subseteq ker\pi$ and $ker\pi$ is a locally δ-small, then $ker\pi|_A$ is also locally δ-small. Since A is projective then the homomorphism $h : Q \to A$ splits. So, there exists $g : A \to Q$ such that $h \circ g = I_A$. Therefore, $Q = X \oplus Y$ with $Y = kerh$ and $X = g(A)$. Thus, $X \cong A$ and $ker\pi|_A$ is locally δ-small in A, the we get $kerq|_X = g(ker\pi|_A)$ is locally δ-small in $g(A) = X$. Note that $q(X) = (\pi \circ h)(X) = (\pi \circ h)(X + Y) = (\pi \circ h)(Q) = q(Q) = M$. Thus $q|_X : X \to M$ is a locally projective δ-cover. □

Finally we give the following theorem.

Theorem 4.11. *Suppose that M is an $R-$module and N is a submodule of M. Then the following are equivalent:*

1. *$\frac{M}{N}$ has a locally projective δ-cover.*
2. *$M = M_1 \oplus M_2$, where $M_1 \subseteq N$ is a projective and $M_1 \cap N$ is a locally δ-samll in M.*

Proof. $(1 \Rightarrow 2)$ Let $q : Q \to \frac{M}{N}$ be a locally projective δ-cover and $\pi : M \to \frac{M}{N}$ the canonical mapping. Then there exists a decomposition $M = X \oplus Y$ such that $\pi|_X : X \to \frac{M}{N}$ is a locally projective δ-cover and $Y \subseteq ker\pi = N$. So, $X \cap N = ker\pi|_X$ which is locally δ-small in X. Therefore, $X \cap N$ is a locally δ-small in M.

$(2 \Rightarrow 1)$ Obvious. □

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