# Fuzzy Γ-ideals in regular and intra-regular Γ-AG-groupoids

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**Abstract.** This paper is devoted to the concepts of regular and intra-regular Γ-AG-groupoids. We investigate some characteristics of Γ-AG-groupoids by the properties of fuzzy Γ-quasi-ideals, fuzzy Γ-interior-ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals.

**Keywords:** Γ-AG-groupoids, regular and intra-regular Γ-AG-groupoids, fuzzy Γ-quasi-ideals, fuzzy Γ-bi-ideals and fuzzy Γ-interior-ideals.

## 1. Introduction

Abel-Grassmann's groupoid [1, 2], abbreviated as AG-groupoid, is a groupoid S whose elements satisfy the left invertive law $a(bc) = (cb)a$. This structure is also known as left almost semigroup [3, 4]. It is a non-associative structure which has wide applications in the theory of flocks and the theory of loops. Sen and Saha in [5, 6], defined Γ-semigroups as a generalization of semigroups.

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Γ-semigroups have been studied by a lot of mathematicians, for instance Dutta and Adhikari [7], Hila [8], Chinram [9], Sen et al., [5, 6, 10] and Seth [11]. After the introduction of fuzzy sets by Zadeh [12], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics as well as its application in many areas, the notion of fuzzy subgroups was introduced by Rosenfeld [13] and its structure was investigated. Several other mathematicians started the fuzzification of different branches of algebras [14, 15, 16, 17, 18]. The notion of fuzzy ideal in Γ-rings was first introduced by Jun and Lee [19]. They studied some preliminary properties of fuzzy ideals of Γ-rings. In [20], Dutta and Chanda studied the structures of fuzzy ideals of a -ring and characterized -field, Noetherian -ring with the help of fuzzy ideals via operator rings of Γ-ring. Jun [21] defined prime fuzzy ideal of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a prime fuzzy ideal.

We remark that the Γ-semigroup given in [5] by Sen and Saha may be called a one-sided Γ-semigroup. Later on in [7], Dutta and Adhikari introduced a both sided Γ-semigroup in which the operation $\Gamma \times S \times \Gamma \to \Gamma$ was also taken into consideration. In [22], Shah et al., introduced the notion of -AG-groupoids which is in fact a generalization of AG-groupoids and discussed some properties of Γ-ideals and bi-Γ-ideals in Γ-AG-groupoids. In continuation, Shah et al., studied M-systems in Γ-AG-groupoids [23]. Recently in [24], Shah, Rehman and Khan discussed fuzzy Γ-ideals, prime (res. semiprime) fuzzy Γ-ideals in Γ-AG-groupoids.

The aim of our present study is to characterize regular and intra-regular Γ-AG-groupoids by the properties of fuzzy Γ-quasi-ideals, fuzzy Γ-interior ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals.

## 2. Preliminaries

Throughout this study will denote a Γ-AG-groupoid unless otherwise stated. For the convenience of readers we recall certain definitions and results from [22, 24].

**Definition 2.1** ([22]). *Let $S$ and $\Gamma$ be nonempty sets. We call $S$ to be a $\Gamma$-AG-groupoid if there exists a mapping $S \times \Gamma \times S \to S$ , written as $(a, \gamma, c)$ and denoted by $a\gamma c$ such that $S$ satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.*

**Definition 2.2** ([22]). *An element $e$ of a $\Gamma$-AG-groupoid $S$ is called left identity if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.*

**Definition 2.3** ([22]). *A non-empty subset $I$ of a $\Gamma$-AG-groupoid is called a left (right) $\Gamma$-ideal of $S$ if $S\Gamma I \subseteq I (I\Gamma S \subseteq I)$. A non-empty subset $I$ of a $\Gamma$-AG groupoid $S$ is called a $\Gamma$-ideal if it is both a left and a right $\Gamma$-ideal of $S$.*

**Definition 2.4** ([24]). *Let $S$ be a $\Gamma$-AG-groupoid and $\phi \neq A \subseteq S$. Then the characteristic function $\chi_A$ of $A$ is defined by*

$$\chi_A(x) = \begin{cases} 1, & if x \in A \\ 0, & if x \notin A \end{cases}$$

**Definition 2.5** ([24]). *A function $f$ from a non empty set $X$ to the unit interval $[0,1]$ is called a fuzzy subset of $S$. For fuzzy subsets $f, g$ of $X$, $f \subseteq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The fuzzy subsets $f \wedge g$ and $f \vee g$ of $X$ are defined as*

$$(f \wedge g)(x) = f(x) \wedge g(x) = \min\{f(x), g(x)\},$$

$$(f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\} \text{ for all } x, y \in S.$$

*Let $f, g$ be any fuzzy subsets of $S$. We define the product $f\Gamma g$ of $f$ and $g$ as follows $(f\Gamma g)(x) = \sup\{\min\{f(y), g(z)\}$ if $\exists\, x, y \in S$ and $\gamma \in \Gamma$ such that $x = y\gamma z\}$ and $(f\Gamma g)(x) = 0$,* if *$x \neq y\gamma z$.*

**Definition 2.6** ([24]). *A fuzzy subset $f$ of $S$ is called a fuzzy sub$\Gamma$-AG groupoid if $f(x\gamma y) \geq min\{f(x), f(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$ and $f$ is called a fuzzy left (right) $\Gamma$-ideal of $S$ if $f(x\gamma y) \geq f(y)(f(x\gamma y) \geq f(x))$ for all $x, y \in S$ and $\gamma \in \Gamma$. If $f$ is both a fuzzy left $\Gamma$-ideal and a fuzzy right $\Gamma$-ideal of $S$, then $f$ is called a two-sided fuzzy $\Gamma$-ideal of $S$.*

**Lemma 2.1** ([24]). *In a $\Gamma$-AG groupoid $S$ with left identity $e$, every fuzzy right $\Gamma$-ideal of $S$ is a fuzzy left $\Gamma$-ideal.*

**Lemma 2.2** ([24]). *If $S$ is a $\Gamma$-AG groupoid with left identity $e$ and $f, g$ are fuzzy $\Gamma$-ideals of $S$, then $f\Gamma g$ is a fuzzy $\Gamma$-ideal of $S$.*

**Lemma 2.3** ([24]). *If $S$ is a $\Gamma$-AG groupoid and $f, g$ are fuzzy $\Gamma$-ideals of $S$, then $f\Gamma g \subseteq f \cap g$.*

**Definition 2.7** ([24]). *A $\Gamma$-AG-groupoid $S$ is said to be a regular $\Gamma$-AG-groupoid if for each $a$ in $S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.*

## 3. Fuzzy ideals in regular Γ-AG-groupoids

We initiate with the following Lemma

**Lemma 3.1.** *In a regular $\Gamma$-AG-groupoid $S$, $f\Gamma g = f \cap g$, where $f$ is a fuzzy right $\Gamma$-ideal and $g$ is a fuzzy left $\Gamma$-ideal.*

**Proof.** Since $f\Gamma g \subseteq f \cap g$, we only show that $f \cap g \subseteq f\Gamma g$. If $a \in S$, then there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Now, $(f\Gamma g)(a) = \bigvee_{a=y\beta z}\{f(y) \wedge g(z)\} \geq f(a\alpha x) \wedge g(a) \geq f(a) \wedge g(a) = f(a) \wedge g(a)$ $= (f \cap g)(a) \Rightarrow f \cap g \subseteq f\Gamma g$. Hence $f\Gamma g = f \cap g$. □

**Lemma 3.2.** *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is fuzzy Γ-idempotent.*

**Proof.** Let $S$ be a regular Γ-AG-groupoid and $f$ a fuzzy right Γ-ideal of $S$. Since $f\Gamma f\subseteq f$, we only show that $f\subseteq f\Gamma f$. If $a\in S$, then there exist $x$ in $S$ and $\alpha,\beta\in\Gamma$ such that $a=(a\alpha x)\beta a$.

Now, $(f\Gamma f)(a)=\bigvee_{a=y\gamma z}\{f(y)\wedge f(z)\}\geq f(a\alpha x)\wedge f(a)\geq f(a)\wedge f(a)=f(a)\Rightarrow f\subseteq f\Gamma f$. □

**Definition 3.1.** *Let $A$ be a subset of Γ-AG-groupoid $S$. The characteristic function of $A$ is denoted by $C_A$ and is defined by*

$$C_A(a)=\begin{cases}1, \text{ if } a\in A\\ 0, \text{ otherwise}\end{cases}$$

*We note that a Γ-AG-groupoid $S$ can be considered as a fuzzy Γ-subset of itself and we write $S=C_S$, i.e., $S(x)=1$ for all $x\in S$.*

**Proposition 3.1.** *If $A$ and $B$ are any non-empty subsets of Γ-AG-groupoids $S$, then following properties hold.*

(1) *If $A\subseteq B$, then $C_A\subseteq C_B$.*
(2) $C_A\Gamma C_B=C_{A\Gamma B}$.
(3) $C_A\cup C_B=C_{A\cup B}$.
(4) $C_A\cap C_B=C_{A\cap B}$.

**Proof.** (1) Let $a$ be any element of $S$. Suppose $a\in A$, this implies $a\in B$. Thus $C_A(a)=1=C_B(a)$. This implies $C_A\subseteq C_B$. If $a\notin A$, then $a\notin B$. This implies $C_A(a)=0=C_B(a)$. Thus $C_A\subseteq C_B$.

(2) Let $x$ be any element of $S$. Suppose $x\in A\Gamma B$. This implies $x=a\alpha b$ for some $a\in A$, $b\in B$ and $\alpha\in\Gamma$.

$(C_A\Gamma C_B)(x)=\bigvee_{x=y\gamma z}\{C_A(y)\wedge C_B(z)\}\geq C_A(a)\wedge C_B(b)=1\wedge 1=1=C_{A\Gamma B}(x)$.

Suppose $x\notin A\Gamma B$. This implies $x\neq a\alpha b$ for some $a\in A$, $b\in B$ and $\alpha\in\Gamma$.

$$(C_A\Gamma C_B)(x)=\bigvee_{x=y\gamma z}\{C_A(y)\wedge C_B(z)\}=0\wedge 0=0=C_{A\Gamma B}(x)$$

Hence $C_A\Gamma C_B=C_{A\Gamma B}$.

(3) Let $a$ be any element of $S$. Suppose $a\in A\cup B$. Then there are three cases

(i) when $a\in A$ and $a\in B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=1\vee 1=1=C_{A\cup B}(a)$.

(ii) when $a\in A$ and $a\notin B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=1\vee 0=1=C_{A\cup B}(a)$.

(iii) when $a\notin A$ and $a\in B$. $(C_A\cup C_B)(a)=C_A(a)\vee C_B(a)=0\vee 1=1=C_{A\cup B}(a)$.

If $a\notin A\cup B$. This implies $a\notin A$ and $a\notin B$. This implies$(C_A\cup C_B)(a)=C_{A\cup B}(a)$. Hence in all cases $C_A\cup C_B=C_{A\cup B}$.

(4) Let $a$ be any element of $S$. Suppose $a\in A\cap B$. This implies $a\in A$ and $a\in B$.

Now, $(C_A\cap C_B)(a)=C_A(a)\wedge C_B(a)=1\wedge 1=1=C_{A\cap B}(a)$. Suppose $a\notin A\cap B$. This implies $a\notin A$ and $a\notin B$. Now $(C_A\cap C_B)(a)=C_A(a)\wedge C_B(a)=0\wedge 0=0=C_{A\cap B}(a)$. Hence $C_A\cap C_B=C_{A\cap B}$. □

**Definition 3.2.** *A fuzzy $\Gamma$-subAG-groupoid $f$ of a $\Gamma$-AG-groupoid $S$ is called a fuzzy $\Gamma$-bi-ideal of $S$ if $f\left((x\alpha y)\beta z\right) \geq f\left(x\right) \wedge f\left(z\right)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

**Lemma 3.3.** *Every fuzzy right (two-sided) $\Gamma$-ideal of a $\Gamma$-AG-groupoid $S$ is a fuzzy $\Gamma$-bi-ideal of $S$.*

**Proof.** Let $f$ be a fuzzy right $\Gamma$-ideal of $S$. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now, $f\left((x\alpha y)\beta z\right) = f\left(x\alpha y\right) \geq f\left(x\right)$ and $f((x\alpha y)\beta z) = f((z\alpha y)\beta x) \geq f(z\alpha y) \geq f(z)$.

It follows that $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$. Hence $f$ is a fuzzy $\Gamma$-bi-ideal of $S$. Similarly it is fairly easy to prove that every two sided fuzzy $\Gamma$-ideal is also a fuzzy $\Gamma$-bi-ideal of $S$. □

**Lemma 3.4.** *If $S$ is a regular $\Gamma$-AG-groupoid, then for every fuzzy $\Gamma$-bi-ideal $f$, $(f\Gamma S)\Gamma f = f$.*

**Proof.** Since $(f\Gamma S)\Gamma f \subseteq f$, so it follows that there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Now,

$$((f\Gamma S)\Gamma f)(a) = \bigvee_{a=y\gamma z} \{(f\Gamma S)(y) \wedge f(z)\} \geq (f\Gamma S)(a\alpha x) \wedge f(a)$$

$$= \bigvee_{a\alpha x=p\delta q} \{(f(p) \wedge S(q)\} \wedge f(a)\} \geq f(a) \wedge S(x) \wedge f(a) = 1 \wedge f(a) = f(a)$$

$$\Rightarrow f \subseteq (f\Gamma S)\Gamma f.$$

This implies $(f\Gamma S)\Gamma f = f$. □

**Lemma 3.5.** *Every fuzzy right $\Gamma$-ideal of a regular $\Gamma$-AG-groupoid is a fuzzy $\Gamma$-ideal of $S$.*

**Proof.** Let $f$ be a fuzzy right $\Gamma$-ideal of $S$. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

$$f(a\gamma b) = f(((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(b\beta a) \geq f(b). \Rightarrow f(a\gamma b) \geq f(b).$$

This implies $f$ is a fuzzy left $\Gamma$-ideal and hence $f$ is a fuzzy $\Gamma$-ideal of $S$. □

**Definition 3.3.** *A fuzzy subset $f$ of $\Gamma$-AG-groupoid $S$ is called a fuzzy $\Gamma$-interior ideal of $S$ if $f\left((x\alpha y)\beta z\right) \geq f\left(y\right)$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

**Lemma 3.6.** *Let $S$ be a regular $\Gamma$-AG-groupoid. Then any non-empty fuzzy subset $f$ of $S$ is a fuzzy $\Gamma$-interior ideal of $S$ if and only if $f$ is a fuzzy $\Gamma$-ideal of $S$.*

**Proof.** Suppose $f$ is a fuzzy Γ-interior ideal of $S$. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$,

$$f(a\gamma b) = f((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(a) \Rightarrow f(a\gamma b) \geq f(a).$$

It follows that $f$ is a fuzzy Γ-right ideal of $S$ and hence $f$ is a fuzzy Γ-ideal of $S$. Conversely, let $f$ be a fuzzy Γ-ideal of $S$. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f((x\alpha y)\beta z) = f(x\alpha y) \geq f(y)$. Thus $f$ is a fuzzy interior Γ-ideal of $S$. □

**Remark.** Fuzzy interior $\Gamma$-ideal and fuzzy $\Gamma$-ideal coincide if $S$ is regular $\Gamma$-AG-groupoid.

**Definition 3.4.** *A fuzzy subset $f$ of $\Gamma$-AG-groupoid $S$ is called a fuzzy $\Gamma$-quasi-ideal of $S$ if $(f\Gamma S) \cap (S\Gamma f) \subseteq f$.*

**Proposition 3.2.** *In a regular $\Gamma$-AG-groupoid $S$ with left identity $e$, $(f\Gamma S) \cap (S\Gamma f) = f$ for every fuzzy right $\Gamma$-ideal $f$ of $S$.*

**Proof.** Let $f$ be a fuzzy right Γ-ideal of $S$. This implies $(f\Gamma S) \cap (S\Gamma f) \subseteq f$, because every fuzzy right Γ-ideal is fuzzy Γ-quasi-ideal. Let $a \in S$. This implies there exist $x \in S$ and $\alpha, \beta \in \Gamma$, such that $a = (a\alpha x)\beta a$,

$$(f\Gamma S)(a) = \bigvee_{a=y\gamma z} \{f(y) \wedge S(z)\} \geq f(a\alpha x) \wedge S(a) = f(a\alpha x) \wedge 1 = f(a\alpha x) \geq f(a)$$

$$\Rightarrow f \subseteq f\Gamma S (S\Gamma f)(a) = \bigvee_{a=l\delta m} \{S(l) \wedge f(m)\}\}$$

$$\geq S(a\alpha x) \wedge f(a) = 1 \wedge f(a) = f(a) f \subseteq S\Gamma f \Rightarrow f \subseteq (f\Gamma S) \cap (S\Gamma f).$$

Hence $(f\Gamma S) \cap (S\Gamma f) = f$. □

**Lemma 3.7.** *In a $\Gamma$-AG-groupoid $S$ with left identity $e$, every fuzzy $\Gamma$-quasi-ideal is a fuzzy $\Gamma$-bi-ideal of $S$.*

**Proof.** Let $f$ be a fuzzy quasi-ideal of $S$. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. This implies $f((x\alpha y)\beta z) \geq ((f\Gamma S) \cap (S\Gamma f))((x\alpha y)\beta z) = (f\Gamma S)((x\alpha y)\beta z) \wedge (S\Gamma f)((x\alpha y\beta)z)$.
Hence

$$(S\Gamma f)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z=l\gamma m} \{S(l) \wedge f(m)\} = \bigvee_{(x\alpha y)\beta z=l\gamma m} \{1 \wedge f(m)\} \geq f(z)?$$

$$\Rightarrow (S\Gamma f)((x\alpha y)\beta z) \geq f(z).$$

$$(f\Gamma S)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z=p\gamma q} \{f(p) \wedge S(q)\} = \bigvee_{(x\alpha y)\beta z=p\gamma q} \{f(p) \wedge 1\}$$

$$(x\alpha y)\beta z = (x\alpha y)\beta(e\delta z) = (x\alpha e)\beta(y\delta z) \in (x\alpha e)\Gamma S = x\Gamma S.$$

This implies $(x\alpha y)\beta z \in x\Gamma S$, so $(x\alpha y)\beta z = x\eta r$ for some $r \in S$ and $\eta \in \Gamma$. This implies

$$(f\Gamma S)((x\alpha y)\beta z) = \bigvee_{(x\alpha y)\beta z=x\eta r=pq} \{f(p) \wedge S(q)\} = \bigvee_{x\eta r=pq} \{f(p) \wedge 1\} \geq f(x)$$

$$\Rightarrow (f\Gamma S)((x\alpha y)\beta z) \geq f(x). f((x\alpha y)\beta z) \geq (f\Gamma S)((x\alpha y)\beta z) \wedge (S\Gamma f)((x\alpha y)\beta z)$$

$$\geq f(x) \wedge f(z).$$

This implies $f((x\alpha y)\beta z)\geq f(x)\wedge f(z)$. This implies $f$ is a fuzzy $\Gamma$-bi-ideal of $S$. □

**Theorem 3.1.** *If $S$ is a $\Gamma$-AG-groupoid with left identity $e$, then following conditions are equivalent.*

(1) *$S$ is regular.*

(2) *$f\cap g=f\Gamma g$ for every fuzzy right $\Gamma$-ideal $f$ and every fuzzy left $\Gamma$-ideal $g$ of $S$.*

(3) *$h=(h\Gamma S)\Gamma h$ for every fuzzy $\Gamma$-quasi-ideal $h$ of $S$.*

**Proof.** $(1)\Rightarrow(2)$. Obvious

$(1)\Rightarrow(3)$. Let $a\in S$. This implies there exist elements $x$ in $S$ and $\alpha,\beta$ in $\Gamma$ such that $a=(a\alpha x)\beta a$. Now,

$$((h\Gamma S)\Gamma h)(a)=\bigvee_{a=y\gamma z}\{(h\Gamma S)(y)\wedge h(z)\}\geq(h\Gamma S)(a\alpha x)\wedge h(a)$$
$$=(\bigvee_{a\alpha x=l\delta m}\{h(l)\wedge S(m)\})\wedge h(a)\geq h(a)\wedge S(x)\wedge h(a)=h(a)$$
$$\Rightarrow h\subseteq(h\Gamma S)\Gamma h.$$

Since every fuzzy $\Gamma$-quasi-ideal is fuzzy $\Gamma$-bi-ideal of $S$, this implies $(h\Gamma S)\Gamma h\subseteq h$. Hence it follows that $h=(h\Gamma S)\Gamma h$.

$(3)\Rightarrow(2)$. Let $f$ be a fuzzy right $\Gamma$-ideal and $g$ is a fuzzy left $\Gamma$-ideal of $S$. Since $f$ and $g$ are fuzzy $\Gamma$-quasi-ideals of $S$ and intersection of any two fuzzy $\Gamma$-quasi-ideals of $S$ is also a fuzzy $\Gamma$-quasi-ideal of $S$, $f\cap g$ is a fuzzy $\Gamma$-quasi-ideal of $S$. $f\cap g=((f\cap g)\Gamma S)\Gamma(f\cap g)\subseteq(f\Gamma S)\Gamma g\subseteq f\Gamma g\Rightarrow f\cap g\subseteq f\Gamma g$. Since $f\Gamma g\subseteq f\cap g$. Hence $f\Gamma g=f\cap g$.

$(2)\Rightarrow(1)$. Let $a\in S$. Since $S\Gamma a$ is a left $\Gamma$-ideal of $S$ generated by $a$ and $a\Gamma S\cup S\Gamma a$ is a right $\Gamma$-ideal of $S$ containing $a$. This implies $C_{S\Gamma a}$ and $C_{a\Gamma S\cup S\Gamma a}$ are fuzzy $\Gamma$-left and fuzzy $\Gamma$-right ideals of $S$ respectively and by hypothesis $C_{a\Gamma S\cup S\Gamma a}\cap C_{S\Gamma a}=C_{a\Gamma S\cup S\Gamma a}\Gamma C_{S\Gamma a}$.

So, by Proposition 1, we have $C_{(a\Gamma S\cup S\Gamma a)\cap S\Gamma a}=C_{(a\Gamma S\cup S\Gamma a)S\Gamma a}$. Consequently $(a\Gamma S\cup S\Gamma a)\cap S\Gamma a=(a\Gamma S\cup S\Gamma a)S\Gamma a$.

Now, as $a\in(a\Gamma S\cup S\Gamma a)\cap S\Gamma a$, so $a\in(a\Gamma S\cup S\Gamma a)S\Gamma a$. This implies $a$ is regular and hence $S$ is regular. □

**Theorem 3.2.** *For a $\Gamma$-AG-groupoid with left identity $e$, the following conditions are equivalent.*

(1) *$S$ is regular.*

(2) *$f=(f\Gamma S)\Gamma f$ for every quasi-ideal $f$ of $S$.*

(3) *$f=(f\Gamma S)\Gamma f$ for every bi-ideal $f$ of $S$.*

**Proof.** $(1)\Rightarrow(3)$. Straightforward.

$(3)\Rightarrow(2)$. Because every fuzzy $\Gamma$-quasi -ideal is fuzzy $\Gamma$-bi-ideal.

$(2)\Rightarrow(1)$. Obvious. □

**Theorem 3.3.** *If $S$ is a $\Gamma$-AG-groupoid with left identity $e$, then following conditions are equivalent.*
(1) *$S$ is regular.*
(2) *$f\cap g=(f\Gamma g)\Gamma f$ for every fuzzy $\Gamma$-quasi-ideal $f$ and every fuzzy $\Gamma$-ideal $g$ of $S$.*
(3) *$h\cap k=(h\Gamma k)\Gamma h$ for every fuzzy $\Gamma$-bi-ideal $h$ and every fuzzy $\Gamma$-ideal $k$ of $S$.*

**Proof.** $(1)\Rightarrow(3)$. If $h$ is a fuzzy Γ-bi-ideal and $k$ be a fuzzy Γ-ideal of $S$, then $(h\Gamma k)\Gamma h\subseteq(S\Gamma k)\Gamma S\subseteq k\Gamma S\subseteq k$, $(h\Gamma k)\Gamma h\subseteq(h\Gamma S)\Gamma h\subseteq h$.

It follows that $(h\Gamma k)\Gamma h\subseteq h\cap k$. Now,

$$((h\Gamma k)\Gamma h)(a)=\bigvee_{a=l\eta m}\{(h\Gamma k)(l)\wedge h(m)\}\geq(h\Gamma k)(a\alpha((x\beta a)\alpha x))\wedge h(a)$$
$$=(\bigvee_{a\alpha((x\beta a)\alpha x)=y\delta z}\{h(y)\wedge k(z)\})\wedge h(a)\geq h(a)\wedge k((x\beta a)\alpha x)\wedge h(a)$$
$$\geq h(a)\wedge k(a)=(h\cap k)(a)\Rightarrow h\cap k\subseteq(h\Gamma k)\Gamma h.$$

Consequently $h\cap k=(h\Gamma k)\Gamma h$.
$(3)\Rightarrow(2)$. Straightforward because every fuzzy Γ-quasi-ideal of $S$ is a fuzzy Γ-bi-ideal of $S$.
$(2)\Rightarrow(1)$. Since $S$ is fuzzy Γ-ideal, so $f\cap S=(f\Gamma S)\Gamma f$.

It follows that $f=(f\Gamma S)\Gamma f$, where $f$ is a fuzzy Γ-quasi-ideal of $S$.

It follows that $S$ is regular. □

**Theorem 3.4.** *For a $\Gamma$-AG-groupoid with left identity $e$, the following conditions are equivalent.*
(1) *$S$ is regular.*
(2) *$f\cap g\subseteq g\Gamma f$ for every fuzzy $\Gamma$-quasi-ideal $f$ and every fuzzy right $\Gamma$-ideal $g$ of $S$.*
(3) *$h\cap g\subseteq g\Gamma h$ for every fuzzy $\Gamma$-bi-ideal $h$ and every fuzzy right $\Gamma$-ideal $g$ of $S$.*
(4) *$k\cap g\subseteq g\Gamma k$ for every fuzzy generalized $\Gamma$-bi-ideal $k$ and every fuzzy right $\Gamma$-ideal $g$ of $S$.*

**Proof.** $(1)\Rightarrow(4)$. Let $a\in S$ and $\alpha,\beta\in\Gamma$, then by hypothesis, $a=(a\alpha x)\beta a$. Now, $(g\Gamma k)(a)=\bigvee_{a=l\gamma m}\{(g)(l)\wedge k(m)\}\geq g(a\alpha x)\wedge k(a)\geq g(a)\wedge k(a)=k(a)\wedge g(a)$ $=(k\wedge g)(a)$. Consequently, $k\cap g\subseteq g\Gamma k$ for every fuzzy generalized Γ-bi-ideal $k$ and every fuzzy right Γ-ideal $g$ of $S$.
$(4)\Rightarrow(3)\Rightarrow(2)$ are straightforward.
$(2)\Rightarrow(1)$. Let $f$ be a fuzzy right Γ-ideal and $g$ be a fuzzy left Γ-ideal of $S$. Then clearly $g\Gamma f\subseteq f\cap g$. Also since every fuzzy left Γ- ideal is fuzzy Γ-quasi-ideal of $S$, so by hypothesis $f\cap g\subseteq g\Gamma f$. It follows that $f\cap g=g\Gamma f$ for every fuzzy right Γ-ideal $f$ and $g$ is a fuzzy left Γ-ideal of $S$. This implies that $g\cap f=g\Gamma f$. Hence $S$ is regular. □

## 4. Fuzzy ideals in intra-regular Γ-AG-groupoids

**Definition 4.1.** *A Γ-AG-groupoid $S$ is called intra-regular if for each $a \in S$, there exist $x$, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$.*

**Lemma 4.1.** *In an intra-regular Γ-AG-groupoid $S$, every fuzzy Γ-ideal is fuzzy Γ-idempotent.*

**Proof.** Let $f$ be a any fuzzy Γ-ideal of $S$. Since $f\Gamma f \subseteq f$, we only show that $f \subseteq f\Gamma f$. Let $a \in S$. So, by definition $a = (x\alpha a)\beta(a\gamma y)$. Now, $(f\Gamma f)(a) = \bigvee_{a=y\delta z}\{f(y) \wedge f(z)\} \geq f(x\alpha a) \wedge f(a\gamma y) \geq f(a) \wedge f(a) = f(a) \Rightarrow f \subseteq f\Gamma f$. Hence $f = f\Gamma f$. □

**Lemma 4.2.** *If $S$ is an intra-regular Γ-AG-groupoid $S$, then $g \cap f \subseteq f\Gamma g$ for every fuzzy left Γ-ideal $f$ and for every fuzzy right Γ-ideal $g$ of $S$.*

**Proof.** Let $a \in S$. This implies there exist $x$, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$. Now, $(f\Gamma g)(a) = \bigvee_{a=y\delta z}\{f(y) \wedge g(z)\} \geq f(x\alpha a) \wedge g(a\gamma y) \geq f(a) \wedge g(a) = g(a) \wedge f(a) = (g \cap f)(a) \Rightarrow g \cap f \subseteq f\Gamma g$. □

**Lemma 4.3.** *Every fuzzy right Γ-ideal of an intra-regular Γ-AG-groupoid $S$ is a fuzzy Γ-ideal of $S$.*

**Proof.** Let $f$ be a fuzzy right Γ-ideal of $S$. Let $a, b \in S$. Then by definition, $a = (x\alpha a)\beta(a\gamma y)$. Now, $f(a\delta b) = f((x\alpha a)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(x\alpha a)) \geq f(b\beta(a\gamma y)) \geq f(b)$. This implies $f$ is a fuzzy left Γ-ideal of $S$ and consequently $f$ is a fuzzy Γ-ideal of $S$. □

**Theorem 4.1.** *If $S$ is an intra-regular Γ-AG-groupoid with left identity $e$, then any non-empty fuzzy subset $f$ of $S$ is a fuzzy Γ-interior ideal of $S$ if and only if $f$ is a fuzzy Γ-ideal of $S$.*

**Proof.** Suppose $f$ is a fuzzy Γ-interior ideal of $S$. Let $a, b \in S$. Then, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$, such that $a = (x\alpha a)\beta(a\gamma y)$, $f(a\delta b) = f((a\alpha x)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(a\alpha x)) \geq f(a\gamma y) \geq f((e\eta a)\gamma y) \geq f(a) \Rightarrow f(a\delta b) \geq f(a)$. It follows that $f$ is a fuzzy right Γ-ideal of $S$ and hence $f$ becomes a fuzzy Γ-ideal of $S$. Conversely, let $f$ be a fuzzy Γ-ideal of $S$. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f((x\alpha y)\beta z) = f(x\alpha y) \geq f(y)$. Thus $f$ is a fuzzy interior Γ-ideal of $S$. □

**Remark.** In an intra-regular Γ-AG-groupoid $S$, fuzzy Γ-interior ideal and fuzzy Γ-ideal coincide.

**Theorem 4.2.** *If $S$ is a Γ-AG-groupoid with left identity $e$, then following conditions are equivalent.*

(1) *$S$ is intra-regular.*

(2) *$f \cap g = (f\Gamma g)\Gamma f$ for every fuzzy Γ-quasi-ideal $f$ and for every fuzzy Γ-ideal $g$ of $S$.*

(3) *$h \cap k = (h\Gamma k)\Gamma h$ for every fuzzy Γ-bi-ideal $h$ and for every fuzzy Γ-ideal $k$ of $S$.*

**Proof.** $(1)\Rightarrow(3)$. Let $x\in S$, so there exist $s,\ t\in S$ and $\alpha,\beta,\gamma\in\Gamma$ such that $x=(s\alpha x)\beta(x\gamma t)$.

Consider

$$\begin{aligned}x\gamma t &= ((s\alpha x)\beta(x\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma(e\eta t)\\ &= (x\beta e)\gamma(((s\alpha x)\gamma t)\eta t) = (x\beta e)\gamma((t\gamma t)\eta(s\alpha x)) = (x\beta e)\gamma(((e\eta t)\gamma t)\eta(s\alpha x)\\ &= (x\beta e)\gamma(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta(e\alpha x)) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta x)\\ &= (x\beta((t\eta t)\gamma s))\gamma(e\eta x) = (x\beta((t\eta t)\gamma s))\gamma x\end{aligned}$$

and $s\alpha x = s\alpha((s\alpha x)\beta(x\gamma t)) = s\alpha(x\beta((s\alpha x)\gamma t)) = (e\eta s)\alpha(x\beta((s\alpha x)\gamma t)) = (e\eta x)\beta(s\alpha((s\alpha x)\gamma t)) = x\beta(s\alpha((s\alpha x)\gamma t))$.

Now,

$$\begin{aligned}((h\Gamma k)\Gamma h)(x) &= \bigvee_{x=y\gamma z}(h\Gamma k)(y)\wedge h(z) \geq (h\Gamma k)(s\alpha x)\wedge h(x\gamma t)\\ &= (\bigvee_{s\alpha x=l\varepsilon m} h(l)\wedge k(m))\wedge h((x\beta((t\eta t)\gamma s))\gamma x)\\ &\geq h(x)\wedge k(s\alpha((s\alpha x)\gamma t))\wedge h(x)\geq h(x)\wedge k(x)=h\cap k(x)\Rightarrow h\cap k\subseteq (h\Gamma k)\Gamma h.\end{aligned}$$

Also, $(h\Gamma k)\Gamma h\subseteq (S\Gamma k)\Gamma S\subseteq k\Gamma S\subseteq k\Rightarrow (h\Gamma k)\Gamma h\subseteq k(h\Gamma k)\Gamma h\subseteq (h\Gamma S)\Gamma h\subseteq h\Rightarrow (h\Gamma k)\Gamma h\subseteq h\Rightarrow (h\Gamma k)\Gamma h\subseteq h\cap k$. Hence, $h\cap k=(h\Gamma k)\Gamma h$.

$(3)\Rightarrow(2)$. Straightforward because every fuzzy Γ-quasi-ideal of $S$ is a fuzzy Γ-bi-ideal of $S$.

$(2)\Rightarrow(1)$. Let $E$ be a fuzzy right Γ-ideal and $J$ be a fuzzy two-sided Γ-ideal of $S$. Since every fuzzy right Γ-ideal of $S$ is a fuzzy Γ-quasi-ideal of $S$. Then by hypothesis $E\cap J=(E\Gamma J)\Gamma E\subseteq (S\Gamma J)\Gamma E\subseteq J\Gamma E$. Since $E$ is a fuzzy right Γ-ideal and $J$ is also a fuzzy left Γ-ideal. Hence $S$ is intra-regular. □

**Theorem 4.3.** *In a Γ-AG-groupoid $S$ with left identity $e$, the following conditions are equivalent.*
(1) *$S$ is intra-regular.*
(2) *$f\cap g\subseteq g\Gamma f$ for every fuzzy Γ-quasi-ideal $f$ and every fuzzy left Γ-ideal $g$ of $S$.*
(3) *$h\cap g\subseteq g\Gamma h$ for every fuzzy Γ-bi-ideal $h$ and every fuzzy left Γ-ideal $g$ of $S$.*

**Proof.** $(1)\Rightarrow(3)$. Let $h$ be fuzzy Γ-bi-ideal and $g$ be a fuzzy left Γ-ideal of $S$. Let $x\in S$. Then by definition $x=(s\alpha x)\beta(x\gamma t)$. Now,

$$\begin{aligned}x\gamma t &= ((s\alpha x)\beta(x\gamma t))\delta t = (x\beta((s\alpha x)\gamma t))\delta t = (x\beta((s\alpha x)\gamma t))\delta(e\eta t)\\ &= (x\beta e)\delta(((s\alpha x)\gamma t)\eta t) = (x\beta e)\delta((t\gamma t)\eta(s\alpha x)) = (x\beta e)\delta(((e\eta t)\gamma t)\eta(s\alpha x))\\ &= (x\beta e)\delta(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\delta(((t\eta t)\gamma s)\eta(e\alpha x)) = (x\beta e)\delta(((t\eta t)\gamma s)\eta x)\\ &= (x\beta((t\eta t)\gamma s))\delta(e\eta x) = (x\beta((t\eta t)\gamma s))\delta x.\end{aligned}$$

Now,

$$\begin{aligned}(g\Gamma h)(x) &= \bigvee_{x=a\mu b}\gamma(a)\wedge h(b)\geq g(s\alpha x)\wedge h((x\beta((t\eta t)\gamma s))\delta x)\\ &\geq g(x)\wedge h(x)=h(x)\wedge g(x)=(h\cap g)(x)\Rightarrow h\cap g\subseteq g\Gamma h.\end{aligned}$$

$(3) \Rightarrow (2)$. Obvious, because every fuzzy Γ-quasi-ideal of $S$ is fuzzy Γ-bi-ideal of $S$.
$(2) \Rightarrow (1)$. Let $R$ be a fuzzy right Γ-ideal of $S$ and $L$ be a fuzzy left Γ-ideal of $S$. Since every fuzzy right Γ-ideal of $S$ is fuzzy Γ-quasi-ideal of $S$. Consequently $R$ is a fuzzy Γ-quasi-ideal of $S$. So by hypothesis $R \cap L \subseteq L\Gamma R$. Hence $S$ is intra-regular. □

## 5. Fuzzy idempotent ideals

**Theorem 5.1.** *In a Γ-AG-groupoid $S$ with left identity $e$, the following conditions are equivalent.*
(1) *$S$ is Regular and intra-regular.*
(2) *$f\Gamma f = f$ for all fuzzy Γ-bi-ideals of $S$.*
(3) *$f_1 \cap f_2 = (f_1\Gamma f_2) \cap (f_2\Gamma f_1)$ for all fuzzy Γ-bi-ideals $f_1$, $f_2$ of $S$.*

**Proof.** $(1) \Rightarrow (2)$. Let $x \in S$. Since $S$ is regular, so there exist elements $a \in R$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha a)\beta x$, also $S$ is intra-regular, so by definition $x = (s\alpha x)\beta(x\gamma t)$. Now,

$$\begin{aligned}
x &= (x\alpha a)\beta x = (x\alpha a)\beta((x\alpha a)\beta x) = ((s\alpha x)\beta(x\gamma t))\alpha(e\eta a) \\
&= ((s\alpha x)\beta e)\alpha((x\gamma t)\eta a) = ((e\alpha x)\beta s)\alpha((x\gamma t)\eta a) = (x\beta s)\alpha((x\gamma t)\eta a) \\
&= (x\beta s)\alpha((((x\alpha a)\beta x)\gamma t)\eta a) = (x\beta s)\alpha(((t\beta x)\gamma(x\alpha a))\eta a) \\
&= (x\beta s)\alpha((a\gamma(x\alpha a))\eta(t\beta x)) = (x\beta s)\alpha((x\gamma(a\alpha a))\eta(t\beta x)) \\
&= (x\beta s)\alpha(((e\mu x)\gamma(a\alpha a))\eta(t\beta x)) = (x\beta s)\alpha((((a\alpha a)\mu x)\gamma e)\eta(t\beta x)) \\
&= (x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta(e\beta x)) = (x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta x) \\
&= (x\beta s)\alpha(((t\mu x)\gamma(a\alpha a))\eta x) = (x\beta s)\alpha((((e\mu t)\mu x)\gamma(a\alpha a))\eta x) \\
&= (x\beta s)\alpha((((x\mu t)\mu e)\gamma(a\alpha a))\eta x) = (x\beta s)\alpha((((a\alpha a)\mu e)\gamma(x\mu t))\eta x) \\
&= (x\beta s)\alpha((((e\alpha a)\mu a)\gamma(x\mu t))\eta x) = (x\beta s)\alpha(((a\mu a)\gamma(x\mu t))\eta x) \\
&= (x\beta s)\alpha((x\gamma((a\mu a)\mu t))\eta x) = (x\beta s)\alpha((x\gamma z)\eta x), where z = (a\mu a)\mu t \\
&= (((x\gamma z)\eta x)\beta s)\alpha x = ((s\eta x)\beta(x\gamma z))\alpha x = (x\beta((s\eta x)\gamma z))\alpha x.
\end{aligned}$$

Now, consider

$$\begin{aligned}
(f\Gamma f)(x) &= \bigvee_{x = a\rho b} \{f(a) \wedge f(b)\} \geq f((x\beta((s\eta x)\gamma z))\alpha x) \wedge f((x\alpha a)\beta x) \\
&\geq (f(x) \wedge f(x) = f(x).
\end{aligned}$$

So, $f \subseteq f\Gamma f$. Also $f\Gamma f \subseteq f$. Hence it follows that $f = f\Gamma f$.
$(2) \Rightarrow (3)$. Let $f_1$, $f_2$ are fuzzy Γ-bi-ideals of $S$. Then obviously $f_1 \cap f_2$ is also a fuzzy Γ-bi-ideal of $S$. So by hypothesis, we have $f_1 \cap f_2 = (f_1 \cap f_2)\Gamma(f_1 \cap f_2) \subseteq f_1\Gamma f_2$. Also $f_1 \cap f_2 = (f_1 \cap f_2)\Gamma(f_1 \cap f_2) \subseteq f_2\Gamma f_1$. It follows that $f_1 \cap f_2 \subseteq (f_1\Gamma f_2) \cap (f_2\Gamma f_1)$. Now we claim that $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of $S$. For this, we show that

$((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) \subseteq (f_1\Gamma f_2)$,

$$((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) = ((f_1\Gamma f_2)\Gamma(S\Gamma S))\Gamma(f_1\Gamma f_2)$$
$$= ((f_1\Gamma S)\Gamma(f_2\Gamma S))\Gamma(f_1\Gamma f_2) = ((f_1\Gamma S)\Gamma f_1)((f_2\Gamma S)\Gamma f_2) \subseteq f_1\Gamma f_2.$$

Consequently, $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of $S$. Similarly $f_2\Gamma f_1$ is also a fuzzy Γ-bi-ideal of $S$. As intersection of fuzzy Γ-bi-ideals of $S$ is also fuzzy Γ-bi-ideal of $S$, so $(f_1\Gamma f_2)\cap(f_2\Gamma f_1)$ is a fuzzy Γ-bi-ideal. Then, by hypothesis

$$(f_1\Gamma f_2) \cap (f_2\Gamma f_1) = ((f_1\Gamma f_2) \cap (f_2\Gamma f_1))\Gamma((f_1\Gamma f_2) \cap (f_2\Gamma f_1))$$
$$\subseteq (f_1\Gamma f_2)\Gamma(f_2\Gamma f_1) \subseteq (f_1\Gamma S)\Gamma(S\Gamma f_1) = ((S\Gamma f_1)\Gamma S)\Gamma f_1 = (((Se)\Gamma f_1)\Gamma S)\Gamma f_1$$
$$= (((f_1 e)\Gamma S)\Gamma S)\Gamma f_1 = ((f_1\Gamma S)\Gamma S)\Gamma f_1 = ((S\Gamma S)\Gamma f_1)\Gamma f_1 = (S\Gamma f_1)\Gamma f_1$$
$$= ((Se)\Gamma f_1)\Gamma f_1 = ((f_1 e)\Gamma S)\Gamma f_1 = (f_1\Gamma S)\Gamma f_1 \subseteq f_1.$$

Also, $(f_1\Gamma f_2) \cap (f_2\Gamma f_1) \subseteq f_2$, hence it follows that $(f_1\Gamma f_2) \cap (f_2\Gamma f_1) \subseteq f_1 \cap f_2$. Hence $f_1 \cap f_2 = (f_1\Gamma f_2) \cap (f_2\Gamma f_1)$.
$(3) \Rightarrow (1)$. Let $R$ be a fuzzy right Γ- ideal and $L$ a fuzzy Γ-ideal of $S$. Then $R$ and $L$ are fuzzy Γ-bi-ideals of $S$. As every fuzzy right Γ-ideal and fuzzy two-sided Γ-ideal is a fuzzy Γ-bi-ideal of $S$. By hypothesis we have $R\cap L = (R\Gamma L)\cap(L\Gamma R)$. This implies $R\cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$ and $R\cap L \subseteq L\Gamma R \Rightarrow R\cap L \subseteq R\Gamma L$. Since $R\Gamma L \subseteq R\cap L$ always true. Hence it follows that $R\cap L = R\Gamma L$ and $R\cap L \subseteq L\Gamma R$. Hence $S$ is regular and intra-regular. □

**Theorem 5.2.** *If $S$ is a Γ-AG-groupoid with left identity $e$, then following conditions are equivalent.*
(1) *$S$ is regular and intra-regular.*
(2) *Every fuzzy Γ-quasi-ideal of $S$ is Γ-idempotent.*

**Proof.** $(1) \Rightarrow (2)$. Let $g$ be a fuzzy Γ-quasi-ideal of $S$. Since $g\Gamma g \subseteq g$, so we only show that $g \subseteq g\Gamma g$. If $x \in S$, then we have by definition $x = (x\alpha a)\beta x$, also $S$ is intra-regular, so there exist $s, t \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = (s\alpha x)\beta(x\gamma t)$. Consider

$$x = (x\alpha a)\beta x = (((s\alpha x)\beta(x\gamma t))\alpha a)\beta x = ((a\beta(x\gamma t))\alpha(s\alpha x))\beta x$$
$$= ((x\beta(a\gamma t))\alpha(s\alpha x))\beta x = (((s\alpha x)\beta(a\gamma t))\alpha x)\beta x$$
$$= ((((e\eta s)\alpha x)\beta(a\gamma t))\alpha x)\beta x = ((((x\eta s)\alpha e)\beta(a\gamma t))\alpha x)\beta x$$
$$= ((((a\gamma t)\alpha e)\beta(x\eta s))\alpha x)\beta x = ((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x)\beta x.$$

Now, $(g\Gamma g)(x) = \bigvee_{x=y\mu z}\{g(y) \wedge g(z)\} \geq g((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x) \wedge g(x) \geq (g(x) \wedge g(x)) \wedge g(x) = g(x) \Rightarrow g \subseteq g\Gamma g$.

Hence, it follows that $g = g\Gamma g$.
$(2) \Rightarrow (1)$ If $a \in S$, then $S\Gamma a$ is a right Γ-ideal of $S$ containing $a$. Since every right Γ-ideal is Γ-quasi ideal of $S$, so $S\Gamma a$ is a Γ-quasi-ideal of $S$. This implies $C_{S\Gamma a}$ a fuzzy Γ-quasi- ideal of $S$. Then by (2), $C_{S\Gamma a} = C_{S\Gamma a}\Gamma C_{S\Gamma a} = C_{(S\Gamma a)(S\Gamma a)}$. It follows that $S\Gamma a = (S\Gamma a)(S\Gamma a)$. As $a \in S\Gamma a$, so $a \in (S\Gamma a)(S\Gamma a)$. Hence $a$ is regular and intra-regular. Consequently $S$ is regular and intra-regular. □

**Theorem 5.3.** *In a Γ-AG-groupoid S with left identity e, the following conditions are equivalent.*
(1) *$S$ is regular and intra-regular.*
(2) *Every fuzzy $\Gamma$-quasi-ideal of $S$ is $\Gamma$-idempotent.*
(3) *Every fuzzy $\Gamma$-bi-ideal of $S$ is $\Gamma$-idempotent.*

**Proof.** $(1) \Rightarrow (3)$. Obvious.
$(3) \Rightarrow (2)$. Straightforward because every fuzzy Γ-quasi-ideal of $S$ is a fuzzy Γ-bi-ideal of $S$.
$(2) \Rightarrow (1)$. Obvious. □

**Theorem 5.4.** *If S is a Γ-AG-groupoid with left identity e, then following conditions are equivalent.*
(1) *$S$ is regular and intra-regular.*
(2) *$f_1 \cap f_2 \subseteq f_1 \Gamma f_2$ for all fuzzy $\Gamma$-quasi-ideals $f_1$, $f_2$ of $S$.*
(3) *$f \cap g \subseteq f \Gamma g$ for every fuzzy $\Gamma$-quasi-ideal $f$ and every fuzzy $\Gamma$-bi-ideal $g$ of $S$.*
(4) *$g \cap f \subseteq g \Gamma f$ for every fuzzy $\Gamma$- bi-ideal $g$ and every fuzzy $\Gamma$-quasi-ideal $f$ of $S$.*
(5) *$g_1 \cap g_2 \subseteq g_1 \Gamma g_2$ for all fuzzy $\Gamma$-bi-ideals $g_1, g_2$ of $S$.*

**Proof.** $(1) \Rightarrow (5)$. Let $g_1$, $g_2$ are fuzzyΓ-bi-ideals of $S$. Then $g_1 \cap g_2$ is also a fuzzy Γ-bi-ideal of $S$. Also by Theorem 9, every fuzzy Γ-bi-ideal in $S$ is Γ-idempotent. It follows that

$$g_1 \cap g_2 = (g_1 \cap g_2)\Gamma(g_1 \cap g_2) \subseteq g_1 \Gamma g_2.$$

$(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$. Straightforward.
$(2) \Rightarrow (1)$. Let $R$ be a fuzzy right Γ- ideal and $L$ is a fuzzy left Γ-ideal of $S$. Since every fuzzy right and fuzzy left Γ-ideal of $S$ is fuzzy Γ-quasi-ideal of $S$. By hypothesis, we have $R \cap L \subseteq R \Gamma L$. Also it is always true that $R \Gamma L \subseteq R \cap L$. Thus $R \cap L = R \Gamma L$. Hence $S$ is regular by Theorem 1. Again by (2), $R \cap L = L \cap R \subseteq L \Gamma R$. It follows that $R \cap L \subseteq L \Gamma R$. Hence $S$ is intra-regular. □

## 6. Conclusion

Due to importance of the concept of non-associativity, it is a general prediction that in coming years, non-associativity will govern mathematics and applied sciences. If we look into literature, we can observe that the study of fuzzy sets was previously mostly restricted to associative algebraic structures, but on the other hand there are many sets which possess non-associative binary operations and we can dig out examples which investigate a non-associative structure in the context of fuzzy sets. In this paper we made an effort to use the theory of fuzzy sets to a non-associative algebraic structure (*Γ*-AG-groupoid). We introduced the notions of regular and intra-regular Γ-AG-groupoids. By defining fuzzy Γ-quasi-ideals, fuzzy Γ-interior ideals, fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals, we investigated various related properties. To extend this work, one could study the characterization of *Γ*-AG-groupoids in terms of fuzzy M-Systems.

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