On a new class of derivations on residuated lattices

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Abstract. In this paper, as a generalization of a derivation in a residuated lattice, the notion of an f-derivation of a residuated lattice is proposed, and some related properties of isotone (resp. contractive) f-derivations and ideal f-derivations are investigated. The properties of principal ideal f-derivations are also investigated. We obtain that the fixed point set of principal ideal f-derivations and their implicative f-derivation are order isomorphism. Finally, by using the fixed point set of principal ideal f-derivations, we give a characterization of Heyting algebras.
Keywords: residuated lattice, derivation, f-derivation, principal ideal f-derivation.

1. Introduction

Derivation is helpful to the research of structure and property in algebraic systems, which was introduced from analytic theory. They were many authors who studied derivations in various algebraic structures. In 1957, Posner [12] proposed the notion of derivations in a prime ring $(R, +, -)$. In 2004, Jun and Xin [8] applied the notion of derivations to BCI-algebras. Based on [8], Zhan and Liu [18] introduced the notion of f-derivations of BCI-algebras. In 2008, Xin et al. [17] proposed the concept of a derivation on a lattice $(L, \wedge, \vee)$. They studied some properties of derivations and characterized modular and distributive lattices by some special derivations. Based on [17], Çeven and Özturk [4] introduced the notion of an f-derivation on a lattice. In 2010, Alshehri [1] introduced the concept of derivations in MV-algebra, and some related properties are investigated. In 2016, Xiao and Liu [16] introduced the notion of derivations for a quantale. In the same year, He et al. [7] introduced the concept of derivations in a residuated lattice, and then they characterized some special types of residuated lattices in terms of derivations. In 2018, Rachůnek and Šalounová [13] introduced the

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concept of derivations on non-commutative generalization of MV-algebras, and a complete description of all derivations on any non-commutative generalization of MV-algebras was given. In the same year, Liang et al. [9] introduced the notions of derivations on EQ-algebras and obtained several special types of them. Further, Wang et al. [14] introduced the notion of derivations of commutative multiplicative semilattices, they investigated related properties of some particular derivations and gave some characterizations of regular derivations in commutative multiplicative semilattices.

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In this paper, as a generalization of a derivation in a residuated lattice, the notion of an f-derivation is introduced, and some related properties are investigated. As important results of f-derivations on residuated lattices, fixed point sets are characterized by f-derivations under some conditions.

This paper is organized as follows. In Section 2, we recall some concepts and results on residuated lattices. In Section 3, we investigate and characterize some particular f-derivations in residuated lattices. Also, we discuss some properties of the fixed point set for ideal f-derivations and some operations of the kernel for f-derivations. In Section 4, we investigate principal ideal f-derivations and discuss their implicative f-derivations. In particular, we characterize Heyting algebras in terms of principal ideal f-derivations.

2. Preliminaries

In this section, we recall some basic notions and results about residuated lattices, which will be used in the further.

Definition 2.1 ([15]). A residuated lattice is an algebraic structure $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following conditions:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(L, \otimes, 1)$ is a commutative monoid,

(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$.

In what follows, we denote by L a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, unless otherwise specified.

For any $x \in L$ and a natural number n, we define $x' = x \rightarrow 0, x'' = (x')' = (x \rightarrow 0) \rightarrow 0, x^0 = 1$ and $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

Proposition 2.2 ([15]). For all $x, y, z, w \in L$, the following properties hold.

(1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$.

(2) $x \leq y$ if and only if $x \rightarrow y = 1$.

(3) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

(4) if $x \leq z$ and $y \leq w$ then $x \otimes y \leq z \otimes w$.

(5) $x \otimes y \leq x \wedge y$.

(6) $x \to (y \to z) = x \otimes y \to z = y \to (x \to z)$.
(7) $0' = 1, 1' = 0, x \leq x''$.
(8) $x \otimes y = 0$ if and only if $x \leq y'$.
(9) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

Definition 2.3 ([6]). L is called to be:
(1) divisible if $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$,
(2) idempotent if $x \otimes x = x$ for all $x \in L$.

In what follows, we recall the concept of Heyting algebras.

Definition 2.4 ([3]). A lattice $(L, \vee, \wedge)$ is called to be a Heyting algebra if for any $x, y \in L$, there exists $x \to y \in L$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$ for all $z \in L$.

Theorem 2.5 ([11]). Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice. Then, the following statements are equivalent:
(1) L is a Heyting algebra.
(2) $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$.

In what follows, we recall the notion of filters on residuated lattices, see [11].

Definition 2.6 ([11]). Let $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if:
(1) for any $x, y \in F$, $x \otimes y \in F$,
(2) for any $x, y \in L$, $x \leq y$ and $x \in F$ imply $y \in F$.

An equivalent definition for a filter in a residuated lattice L is: (1) $1 \in F$, (2) if $x, x \to y \in F$, then $y \in F$. Hence, a filter is also called a deductive system. We denote by $F(L)$ the set of all filters in $(L, \vee, \wedge, \otimes, \to, 0, 1)$.

For a nonempty subset W of L, we denote by $\langle W \rangle$ the filter generated by W. One can check that $\langle W \rangle = \{x \in L | x \geq x_1 \otimes x_2 \otimes \cdots \otimes x_n, x_i \in W, i = 1, 2 \cdots, n\}$.

Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice. We denote by $B(L)$ the set of all complement element of the lattice $(L, \wedge, \vee, 0, 1)$, see [2]. The set $B(L)$ is called the Boolean center of L. For any $t \in L$, $t \in B(L)$ if and only if $t \vee t' = 1$.

A mapping $f : L \longrightarrow L$ is called a homomorphism if it satisfies the following conditions:
(1) $f(0) = 0$, $f(1) = 1$,
(2) $f(x * y) = f(x) * f(y)$, for all $x, y \in L$ and $* \in \{\vee, \wedge, \otimes, \to\}$.
Moreover, f is called an isomorphism if it is bijective.

Proposition 2.7 ([10]). For any $t \in B(L)$, the following statements hold:
(1) $t \otimes t = t$.
(2) $t \otimes x = t \wedge x$ for any $x \in L$.

A nonempty subset I of L is called a lattice ideal of L if it satisfies: (1) for all $x, y \in I, x \vee y \in I$, (2) for all $x, y \in L$, if $x \in I$ and $y \leq x$, then $y \in I$, i.e., a lattice ideal of L is the notion of ideal in the underlying lattice $(L, \vee, \wedge)$.

For a nonempty subset A of L, the smallest lattice ideal containing A is called the lattice ideal generated by A. The lattice ideal generated by A will be denoted by $(A]$. In particular, if $A = \{a\}$, we write $(a]$ for $\{(a]\}$, $(a]$ is called a principal lattice ideal of L. It is easy to check that $(a] = \downarrow a = \{x \in L | x \leq a\}$, see [5].

At the end of this section, we recall the notion of a derivation in a residuated lattice [7] .

Definition 2.8 ([7])**.** A mapping $d : L \longrightarrow L$ is called a derivation on L if it satisfies the following conditions: for any $x, y \in L$,

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

3. f-derivations on residuated lattices

By means of the idea in [4], [7] and [18], in this section, as a generalization of a derivation on a residuated lattice, the notion of an f-derivation for a residuated lattice is introduced and some related properties are investigated. Firstly, we give the concept of an f-derivation in a residuated lattice as follows.

Definition 3.1. Let $f : L \longrightarrow L$ be a homomorphism. A mapping $d_f : L \longrightarrow L$ is called a multiplicative f-derivation on L if

$$d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)),$$

for any $x, y \in L$.

In what follows, unless otherwise stated, a multiplicative f-derivation is called an f-derivation on L. The pair (L, d_f) is said to be a residuated lattice with f-derivations.

Remark 3.2. It is obvious in Definition 3.1 that if f is an identity mapping, i.e., $f(x) = x$ for all $x \in L$, then d_f is a derivation on L. Under such a condition, f-derivation coincide with derivation, see Definition 2.8. Moreover, according to Definition 3.1, a mapping d_f on L is an f-derivation only when a homomorphism f satisfying the above equation. However, to obtain some results, d_f or f must satisfy some additional conditions in the following propositions and theorems.

In what follows, $f : L \longrightarrow L$ is always a homomorphism, unless otherwise specified. Next, we show two examples for f-derivations on residuated lattices.

Example 3.3. Let $t \in L$. Define a mapping $d_f : L \longrightarrow L$ by $d_f(x) = f(x) \otimes t$ for all $x \in L$. One can check that d_f is an f-derivation on L.

Example 3.4. Let $L = \{0, a, b, 1\}$ be a chain and operations $\otimes$ and $\rightarrow$ be defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that $L = \{0, a, b, 1\}$ is a residuated lattice. Now, we define a mapping $f : L \longrightarrow L$ as follows: for all $x \in L$,

$$f(x) = \begin{cases} 0, & x = 0, a, \\ b, & x = b, \\ 1, & x = 1. \end{cases}$$

Further, we define a mapping $d_f : L \longrightarrow L$ determined by f as follows: for all $x \in L$,

$$d_f(x) = \begin{cases} 0, & x = 0, a, \\ b, & x = b, 1. \end{cases}$$

One can check that d_f is an f-derivation on L.

Next, we present some properties of f-derivations in residuated lattices.

Proposition 3.5. Let d_f be an f-derivation on L. Then the following statements hold.

(1) $d_f(0) = 0$.

(2) $f(x) \otimes d_f(1) \leq d_f(x)$ for all $x \in L$.

(3) If $x \leq y'$, then $d_f(x) \leq f(y)'$ and $d_f(y) \leq f(x)'$ for all $x, y \in L$.

Proof. (1) It follows from Definition 3.1 that $d_f(0) = d_f(0 \otimes 0) = (d_f(0) \otimes f(0)) \vee (f(0) \otimes d_f(0)) = d_f(0) \otimes f(0)$. Since $f(0) = 0$, we have $d_f(0) = 0$.

(2) Let $x \in L$. Then we have $d_f(x) = d_f(x \otimes 1) = (d_f(x) \otimes f(1)) \vee (f(x) \otimes d_f(1)) = d_f(x) \vee (f(x) \otimes d_f(1))$, which implies $f(x) \otimes d_f(1) \leq d_f(x)$.

(3) For all $x, y \in L$, suppose that $x \leq y'$. It follows from Proposition 2.2 (8) that $x \otimes y = 0$. Then $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = 0$, which implies $d_f(x) \otimes f(y) = 0$ and $f(x) \otimes d_f(y) = 0$. Therefore, $d_f(x) \leq f(y)'$ and $d_f(y) \leq f(x)'$ for all $x, y \in L$. □

Proposition 3.6. Let d_f be an f-derivation on L. If $d_f(1) = 1$, then $f(x) \otimes f(y) \leq d_f(x \otimes y)$ for all $x, y \in L$.

Proof. It follows from Definition 3.1 that $d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. If $d_f(1) = 1$, then it follows from Proposition 3.5 (2) that $f(x) \leq d_f(x)$. Therefore, $f(x) \otimes f(y) \leq d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. □

In order to discuss some related properties of f-derivations, in what follows, we first introduce three kinds of special f-derivations.

Definition 3.7. Let d_f be an f-derivation on L. Then for all $x, y \in L$,

(1) if $x \leq y$ implies $d_f(x) \leq d_f(y)$, we call d_f an isotone f-derivation,

(2) if $d_f(x) \leq f(x)$, we call d_f a contractive f-derivation.

In particular, if d_f is both isotone and contractive, we call d_f an ideal f-derivation.

Example 3.8. Consider the Example 3.3. One can check that d_f is an ideal f-derivation on L.

Now, some properties of isotone f-derivations and contractive f-derivations are investigated, respectively, most of which are obtained from the adjoint pair in residuated lattices.

Proposition 3.9. Let d_f be an isotone f-derivation on L. Then the following statements hold.

(1) If $z \leq x \to y$, then $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$ for all $x, y, z \in L$.

(2) $f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$.

(3) $f(x) \leq d_f(y) \to d_f(x)$ and $f(y) \leq d_f(x) \to d_f(y)$ for all $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since d_f is an isotone f-derivation on L, we have $d_f(x \otimes z) \leq d_f(y)$. It follows from Definition 3.1 that $d_f(x \otimes z) = (d_f(x) \otimes f(z)) \vee (f(x) \otimes d_f(z))$. Thus, $(d_f(x) \otimes f(z)) \vee (f(x) \otimes d_f(z)) \leq d_f(y)$, which implies $d_f(x) \otimes f(z) \leq d_f(y)$ and $f(x) \otimes d_f(z) \leq d_f(y)$. Therefore, $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$.

(2) Since $x \otimes (x \to y) \leq y$ for all $x, y \in L$, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. It follows from Definition 3.1 that $d_f(x \otimes (x \to y)) = (d_f(x) \otimes f(x \to y)) \vee (f(x) \otimes d_f(x \to y))$, which implies $d_f(x) \otimes f(x \to y) \leq d_f(y)$ and $f(x) \otimes d_f(x \to y) \leq d_f(y)$. Therefore, $f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $d_f(x \otimes y) \leq d_f(x)$. It follows from Definition 3.1 that $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y))$. Thus, $f(x) \leq d_f(y) \to d_f(x)$. In a similar way, we have $f(y) \leq d_f(x) \to d_f(y)$. □

Proposition 3.10. Let d_f be a contractive f-derivation on L. Then the following statements hold.

(1) $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$ for all $x, y \in L$.

(2) If d_f is isotone, then $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$ for all $x, y \in L$.

(3) If $d_f(1) = 1$, then $d_f(x) = f(x)$ for all $x \in L$.

Proof. (1) Since d_f is a contractive f-derivation on L, we have $d_f(x) \otimes d_f(y) \leq f(x) \otimes d_f(y)$ and $d_f(x) \otimes d_f(y) \leq d_f(x) \otimes f(y)$ for all $x, y \in L$. Thus, for all $x, y \in L$

$$d_f(x) \otimes d_f(y) \leq (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = d_f(x \otimes y).$$

On the other hand, since $d_f(x) \otimes f(y) \leq d_f(x)$ and $f(x) \otimes d_f(y) \leq d_f(y)$, we have

$$d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) \leq d_f(x) \vee d_f(y).$$

Therefore, $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$.

(2) For all $x, y \in L$, since $x \otimes (x \to y) \leq y$ and d_f is isotone, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. It follows from the statement (1) that $d_f(x \to y) \otimes d_f(x) \leq d_f(x \otimes (x \to y))$, which implies $d_f(x \to y) \otimes d_f(x) \leq d_f(y)$, i.e., $d_f(x \to y) \leq d_f(x) \to d_f(y)$. On the other hand, since $d_f(y) \leq f(y)$, we have $d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$. Therefore, $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$.

(3) It follows from Proposition 3.5 (2) that $f(x) \otimes d_f(1) \leq d_f(x)$ for all $x \in L$. If $d_f(1) = 1$, then we have

$$f(x) = f(x) \otimes d_f(1) \leq d_f(x) \leq f(x),$$

which implies $d_f(x) = f(x)$ for all $x \in L$. □

Theorem 3.11. Let d_f be an f-derivation on L. For all $x, y \in L$, if d_f, f satisfy $d_f(x) \to d_f(y) = d_f(x) \to f(y)$, then d_f is an ideal f-derivation on L.

Proof. Let $d_f(x) \to d_f(y) = d_f(x) \to f(y)$ for all $x, y \in L$. Since $d_f(x) \otimes 1 \leq d_f(x)$, we have $1 \leq d_f(x) \to d_f(x) = d_f(x) \to f(x)$. Thus $d_f(x) \otimes 1 \leq f(x)$ for all $x \in L$, which implies d_f is contractive. On the other hand, let $x \leq y, x, y \in L$. Since f is a homomorphism, we have $f(x) \leq f(y)$. Thus $d_f(x) \otimes 1 \leq d_f(x) \leq f(x) \leq f(y)$, i.e., $d_f(x) \otimes 1 \leq f(y)$, which implies $1 \leq d_f(x) \to f(y) = d_f(x) \to d_f(y)$, i.e., $d_f(x) \otimes 1 \leq d_f(y)$. Thus, $d_f(x) \leq d_f(y)$, i.e., d_f is isotone. Therefore, d_f is an ideal f-derivation on L. □

Theorem 3.12. Let d_f be a contractive f-derivation on L, which satisfies $d_f(1) \in B(L)$. Then the following statements are equivalent.

(1) d_f is an ideal f-derivation on L.

(2) $d_f(x) \leq d_f(1)$ for all $x \in L$.

(3) $d_f(x) = d_f(1) \otimes f(x)$ for all $x \in L$.

(4) $d_f(x \wedge y) = d_f(x) \wedge d_f(y)$ for all $x, y \in L$.

(5) $d_f(x \vee y) = d_f(x) \vee d_f(y)$ for all $x, y \in L$.

(6) $d_f(x \otimes y) = d_f(x) \otimes d_f(y)$ for all $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Let $d_f(x) \leq d_f(1)$ for all $x \in L$. If $d_f(1) \in B(L)$, then it follows from Proposition 2.7 (2) that

$$d_f(x) = d_f(1) \wedge d_f(x) = d_f(1) \otimes d_f(x) \leq d_f(1) \otimes f(x).$$

On the other hand, it follows from Proposition 3.5 (2) that $f(x) \otimes d_f(1) \leq d_f(x)$. Therefore, $d_f(x) = d_f(1) \otimes f(x)$.

(3) ⇒ (4) Let $d_f(x) = d_f(1) \otimes f(x)$ for all $x \in L$. Then for all $x, y \in L$,

$$\begin{aligned} d_f(x \wedge y) &= d_f(1) \otimes f(x \wedge y) \\ &= d_f(1) \otimes (f(x) \wedge f(y)) \\ &= (d_f(1) \wedge f(x)) \wedge (d_f(1) \wedge f(y)) \\ &= (d_f(1) \otimes f(x)) \wedge (d_f(1) \otimes f(y)) \\ &= d_f(x) \wedge d_f(y). \end{aligned}$$

(4) ⇒ (1) Let $x \leq y, x, y \in L$. It follows from (4) that $d_f(x) = d_f(x \wedge y) = d_f(x) \wedge d_f(y)$, which implies $d_f(x) \leq d_f(y)$ for all $x, y \in L$. Since d_f is a contractive f-derivation, we have d_f is an ideal f-derivation on L.

(3) ⇒ (5) For all $x, y \in L$, it follows from (3) and Proposition 2.2 (9) that

$$\begin{aligned} d_f(x \vee y) &= d_f(1) \otimes f(x \vee y) \\ &= d_f(1) \otimes (f(x) \vee f(x)) \\ &= (d_f(1) \otimes (f(x))) \vee (d_f(1) \otimes f(x)) \\ &= d_f(x) \vee d_f(y). \end{aligned}$$

(5) ⇒ (1) Let $x, y \in L$ and $x \leq y$. It follows from (5) that $d_f(y) = d_f(x \vee y) = d_f(x) \vee d_f(y)$, which implies $d_f(x) \leq d_f(y)$, i.e., D is isotone. Since d_f is a contractive f-derivation, we have d_f is an ideal f-derivation on L.

(3) ⇒ (6) For all $x, y \in L$, it follows from (3) that

$$\begin{aligned} d_f(x \otimes y) &= d_f(1) \otimes f(x \otimes y) \\ &= d_f(1) \otimes (f(x) \otimes f(y)) \\ &= (d_f(1) \otimes f(x)) \otimes (d_f(1) \otimes f(x)) \\ &= d_f(x) \otimes d_f(y). \end{aligned}$$

(6) ⇒ (2) For all $x \in L$, it follows from (6) that

$$d_f(x) = d_f(x \otimes 1) = d_f(x) \otimes d_f(1) = d_f(x) \wedge d_f(1).$$

Thus, $d_f(x) \leq d_f(1)$. □

Let d_f be an f-derivation on L. Define a set $Fix_{d_f}(L) = \{x \in L | d_f(x) = f(x)\}$.

Now, we investigate some operations on $Fix_{d_f}(L)$.

Proposition 3.13. Let d_f be an f-derivation on L. If $x, y \in Fix_{d_f}(L)$, then $x \otimes y \in Fix_{d_f}(L)$.

Proof. Let $x, y \in Fix_{d_f}(L)$. Then $d_f(x) = f(x)$, $d_f(y) = f(y)$. It follows from Definition 3.1 that

$$\begin{aligned} d_f(x \otimes y) &= (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) \\ &= (f(x) \otimes f(y)) \vee (f(x) \otimes f(y)) \\ &= f(x) \otimes f(y) \\ &= f(x \otimes y). \end{aligned}$$

Therefore, $x \otimes y \in Fix_{d_f}(L)$. □

Proposition 3.14. Let d_f be an ideal f-derivation on L. If $x, y \in Fix_{d_f}(L)$, then $x \vee y \in Fix_{d_f}(L)$.

Proof. Let $x, y \in Fix_{d_f}(L)$. Then $d_f(x) = f(x)$, $d_f(y) = f(y)$. Thus

$$f(x \vee y) = f(x) \vee f(y) = d_f(x) \vee d_f(y) \leq d_f(x \vee y) \leq f(x \vee y),$$

which implies $d_f(x \vee y) = f(x \vee y)$. Therefore, $x \vee y \in Fix_{d_f}(L)$. □

Next, we discuss some properties of $Fix_{d_f}(L)$.

Theorem 3.15. Let d_f be an ideal f-derivation on L. For all $x, y \in L$, if L satisfies the divisibility condition $x \wedge y = x \otimes (x \to y)$, then $Fix_{d_f}(L)$ is a lattice ideal of L.

Proof. It follows from Proposition 3.14 that $Fix_{d_f}(L)$ is closed under $\vee$, i.e., for any $x, y \in Fix_{d_f}(L)$, $x \vee y \in Fix_{d_f}(L)$. Now let $x \in Fix_{d_f}(L)$, $y \in L$ and $y \leq x$. Since d_f is an ideal f-derivation on L, we have

$$\begin{aligned} d_f(y) &= d_f(x \wedge y) \\ &= d_f(x \otimes (x \to y)) \\ &= (d_f(x) \otimes f(x \to y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x) \otimes f(x \to y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x \otimes (x \to y))) \vee (d_f(x \to y) \otimes f(x)) \\ &= (f(x \wedge y)) \vee (d_f(x \to y) \otimes f(x)) \\ &= f(y) \vee (d_f(x \to y) \otimes f(x)), \end{aligned}$$

which implies $f(y) \leq d_f(y)$. Since $d_f(y) \leq f(y)$, we have $d_f(y) = f(y)$, i.e., $y \in Fix_{d_f}(L)$. Therefore, $Fix_{d_f}(L)$ is a lattice ideal of L. □

Let d_f be an f-derivation on L. Define a set $Kerd_f(L) = \{x \in L | d_f(x) = 0\}$. Now, we investigate some operations on $Kerd_f(L)$.

Proposition 3.16. Let d_f be an isotone f-derivation on L. If $x, y \in Kerd_f(L)$, then $x \wedge y \in Kerd_f(L)$.

Proof. Let $x, y \in Kerd_f(L)$. Then $d_f(x) = 0$ and $d_f(y) = 0$. Since d_f is an isotone f-derivation on L, we have $d_f(x \wedge y) \leq d_f(x) \wedge d_f(y) = 0$, i.e., $d_f(x \wedge y) = 0$. Therefore, $x \wedge y \in Kerd_f(L)$. □

Proposition 3.17. Let d_f be an f-derivation on L. If $x, y \in Kerd_f(L)$, then $x \otimes y \in Kerd_f(L)$.

Proof. Let $x, y \in Kerd_f(L)$. Then $d_f(x) = 0$ and $d_f(y) = 0$. It follows from Definition 3.1 that $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = 0$. Therefore, $x \otimes y \in Kerd_f(L)$. □

Proposition 3.18. Let d_f be an isotone f-derivation on L. If $x \leq y$ and $y \in Kerd_f(L)$, then $x \in Kerd_f(L)$.

Proof. Let $x \leq y$ and $y \in Kerd_f(L)$. Then $d_f(y) = 0$. Since d_f is an isotone f-derivation on L, we have $d_f(x) \leq d_f(y) = 0$, i.e., $d_f(x) = 0$. Therefore, $x \in Kerd_f(L)$. □

Finally, we introduce a special kind of an f-derivation on L.

Definition 3.19. Let d_f be an f-derivation on L. If $d_f(x \otimes y) = d_f(x) \otimes d_f(x)$ for all $x, y \in L$, then d_f is called a $\otimes$-f-derivation on L.

Example 3.20. Define an f-derivation by $d_f(x) = x$. One can check that d_f is a $\otimes$-f-derivation on L.

For a residuated lattice L, we define a set $Ide(L) = \{x \in L | x \otimes x = x\}$.

Theorem 3.21. Let d_f be a $\otimes$-f-derivation on L. If $d_f(x) \neq 0$ for all $x \in L$, then $d_f(Ide(L)) \subseteq Ide(L)$.

Proof. Let $y \in d_f(Ide(L))$. Then $y = d_f(x)$ for some $x \in Ide(L)$. Thus

$$y \otimes y = d_f(x) \otimes d_f(x) = d_f(x \otimes x) = d_f(x) = y.$$

Therefore, $y \in Ide(L)$, i.e., $d_f(Ide(L)) \subseteq Ide(L)$. □

4. Principal ideal f-derivations and their applications on residuated lattices

In this section, we investigate principal ideal f-derivations and discuss their applications. In particular, we characterize Heyting algebras in terms of principal ideal f-derivations.

In what follows, let $a \in L$. We define a mapping $d_{(f,a)} : L \longrightarrow L$ as follows: $d_{(f,a)}(x) = a \otimes f(x)$ for all $x \in L$.

Theorem 4.1. Let $a \in L$. Then the mapping $d_{(f,a)}$ is an ideal f-derivation on L.

Proof. Let $x, y \in L$. Then

$$\begin{aligned} d_{(f,a)}(x \otimes y) &= a \otimes f(x \otimes y) \\ &= (a \otimes f(x \otimes y)) \vee (a \otimes f(x \otimes y)) \\ &= (a \otimes f(x) \otimes f(y)) \vee (a \otimes f(x) \otimes f(y)) \\ &= ((a \otimes f(x)) \otimes f(y)) \vee (f(x) \otimes (a \otimes f(y))) \\ &= (d_{(f,a)}(x) \oplus f(y)) \vee (f(x) \otimes d_{(f,a)}(y)). \end{aligned}$$

Let $x \leq y$. Since f is a homomorphism, we have $f(x) \leq f(y)$. Thus $d_{(f,a)}(x) = a \otimes f(x) \leq a \otimes f(y) = d_{(f,a)}(y)$, which implies $d_{(f,a)}$ is isotone. Moreover, it follows from Proposition 2.1 (6) that $d_{(f,a)}(x) = a \otimes f(x) \leq f(x)$ for all $x \in L$, which implies $d_{(f,a)}$ is contractive. Therefore, $d_{(f,a)}$ is an ideal f-derivation on L. □

Remark 4.2. In Theorem 4.1, $d_{(f,a)}$ is called a principal ideal f-derivation on L.

Now, we discuss the properties of principal ideal f-derivations. First of all, we introduce a new derivation in residuated lattices.

Definition 4.3. A mapping $s_f : L \longrightarrow L$ is called an implicative f-derivation on L if it satisfies the following condition: for any $x, y \in L$,

$$s_f(x \to y) = (s_f(x) \to f(y)) \vee (f(x) \to s_f(y)).$$

Example 4.4. Let $a \in L$. Define a mapping $s_{(f,a)} : L \longrightarrow L$ by $s_{(f,a)}(x) = a \to f(x)$ for all $x \in L$. One can check that that $s_{(f,a)}$ is an implicative f-derivation on L.

In order to discuss the properties of principal ideal f-derivations, we recall the notion of Galois connection, see [3]. Let M and N be two posets. We say that the pair (d, s) of mappings: $d : M \to N$ and $s : N \to M$ is a Galois connection between M and N if d and s satisfy: (1) both d and s are isotone, (2) $n \leq d(m)$ if and only if $s(n) \leq m$ for all $m \in M, n \in N$.

Based on the notion of Galois connection, we introduce the notion of f-connection based on residuated lattices.

Definition 4.5. We say that the pair (d_f, s_f) of mappings: $d_f : L \longrightarrow L$ and $s_f : L \to L$ is an f-connection if d_f and s_f satisfy:
(1) both d_f and s_f are isotone,
(2) $d_f(m) \leq f(n)$ if and only if $f(m) \leq s_f(n)$ for all $m, n \in L$.

Example 4.6. For a given $a \in L$, consider the implicative f-derivation $s_{(f,a)}$ in Example 4.4 and define a mapping $d_{(f,a)} : L \longrightarrow L$ by $d_{(f,a)}(x) = a \otimes f(x)$ for all $x \in L$. One can check that the pair $(d_{(f,a)}, s_{(f,a)})$ is an f-connection.

Definition 4.7. Let d_f be a multiplicative f-derivation on L. The derivation d_f is called to be residuated if there exists an implicative f-derivation s_f on L such that a pair (d_f, s_f) forms an f-connection.

Remark 4.8. It follows from the notion of f-connections, we know that if the f-derivation d_f is residuated, then d_f and s_f must be isotone. Moreover, if the f-derivation d_f has an implicative f-derivation s_f, then s_f is unique. We denote this unique by d_f^*.

Next, we discuss some properties of d_f^*.

Proposition 4.9. Let d_f be a multiplicative f-derivation on L. Then the following statements hold.

(1) $d_f^*(1) = 1$.
(2) $d_f^*(x) \geq f(x)$ for all $x \in L$.
(3) $d_f^*(x) \vee d_f^*(y) \leq d_f^*(x \to y) \vee d_f^*(y \to x)$ for all $x, y \in L$.
(4) $d_f^*(x \wedge y) = d_f^*(x) \wedge d_f^*(y)$ for all $x, y \in L$.

Proof. (1) It follows from Definition 4.3 that

$$\begin{aligned} d_f^*(1) &= d_f^*(1 \to 1) \\ &= (d_f^*(1) \to f(1)) \vee (f(1) \to d_f^*(1)) \\ &= (d_f^*(1) \to 1) \vee (f(1) \to d_f^*(1)) \\ &= 1 \vee (f(1) \to d_f^*(1)) \\ &= 1. \end{aligned}$$

(2) Let $x \in L$. Then

$$\begin{aligned} d_f^*(x) &= d_f^*(1 \to x) \\ &= (d_f^*(1) \to f(x)) \vee (f(1) \to d_f^*(x)) \\ &= (1 \to f(x)) \vee (f(1) \to d_f^*(x)) \\ &= f(x) \vee (f(1) \to d_f^*(x)), \end{aligned}$$

which implies $d_f^*(x) \geq f(x)$.

(3) Let $x, y \in L$. Since $f(y) \leq d_f^*(x) \to f(y)$, $d_f^*(y) \leq f(x) \to d_f^*(y)$. It follows from Definition 4.3 and (2) that

$$\begin{aligned} d_f^*(y) &= d_f^*(y) \vee f(y) \\ &\leq (d_f^*(x) \to f(y)) \vee (f(x) \to d_f^*(y)) \\ &= d_f^*(x \to y). \end{aligned}$$

In a similar way, we have $d_f^*(x) \leq d_f^*(y \to x)$. Therefore, $d_f^*(x) \vee d_f^*(y) \leq d_f^*(x \to y) \vee d_f^*(y \to x)$.

(4) Since d_f^* is isotone, we have $d_f^*(x \wedge y) \leq d_f^*(x), d_f^*(y)$ for all $x, y \in L$. Hence, $d_f^*(x \wedge y)$ is a lower bound of $d_f^*(x)$ and $d_f^*(y)$. Now, let $f(n) \in L$ be any lower bound of $d_f^*(x)$ and $d_f^*(y)$. Then we have $f(n) \leq d_f^*(x)$ and $f(n) \leq d_f^*(y)$. It follows from the notion of f-connections that $d(n) \leq f(x)$, $d(n) \leq f(y)$, which implies $d(n) \leq f(x) \wedge f(y) = f(x \wedge y)$, i.e., $f(n) \leq d_f^*(x \wedge y)$. Thus, $d_f^*(x \wedge y)$ is the infimum of $d_f^*(x)$ and $d_f^*(y)$. Therefore, $d_f^*(x \wedge y) = d_f^*(x) \wedge d_f^*(y)$. □

In what follows, we denote by $Fix_{d_{(f,a)}}(L)$ the set of all fixed points of L for $d_{(f,a)}$ and $Fix_{s_{(f,a)}}(L)$ the set of all fixed points of L for $s_{(f,a)}$, respectively, i.e., $Fix_{d_{(f,a)}}(L) = \{x \in L | d_{(f,a)}(x) = f(x)\}$ and $Fix_{s_{(f,a)}}(L) = \{x \in L | s_{(f,a)}(x) = f(x)\}$.

In order to show the relationship between $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$, we first give the following lemma.

Lemma 4.10. For all $x, y \in L$, $x \to (x \otimes y) = y$ if and only if there exists $z \in L$ such that $x \to z = y$.

Proof. It can be obtained immediately from Proposition 2.2 (3). □

Theorem 4.11. If $f : L \to L$ is an isomorphism, then ordered sets $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$ are isomorphic.

Proof. For all $a \in L$, let $p : Fix_{d_{(f,a)}}(L) \to Fix_{d_{(f,a)}}(L)$ be defined by $p(x) = a \to f(x)$ for all $x \in Fix_{d_{(f,a)}}(L)$. Then it is easy to see that p is a mapping from $Fix_{d_{(f,a)}}(L)$ to $Fix_{d_{(f,a)}}(L)$, i.e., p is well defined.

Now, we complete the proof by three steps.

(1) First, we show that p is injective. Let $x, y \in Fix_{d_{(f,a)}}(L)$. Then $d_{(f,a)}(x) = f(x) = a \otimes f(x)$, $d_{(f,a)}(y) = f(y) = a \otimes f(x)$. Suppose that $p(x) = p(y)$. Then $a \to f(x) = a \to f(y)$. It follows from $a \otimes f(x) \leq f(x)$ that $f(x) \leq a \to f(x)$, then $f(x) \leq a \to f(y)$, i.e., $f(x) \otimes a \leq f(y)$. Therefore, $f(x) \leq f(y)$. In a similar way, we have $f(y) \leq f(x)$. Thus, $f(x) = f(y)$. Since f is isomorphism, we have $x = y$. Therefore, p is injective.

(2) Next, we show that p is surjective. It follows from Lemma 4.10 that $x \to (x \otimes y) = y$ if and only if there exists $z \in L$ such that $x \to z = y$ for all $x, y \in L$. Now, for all $x \in Fix_{\delta_a}(L)$, then $f(x) = d_{(f,a)}(x) = a \otimes f(x)$. Thus, $f(f(x)) = d_{(f,a)}(f(x))$. Therefore, $f(x) \in Fix_{\delta_a}(L)$. Moreover,

$$\begin{aligned} p(a \otimes f(x)) &= a \to (a \otimes f(x)) \\ &= a \to (a \otimes (a \to f(x))) \\ &= a \to f(x) \\ &= f(x). \end{aligned}$$

It follows from f is an isomorphism that g is surjective.

(3) Finally, let $x \leq y$. Then it is easy to know that $f(x) \leq f(y)$. Then

$$p(x) = a \to f(x) \leq a \to f(y) = p(y),$$

which implies p is order preserving. Further, the inverse mapping $p^{-1} : Fix_{s_{(f,a)}}(L) \to Fix_{d_{(f,a)}}(L)$ is also order preserving, where $p^{-1}(x) = a \to f(x)$ for all $x \in Fix_{s_{(f,a)}}(L)$.

Hence, p is an order isomorphism from $Fix_{d_{(f,a)}}(L)$ to $Fix_{s_{(f,a)}}(L)$. Therefore, ordered sets $Fix_{d_{(f,a)}}(L)$ and $Fix_{s_{(f,a)}}(L)$ are isomorphic. □

Finally, we characterize Heyting algebra in terms of principal f-ideal derivations.

Theorem 4.12. Let $a \in L$ and $f(x) \leq x$. If L is a Heyting algebra, then $(a] \subseteq Fix_{d_{(f,a)}}(L)$.

Proof. Suppose that L is a Heyting algebra, we have $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. Taking $y = f(x)$, we have $x \otimes f(x) = x \wedge f(x) = f(x)$ for all $x \in L$. Since $d_{(f,a)}(x) = a \otimes f(x)$, we have $d_{(f,a)}(a) = a \otimes f(a) = f(a)$ for all $a \in L$, which implies $a \in Fix_{d_{(f,a)}}(L)$. Since L is a Heyting algebra, which satisfies $x \wedge y = x \otimes (x \to y)$, it follows from Theorem 3.15 that $Fix_{d_{(f,a)}}(L)$ is a lattice ideal of L, i.e., for all $x \in L$, if $x \leq a$, we have $x \in Fix_{d_{(f,a)}}(L)$, which implies $(a] \subseteq Fix_{d_{(f,a)}}(L)$. □

5. Conclusions

In this paper, as a generalization of a derivation in a residuated lattice, f-derivations is introduced, and some properties of isotone f-derivations, contractive f-derivations are discussed. The main conclusions in this paper and the further work to do are listed as follows.

(1) We investigated isotone (resp. contractive) f-derivations of residuated lattices and discussed some properties of the fixed point set for ideal f-derivations and some operations of the kernel for f-derivations.

(2) We obtained that the fixed point set of $d_{(f,a)}$ and $s_{(f,a)}$ are order isomorphism. In particular, by using the fixed point set of principal ideal f-derivations $d_{(f,a)}$, we gave a characterization of Heyting algebras.

As a further work, the following topics may be considered:

(1) Investigating some more properties related to f-derivations in residuated lattices.

(2) Studying f-derivations in EQ-algebras.

(3) Applying f-derivations to other algebras, such as skew residuated lattices, hyper MV-algebras and hyper BL-algebras and so on.

6. Acknowledgements

The authors would like to express their sincere thanks to the Editors and anonymous reviewers for their most valuable comments and suggestions in improving this paper greatly. This work was supported by the National Natural Science Foundation of China (No. 12001062) and the Natural Science Foundation of Hubei Province (No. 2020CFB368).

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Accepted: 20.08.2019