Some identities and generating functions of third-order recurrence relations

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Abstract. In this paper, we introduce a new generating functions for the product of Narayana numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers by making use of the symmetrizing endomorphism operators $\delta^k_{a_1a_2}$ to the series $\sum_{n=0}^{\infty} S_n(E)\, a_1^n z^n$.
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1. Introduction and notations

Many sequences of numbers can be defined by second-order recurrence relation, such as the Fibonacci sequence, Pell sequence, Lucas sequence, Jacobsthal and Jacobsthal-Lucas sequence. In particular, the Jacobsthal and the Jacobsthal-Lucas sequences are defined by [12]:

$$J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0,\ J_0 = 0,\ J_1 = 1,$$
$$j_{n+2} = j_{n+1} + 2j_n,\ n \geq 0,\ j_0 = 2,\ j_1 = 1,$$

respectively.

There are many studies about these two kinds of sequences, for examples. In [11], Horadam discussed the properties and has given the associated sequence with Jacobsthal numbers [12] and [13]. Moreover, in [15] Köken and Bozkurt have given some matrix properties of Jacobsthal and Jacobsthal-Lucas numbers.

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Consequently, Yilmaz and Bozkurt defined k-Jacobsthal numbers and described Binet's formula for the same [18].

In [10] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by Horadam [12] is expanded and extended to several identities for some of the higher order cases. Furthermore, the authors generalized the Jacobsthal recursion as

$$J_{n+r}^{(r)} = \sum_{s=1}^{r-1} J_{n+r-s}^{(r)} + 2J_n^{(r)},\ n \geq 0,$$

with initial conditions $J_0 = 0$ and $J_s = 1$ for $s = 1, ..., r-1$. For the n-th order-r Jacobsthal Lucas numbers $j_n^{(r)}$ we use the same recursion with initial conditions $j_s^{(r)} = j_s^{(r-1)}$ for $s = 1, ..., r-1$.

In particular, for $r = 3$, the third-order Jacobsthal numbers $\{J_n^{(r)}\}_{n\geq 0}$, are defined by recurrence relation:

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\quad n \geq 0,$$

with initial conditions $J_0^{(3)} = 0$ and $J_1^{(3)} = J_2^{(3)} = 1$ (see [10]).

Also the third-order Jacobsthal-Lucas numbers $\{j_n^{(3)}\}_{n\geq 0}$, is defined by using the following recurrence relation $j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\quad n \geq 0$, with $j_0^{(3)} = 2$, $j_1^{(3)} = 1$ and $j_2^{(3)} = 5$ (see [10]).

In this part, we define k-Fibonacci numbers and k-Jacobsthal, k-Jacobsthal Lucas numbers and Chebyshev polynomials of the third and fourth kinds.

Definition 1.1 ([8])**.** *For any positive real number k, the k-Fibonacci numbers, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 1$; $F_{k,1} = 1$.*

Definition 1.2 ([17])**.** *The Narayana numbers is defined as follows $b_{n+1} = b_n + b_{n-2}$, $n \geq 2$, $b_0 = 0$, $b_1 = 1$, $b_2 = 1$.*

Definition 1.3 ([14])**.** *For any positive real number k, the k-Jacobsthal numbers say $\{J_{k,n}\}_n$ is defined recurrently by $J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}$, for $n \geq 1$, with initial conditions $J_{k,0} = 0$; $J_{k,1} = 1$.*

Definition 1.4 ([14])**.** *For any positive real number k, the k-Jacobsthal -Lucas numbers say $\{j_{k,n}\}_n$ is defined recurrently by $j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$, for $n \geq 1$, with initial conditions $j_{k,0} = 2$, $j_{k,1} = k$.*

Definition 1.5 ([16])**.** *The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are respectively defined by the recurrence relations as follows:*

$$V_{n+2}(x) = 2xV_{n+1}(x) - V_n(x),\quad n \geq 0, V_0(x) = 1,\ V_1(x) = 2x - 1,$$
$$W_{n+2}(x) = 2xW_{n+1}(x) - W_n(x),\quad n \geq 0, W_0(x) = 1,\ W_1(x) = 2x + 1.$$

Let us now start at the following definition.

Definition 1.6. *Let E and A be any two alphabets, then we give $S_n(A-E)$ by the following form:*

$$\frac{\prod_{e\in E}(1-ez)}{\prod\limits_{a\in A}(1-az)}=\sum_{n=0}^{\infty}S_n(A-E)\,z^n, \tag{1.1}$$

with the condition $S_n(A-E)=0$ for $n<0$ (see [1]).

Equation (1.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty}S_n(A-E)\,z^n=\left(\sum_{n=0}^{\infty}S_n(A)\,z^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-E)\,z^n\right),$$

where

$$S_n(A-E)=\sum_{i=0}^{n}S_i(A)\,S_{n-i}(-E).$$

Remark 1.1. Taking $A=\{0,0,...,0\}$ in (1.1) gives

$$\sum_{n=0}^{\infty}S_n(-E)\,z^n=\prod_{e\in E}(1-ez). \tag{1.2}$$

Definition 1.7 ([5])**.** *Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{x_ix_{i+1}}f(x)=\frac{f(x_1,...,x_i,x_{i+1},...,x_n)-f(x_1,...,x_{i-1},x_{i+1},x_i,...,x_n)}{x_i-x_{i+1}}.$$

Definition 1.8. *Let n be positive integer and $E=\{e_1,e_2\}$ are set of given variables, then, the n-th symmetric function $S_n(e_1+e_2)$ is defined by*

$$S_n(E)=S_n(e_1+e_2)=\frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2},$$

with

$$\begin{aligned}S_0(E)&=S_0(e_1+e_2)=1,\\ S_1(E)&=S_1(e_1+e_2)=e_1+e_2,\\ S_2(E)&=S_2(e_1+e_2)=e_1^2+e_1e_2+e_2^2,\\ &\vdots\end{aligned}$$

Definition 1.9 ([5])**.** *Given an alphabet $E=\{e_1,e_2\}$, the symmetrizing operator $\delta_{e_1e_2}^k$ is defined by*

$$\delta_{e_1e_2}^k f(e_1)=\frac{e_1^k f(e_1)-e_2^k f(e_2)}{e_1-e_2},\ k\in\mathbb{N}_0. \tag{1.3}$$

If $f(e_1)=e_1^n$, the operator (1.3) gives us

$$\delta_{e_1e_2}^k(e_1^n)=\frac{e_1^{k+n}-e_2^{k+n}}{e_1-e_2}=S_{k+n-1}(e_1+e_2).$$

2. Principal formulas

In this section, we provide a new theorem in order to derive some new generating functions of the products of some known numbers .

Theorem 2.1. *Let A and E be two alphabets, respectively, $\{e_1, e_2, e_3\}$ and $\{a_1, a_2\}$, then we have*

$$\sum_{n=0}^{\infty} S_n(E) S_{n+k}(A) z^n = \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^{k+1}(a_2^n) z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n z^n\right)},\ k \in \mathbb{N}_0. \tag{2.1}$$

Proof. Applying the operator $\delta_{a_1a_2}^k$ to the series $f(a_1z) = \sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n$, we have

$$\begin{aligned}\delta_{a_1a_2}^k f(a_1z) &= \delta_{a_1a_2}^k\left(\sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n\right)\\ &= \frac{\sum_{n=0}^{+\infty} S_n(E)a_1^{n+k+1}z^n - \sum_{n=0}^{+\infty} S_n(E)a_2^{n+k+1}z^n}{a_1-a_2}\\ &= \sum_{n=0}^{+\infty} S_n(E)\frac{a_1^{n+k+1}-a_2^{n+k+1}}{a_1-a_2}z^n\\ &= \sum_{n=0}^{\infty} S_n(E) S_{n+k}(A) z^n.\end{aligned}$$

On the other part, since $\sum_{n=0}^{+\infty} S_n(E)a_1^{n+1}z^n = \frac{a_1}{\prod_{e\in E}(1-ea_1z)}$ we have

$$\begin{aligned}\delta_{a_1a_2}^k f(a_1z) &= \delta_{a_1a_2}^k f\left(\frac{a_1}{\prod_{e\in E}(1-ea_1z)}\right) = \frac{\frac{a_1^{k+1}}{\prod_{e\in E}(1-ea_1z)} - \frac{a_2^{k+1}}{\prod_{e\in E}(1-ea_2z)}}{a_1-a_2}\\ &= \frac{a_1^{k+1}\prod_{e\in E}(1-ea_2z) - a_2^{k+1}\prod_{e\in E}(1-ea_1z)}{(a_1-a_2)\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1-ea_2z)}.\end{aligned}$$

Using the fact that: $\sum_{n=0}^{+\infty} S_n(-E)a_1^nz^n = \prod_{e\in E}(1-ea_1z)$, then

$$\begin{aligned}\delta_{a_1a_2}^k f(a_1z) &= \frac{a_1^{k+1}\sum_{n=0}^{+\infty} S_n(-E)a_2^nz^n - a_2^{k+1}\sum_{n=0}^{+\infty} S_n(-E)a_1^nz^n}{(a_1-a_2)\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^nz^n\right)}\\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\frac{a_1^{k+1}a_2^n - a_2^{k+1}a_1^n}{a_1-a_2}z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^nz^n\right)}\\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^{k+1}(a_2^n)z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^nz^n\right)}.\end{aligned}$$

This completes the proof. □

3. On the symmetric functions of some numbers and polynomials

In this section, we now derive the new generating functions of the products of some known numbers. For the applications of generating functions of some known functions, we refer the reader to see the references [2, 3, 4, 6, 7, 8], we present some applications for the previous theorem if $k = 0, 1$.

3.1 The case of $E = \{e_1, e_2, 0\}, A = \{1, 0\}$

Lemma 3.1 ([5]). *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(e_1 + e_2) z^n = \frac{1}{(1 - e_1 z)(1 - e_2 z)}. \tag{3.1}$$

From (3.1) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(e_1 + e_2) z^n = \frac{z}{(1 - e_1 z)(1 - e_2 z)}. \tag{3.2}$$

By replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (3.1) and (3.2) and assuming that $e_1 - e_2 = x$ and $4e_1e_2 = -1$, we obtain

$$\sum_{n=0}^{\infty} S_n(2e_1 + [-2e_2]) z^n = \frac{1}{1 - 2xz + z^2}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{z}{1 - 2xz + z^2}, \tag{3.4}$$

respectively.

Subtracting the equation (3.4) from (3.3), we obtain the following proposition and corollary.

Proposition 3.1. *For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} V_n(x)\ z^n = \frac{1 - z}{1 - 2xz + z^2}. \tag{3.5}$$

Corollary 3.1. *The following identity hold true:*

$$V_n(x) = S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2]).$$

By added the equation (3.3) to (3.4), we obtain the following proposition.

Proposition 3.2. *For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of the fourth kind is given by*

$$\sum_{n=0}^{\infty} W_n(x)\ z^n = \frac{1 + z}{1 - 2xz + z^2}. \tag{3.6}$$

By comparing the coefficients z^n on both sides of (3.6) , we have the following corollary.

Corollary 3.2. *The following identity hold true* $W_n(x) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2])$.

3.2 The case of $E = \{e_1, e_2, e_3\}, A = \{1, 0\}$

Lemma 3.2 ([9])**.** *Given an alphabet* $E = \{e_1, e_2, e_3\}$, *we have*

$$\sum_{n=0}^{\infty} S_n(E) z^n = \frac{1}{(1 - e_1z)(1 - e_2z)(1 - e_3z)}. \tag{3.7}$$

From (3.7) *we can deduce*

$$\sum_{n=0}^{\infty} S_{n-1}(E) z^n = \frac{z}{(1 - e_1z)(1 - e_2z)(1 - e_3z)}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E) z^n = \frac{z^2}{(1 - e_1z)(1 - e_2z)(1 - e_3z)}, \tag{3.9}$$

with $(1 - e_1z)(1 - e_2z)(1 - e_3z) = 1 - (e_1 + e_2 + e_3)z + (e_1e_2 + e_1e_3 + e_2e_3)z^2 - e_1e_2e_3z^3 = 1 + S_1(-E)z + S_2(-E)z^2 + S_3(-E)z^3$.

Setting $\begin{cases} S_1(-E) = -1, \\ S_2(-E) = 0, \\ S_3(-E) = -1, \end{cases}$ in (3.8) , we obtain the following proposition and corollary.

Proposition 3.3. *For* $n \in \mathbb{N}$, *the generating function of Narayana numbers is given by*

$$\sum_{n=0}^{\infty} b_n z^n = \frac{z}{1 - z - z^3}.$$

Corollary 3.3. *The following identity hold true:*

$$b_n = S_{n-1}(e_1 + e_2 + e_3).$$

Setting $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1 \\ S_2(-E) = -2 \end{cases}$ in (3.7), (3.8) and (3.9) this gives

$$\sum_{n=0}^{\infty} S_{n-1}(E) z^n = \frac{z}{1 - z - z^2 - 2z^3}, \tag{3.11}$$

which represents a generating function of the third order Jacobsthal numbers, such that $J_n^{(3)} = S_{n-1}(e_1 + e_2 + e_3)$ and

$$\sum_{n=0}^{\infty} S_n(E) z^n = \frac{1}{1 - z - z^2 - 2z^3}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E) z^n = \frac{z^2}{1 - z - z^2 - 2z^3}, \tag{3.13}$$

which represents a new generating functions respectively.

Multiplying the equation (3.12) by 2 and added to (3.11) by (-1) and also added to (3.13) by 2, we obtain the following proposition and corollary.

Proposition 3.4. *For $n \in \mathbb{N}$, the generating function of the third order Jacobsthal-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} j_n^{(3)} z^n = \frac{2 - z + 2z^2}{1 - z - z^2 - 2z^3}.$$

Corollary 3.4. *The following identity hold true:*

$$j_n^{(3)} = 2S_n(e_1 + e_2 + e_3) - S_{n-1}(e_1 + e_2 + e_3) + 2S_{n-2}(e_1 + e_2 + e_3).$$

3.3 The case of $E = \{e_1, e_2, e_3\}$, $A = \{a_1, a_2\}$

Lemma 3.3 ([4]). *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(E) S_n(A) z^n = \frac{1 - a_1 a_2 S_2(-E) z^2 - a_1 a_2 (a_1 + a_2) S_3(-E) z^3}{\prod_{e \in E} (1 - e a_1 z) \prod_{e \in E} (1 - e a_2 z)}. \tag{3.14}$$

From (3.14) *we can deduce*

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(A) z^n = \frac{z - a_1 a_2 S_2(-E) z^3 - a_1 a_2 (a_1 + a_2) S_3(-E) z^4}{\prod_{e \in E}(1 - e a_1 z) \prod_{e \in E}(1 - e a_2 z)}. \tag{3.15}$$

Lemma 3.4 ([9]). *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(E) S_{n+1}(A) z^n = \frac{(a_1 + a_2) + a_1 a_2 S_1(-E) z - a_1^2 a_2^2 S_3(-E) z^3}{\prod_{e \in E}(1 - e a_1 z) \prod_{e \in E}(1 - e a_2 z)}. \tag{3.16}$$

From (3.16) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(A)z^n = \frac{(a_1+a_2)z + a_1a_2S_1(-E)\,z^2 - a_1^2a_2^2S_3(-E)\,z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1-ea_2z)}. \tag{3.17}$$

By replacing a_2 by $(-a_2)$ in (3.14), (3.15) and (3.17) we obtain, respectively, the following relationships

$$\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])z^n = \frac{1 + a_1a_2S_2(-E)\,z^2 + a_1a_2(a_1-a_2)S_3(-E)\,z^3}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])z^n = \frac{z + a_1a_2S_2(-E)\,z^3 + a_1a_2(a_1-a_2)S_3(-E)\,z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])z^n = \frac{(a_1-a_2)z - a_1a_2\ S_1(-E)\,z^2 - a_1^2a_2^2S_3(-E)\,z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}, \tag{3.20}$$

with

$$\begin{aligned}
&\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)\\
&= 1 - (a_1-a_2)(e_1+e_2+e_3)\,z + ((e_1e_2+e_1e_3+e_2e_3)(a_1-a_2)^2\\
&- a_1a_2((e_1+e_2+e_3)^2 - 2(e_1e_2+e_1e_3+e_2e_3)))z^2 - (e_1e_2e_3(a_1-a_2)^3\\
&- a_1a_2(a_1-a_2)((e_1e_2+e_1e_3+e_2e_3)(e_1+e_2+e_3) - 3e_1e_2e_3))z^3\\
&+ (-a_1a_2(a_1-a_2)^2e_1e_2e_3(e_1+e_2+e_3) + a_1^2a_2^2((e_1e_2+e_1e_3+e_2e_3)^2\\
&- 2e_1e_2e_3(e_1+e_2+e_3)))z^4 - e_1e_2e_3a_1^2a_2^2(a_1-a_2)(e_1e_2+e_1e_3+e_2e_3)\,z^5\\
&- e_1^2e_2^2e_3^2a_1^3a_2^3z^6.
\end{aligned}$$

This case consists of three related parts.

Firstly, the substitutions $\begin{cases} S_1(-E) = -1\\ S_2(-E) = 0\\ S_3(-E) = -1\end{cases}$ and $\begin{cases} a_1 - a_2 = k\\ a_1a_2 = 1\end{cases}$ in (3.20), we obtain the following proposition and corollary.

Proposition 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of Narayana numbers and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} b_nF_{k,n}z^n = \frac{kz + z^2 + z^4}{1 - kz - z^2 - (k^3+3k)z^3 - (k^2+2)z^4 - z^6}.$$

Corollary 3.5. *The following identity hold true:*

$$b_n F_{k,n} = S_{n-1}(E)\, S_n(a_1 + [-a_2]).$$

Secondly, the substitutions $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = 0 \\ S_3(-E) = -1 \end{cases}$ and $\begin{cases} a_1 - a_2 = k \\ a_1 a_2 = 2 \end{cases}$ in (3.19), we deduce the following proposition, corollary and theorem.

Proposition 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of Narayana numbers and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} b_n J_{k,n} z^n = \frac{z - 2kz^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}. \tag{3.22}$$

Corollary 3.6. *The following identity hold true:*

$$b_n J_{k,n} = S_{n-1}(E)\, S_{n-1}(a_1 + [-a_2]).$$

Theorem 3.1. *For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal-Luca numbers and Narayana numbers is given by*

$$\sum_{n=0}^{\infty} b_n j_{k,n} z^n = \frac{kz + 4z^2 + (2k^2 + 8)z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}. \tag{3.24}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n j_{k,n} z^n &= \sum_{n=0}^{\infty} S_{n-1}(E)\left(2S_n(a_1 + [-a_2]) - kS_{n-1} + [-a_2])\right) z^n \\
&= \sum_{n=0}^{\infty} 2S_{n-1}(E)S_n(a_1 + [-a_2])z^n - k\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1 + [-a_2])z^n \\
&= 2\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1 + [-a_2])z^n - k\sum_{n=0}^{\infty} b_n J_{k,n} z^n.
\end{aligned}$$

By using

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])z^n = \frac{kz + 2z^2 + 4z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}.$$

We obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n j_{k,n} z^n &= \frac{2kz + 4z^2 + 8z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6} \\
&- k\frac{z - 2kz^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6} \\
&= \frac{kz + 4z^2 + (2k^2 + 8)z^4}{1 - kz - 2z^2 - (k^3 + 6k)z^3 + (-2k^2 - 8)z^4 - 8z^6}.
\end{aligned}$$

This completes the proof. □

Thirdly, the substitutions $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1 \\ S_3(-E) = -2 \end{cases}$ and $\begin{cases} a_1 - a_2 = 2 \\ a_1 a_2 = k \end{cases}$ in (3.19), we deduce the following proposition, corollary and theorem.

Proposition 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of third order Jacobsthal numbers and k-Pell numbers numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n = \frac{z - kz^3 - 4kz^4}{1 - 2z - (4+3k)z^2 - (16+14k)z^3 + (-8k-3k^2)z^4 + 4k^2z^5 - 4k^3z^6}. \tag{3.25}$$

Corollary 3.7. *The following identity hold true:*

$$J_n^{(3)} P_{k,n} = S_{n-1}(E) S_{n-1}(a_1 + [-a_2]).$$

Theorem 3.2. *For $n \in \mathbb{N}$, the new generating function of the product of third order Jacobsthal- Lucas numbers and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n = \frac{z + 8z^2 + (16+7k)z^3 + 4kz^4 + 4k^2z^5}{1 - 2z - (4+3k)z^2 - (16+14k)z^3 + (-8k-3k^2)z^4 + 4k^2z^5 - 4k^3z^6}. \tag{3.26}$$

Proof. We have

$$\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n = \sum_{n=0}^{\infty} \left(2S_n(E) - S_{n-1}(E) + 2S_{n-2}(E)\right) S_{n-1}(a_1 + [-a_2]) z^n$$

$$= 2\sum_{n=0}^{\infty} S_n(E) S_{n-1}(a_1 + [-a_2]) z^n - \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(a_1 + [-a_2]) z^n$$

$$+ 2\sum_{n=0}^{\infty} S_{n-2}(E) S_{n-1}(a_1 + [-a_2]) z^n$$

$$= 2\sum_{n=0}^{\infty} S_n(E) \frac{a_1^n - (-a_2)^n}{a_1 + a_2} z^n - \sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n$$

$$+ 2\sum_{n=0}^{\infty} S_{n-2}(E) \frac{a_1^n - (-a_2)^n}{a_1 + a_2} z^n$$

$$= \frac{2}{a_1 + a_2} \left(\sum_{n=0}^{\infty} S_n(E) a_1^n z^n - \sum_{n=0}^{\infty} S_n(E) (-a_2)^n z^n \right) - \sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n$$

$$+ \frac{2}{a_1 + a_2} \left(\sum_{n=0}^{\infty} S_{n-2}(E) a_1^n z^n - \sum_{n=0}^{\infty} S_{n-2}(E) (-a_2)^n z^n \right).$$

Since

$$\sum_{n=0}^{\infty} S_n\left(e_1+e_2+e_3\right) z^n=\frac{1}{1-z-z^2-2 z^3},$$
$$\sum_{n=0}^{\infty} S_{n-2}\left(e_1+e_2+e_3\right) z^n=\frac{z^2}{1-z-z^2-2 z^3},$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} P_{k,n} z^n \\
&=\frac{2}{a_1+a_2}\left(\frac{1}{1-a_1 z-a_1^2 z^2-2 a_1^3 z^3}-\frac{1}{1+a_2 z-a_2^2 z^2+2 a_2^3 z^3}\right) \\
&-\sum_{n=0}^{\infty} J_n^{(3)} P_{k,n} z^n+\frac{2}{a_1+a_2}\left(\frac{a_1^2 z^2}{1-a_1 z-a_1^2 z^2-2 a_1^3 z^3}\right. \\
&\left.-\frac{a_2^2 z^2}{1+a_2 z-a_2^2 z^2+2 a_2^3 z^3}\right) \\
&=\frac{2 z+4 z^2+4(4+k) z^3}{1-2 z-(4+3 k) z^2-(16+14 k) z^3+(-8 k-3 k^2) z^4+4 k^2 z^5-4 k^3 z^6} \\
&-\frac{z-k z^3-4 k z^4}{1-2 z-(4+3 k) z^2-(16+14 k) z^3+(-8 k-3 k^2) z^4+4 k^2 z^5-4 k^3 z^6} \\
&+\frac{4 z^2+2 k z^3+4 k^2 z^5}{1-2 z-(4+3 k) z^2-(16+14 k) z^3+(-8 k-3 k^2) z^4+4 k^2 z^5-4 k^3 z^6} \\
&=\frac{z+8 z^2+(16+7 k) z^3+4 k z^4+4 k^2 z^5}{1-2 z-(4+3 k) z^2-(16+14 k) z^3+(-8 k-3 k^2) z^4+4 k^2 z^5-4 k^3 z^6}.
\end{aligned}$$

This completes the proof. □

Replacing a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.15) and (3.17), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2 a_1+[-2 a_2]) z^n \\
&=\frac{z+4 a_1 a_2 S_2(-E) z^3+8 a_1 a_2(a_1-a_2) S_3(-E) z^4}{\prod_{e \in E}(1-2 e a_1 z) \prod_{e \in E}(1+2 e a_2 z)}, \qquad (3.27)\\
&\sum_{n=0}^{\infty} S_{n-1}(E) S_n(2 a_1+[-2 a_2]) t^n \\
&=\frac{2(a_1-a_2) z-4 a_1 a_2 S_1(-E) z^2-16 a_1^2 a_2^2 S_3(-E) z^4}{\prod_{e \in E}(1-2 e a_1 z) \prod_{e \in E}(1+2 e a_2 z)}, \qquad (3.28)
\end{aligned}$$

respectively, with

$$\prod_{e \in E}(1-2 e a_1 z) \prod_{e \in E}(1+2 e a_2 z)$$

$$= 1 - 2(a_1 - a_2)(e_1 + e_2 + e_3)z + (4(e_1e_2 + e_1e_3 + e_2e_3)(a_1 - a_2)^2$$
$$- 4a_1a_2((e_1 + e_2 + e_3)^2 - 2(e_1e_2 + e_1e_3 + e_2e_3)))z^2 - (8e_1e_2e_3(a_1 - a_2)^3$$
$$- 8a_1a_2(a_1 - a_2)((e_1e_2 + e_1e_3 + e_2e_3)(e_1 + e_2 + e_3) - 3e_1e_2e_3))z^3$$
$$+ (-16a_1a_2(a_1 - a_2)^2e_1e_2e_3(e_1 + e_2 + e_3) + 16a_1^2a_2^2((e_1e_2 + e_1e_3 + e_2e_3)^2$$
$$- 2e_1e_2e_3(e_1 + e_2 + e_3)))z^4 - 32e_1e_2e_3a_1^2a_2^2(a_1 - a_2)(e_1e_2 + e_1e_3 + e_2e_3)z^5$$
$$- 64e_1^2e_2^2e_3^2a_1^3a_2^3z^6.$$

This case consists two related parts.

Firstly, by making the following restrictions: $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = 0 \\ S_3(-E) = -1 \end{cases}$ and $\begin{cases} a_1 - a_2 = x \\ 4a_1a_2 = -1 \end{cases}$ in (3.27) and (3.28), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(2a_1 + [-2a_2])z^n = \frac{z + 2xz^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}, \tag{3.29}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(2a_1 + [-2a_2])z^n = \frac{2xz - z^2 + z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}. \tag{3.30}$$

which represents a new generating functions respectively.

We deduce the following theorem.

Theorem 3.3. *For $n \in \mathbb{N}$, the new generating function of the product of Narayana numbers and Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} b_n V_n(x) z^n = \frac{(2x - 1)z - z^2 + (1 - 2x)z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}.$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} b_n V_n(x) z^n &= \sum_{n=0}^{\infty} S_{n-1}(E)\left(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2])\right) z^n \\ &= \sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2]) z^n \\ &- \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2]) z^n, \end{aligned}$$

by using the relationships (3.29) and (3.30), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} b_n V_n(x) z^n &= \frac{2xz - z^2 + z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6} \\ &- \frac{z + 2xz^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6} \\ &= \frac{(2x-1)z - z^2 + (1-2x)z^4}{1 - 2xz + z^2 - (8x^3 - 6x)z^3 + (4x^2 - 2)z^4 + z^6}.\end{aligned}$$

This completes the proof. □

Secondly, the substitutions $\begin{cases} S_1(E) = -1 \\ S_2(E) = -1 \\ S_3(E) = -2 \end{cases}$ and $\begin{cases} a_1 - a_2 = x \\ 4a_1 a_2 = -1 \end{cases}$ in (3.27) and (3.28), give

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2]) z^n = \frac{z+z^3+4xz^4}{1-2xz+(-4x^2+3)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}, \tag{3.31}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2]) z^n = \frac{2xz-z^2+2z^4}{1-2xz+(-4x^2+3)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}. \tag{3.32}$$

which represents a new generating functions respectively.

We deduce the following theorem.

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of the product of the third order Jacobsthal numbers and the Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n = \frac{(2x-1)z - z^2 - z^3 + (2-4x)z^4}{1 - 2xz + (-4x^2+3)z^2 - (16x^3 - 14x)z^3 + (8x^2 - 3)z^4 + 4xz^5 + 4z^6}.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n &= \sum_{n=0}^{\infty} S_{n-1}(E)(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]) z^n \\ &= \sum_{n=0}^{\infty} S_{n-1}(E) S_n(2a_1 + [-2a_2]) z^n - \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(2a_1 + [-2a_2]) z^n.\end{aligned}$$

By using (3.31) and (3.32) we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n \\
&= \frac{2xz - z^2 + 2z^4}{1 - 2xz + (-4x^2+3) z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&- \frac{z + z^3 + 4xz^4}{1 - 2xz + (-4x^2+3) z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6} \\
&= \frac{(2x-1) z - z^2 - z^3 + (2-4x) z^4}{1 - 2xz + (-4x^2+3) z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of the third order Jacobsthal-Lucas numbers and the Chebyshev polynomials of the third kind is given by*

$$\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)} V_n(x) z^n \\
&= \frac{2 - (2x+1)z + (1 - 8x + 8x^2)z^2 + (7 + 4x - 16x^2)z^3 + (4x-4)z^4 - 4z^5}{1 - 2xz + (-4x^2+3) z^2 - (16x^3 - 14x)z^3 + (8x^2-3)z^4 + 4xz^5 + 4z^6}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} j_n^{(3)} V_n(x) z^n &= \sum_{n=0}^{\infty} \left(2S_n(E) - S_{n-1}(E) + 2S_{n-2}(E)\right)(S_n(2a_1 + [-2a_2]) \\
&- S_{n-1}(2a_1 + [-2a_2])z^n \\
&= \sum_{n=0}^{\infty} 2S_n(E)(S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]) \\
&- \sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n + \sum_{n=0}^{\infty} 2S_{n-2}(E)(S_n(2a_1 + [-2a_2]) \\
&- S_{n-1}(2a_1 + [-2a_2])z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E) S_n(2a_1 + [-2a_2])z^n - \sum_{n=0}^{\infty} J_n^{(3)} V_n(x) z^n \\
&- 2\sum_{n=0}^{\infty} S_n(E) S_{n-1}(2a_1 + [-2a_2])z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E)(S_n(2a_1 + [-2a_2])z^n \\
&- 2\sum_{n=0}^{\infty} S_{n-2}(E)(S_{n-1}(2a_1 + [-2a_2])z^n
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n - \sum_{n=0}^{\infty} J_n^{(3)}V_n(x)\, z^n \\
&- 2\sum_{n=0}^{\infty} S_n(E)\, \frac{(2a_1)^n-(-2a_2)^n}{2a_1+2a_2} z^n \\
&+ 2\sum_{n=0}^{\infty} S_{n-2}(E)\, \frac{(2a_1)^{n+1}-(-2a_2)^{n+1}}{2a_1+2a_2} z^n \\
&- 2\sum_{n=0}^{\infty} S_{n-2}(E)\, \frac{(2a_1)^n-(-2a_2)^n}{2a_1+2a_2} z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n \\
&- \frac{1}{a_1+a_2}\left(\sum_{n=0}^{\infty} S_n(E)\,(2a_1)^n z^n - \sum_{n=0}^{\infty} S_n(E)\,(-2a_2)^n z^n\right) \\
&- \sum_{n=0}^{\infty} J_n^{(3)}V_n(x)\, z^n + \frac{1}{a_1+a_2}\left(2a_1\sum_{n=0}^{\infty} S_{n-2}(E)\,(2a_1)^n z^n\right. \\
&+ \left. 2a_2\sum_{n=0}^{\infty} S_{n-2}(E)\,(-2a_2)^n z^n\right) \\
&- \frac{1}{a_1+a_2}\left(\sum_{n=0}^{\infty} S_{n-2}(E)\,(2a_1)^n z^n - \sum_{n=0}^{\infty} S_{n-2}(E)\,(-2a_2)^n z^n\right),
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{n=0}^{\infty} S_n(e_1+e_2+e_3)\, z^n &= \frac{1}{1-z-z^2-2z^3}, \\
\sum_{n=0}^{\infty} S_{n-2}(e_1+e_2+e_3)\, z^n &= \frac{z^2}{1-z-z^2-2z^3},
\end{aligned}
$$

and by using

$$
\begin{aligned}
&\sum_{n=0}^{\infty} S_n(E)\, S_n(2a_1+[-2a_2])z^n \\
&= \frac{1+z^2+4xz^3}{1-2xz+(-4x^2+3)\,z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} j_n^{(3)}V_n(x)\, z^n \\
&= \frac{2+2z^2+8xz^3}{1-2xz+(-4x^2+3)\,z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2z+4xz^2+4\left(4x^2-1\right)z^3}{1-2xz+\left(-4x^2+3\right)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}\\
&-\frac{(2x-1)z-z^2-z^3+(2-4x)z^4}{1-2xz+\left(-4x^2+3\right)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}\\
&+\frac{2(4x^2-1)z^2-4xz^3-2z^4}{1-2xz+\left(-4x^2+3\right)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}\\
&-\frac{4xz^2-2z^3+4z^5}{1-2xz+\left(-4x^2+3\right)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}\\
&=\frac{2-(2x+1)z+(1-8x+8x^2)z^2+(7+4x-16x^2)z^3+(4x-4)z^4-4z^5}{1-2xz+\left(-4x^2+3\right)z^2-(16x^3-14x)z^3+(8x^2-3)z^4+4xz^5+4z^6}.
\end{aligned}
$$

So, the proof is completed. □

4. Conclusion

In this paper, by making use of Theorem 2.1, we have derived some new generating functions for the products of Narayana numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers and Chebyshev polynomials of the third and fourth kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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