A full-Newton step IIPM based on new search directions for $P_*(\kappa)$-LCP

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Abstract. In this paper, a new full-Newton step infeasible interior-point algorithm is proposed for solving $P_*(\kappa)$-linear complementarity problem. By using some new analytic tools, we show that the new algorithm is quadratically convergent with iteration complexity $O((1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon})$. This complexity matches the currently best known iteration bound for $P_*(\kappa)$-linear complementarity problem. Some computational results are provided as well.

Keywords: full-Newton step, infeasible interior-point algorithm, $P_*(\kappa)$-linear complementarity problem, iteration complexity.

1. Introduction

In this paper, we consider the standard form of linear complementarity problem (LCP)

$$(P) \qquad s = Mx + q, \quad xs = 0, \quad x \geq 0, \; s \geq 0,$$

where $M \in R^{n\times n}$ is a $P_*(\kappa)$-matrix, $q \in R^n$ and xs denotes the componentwise product (Hadamard product) of vectors x and s. LCP with M being a $P_*(\kappa)$-matrix is denoted by $P_*(\kappa)$-LCP. LCPs have many important applications in the

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field of mathematical programming and equilibrium problems. The interested readers can refer to the book [1].

Because $P_*(\kappa)$-matrix has good structure and $P_*(\kappa)$-LCP is an extension of monotone LCP (i.e., $P_*(0)$-LCP), the study on $P_*(\kappa)$-LCP has become a popular topic in the field of interior point methods (IPMs). In particular, extending feasible (IPMs) and infeasible interior-point methods (IIPMs) for linear optimization (LO) to $P_*(\kappa)$-LCP has been successful in many cases [2-6]. For example, Miao [3] extended the Mizuno-Todd-Ye (MTY) predictor-corrector algorithm to $P_*(\kappa)$-LCP with $O((\kappa+1)\sqrt{n}L)$ iteration complexity. Illes et al [6] gave a version of MTY predictor-corrector IPM for the $P_*(\kappa)$-LCP and obtained iteration complexity as $O((\kappa+1)^{\frac{3}{2}}\sqrt{n}L)$. For an overview these related results, we can refer to [7-14] and its references.

Recently, Mansouri et al [15] changed the definition of the feasibility step in [16] and showed that the iteration complexity enjoys $O(n\log\frac{\varepsilon}{n})$. Later on, Liu and Sun [17] proposed an IIPM that used full-Newton step, which was different from the algorithm given by Roos [16] in the definition of feasibility step, whereas the iteration bound essentially remained the same. It's worth noticing that the definition of the feasibility step in [15] is a special case of definition in [17]. Most recently, Mansouri et al [18] extended Roos's full-Newton IIPM [16] to LCP. Zhu et al [19] proposed a new IIPM for $P_*(\kappa)$-LCP, which is an extension of the work in [15].

Inspired by this series of work [15-19], in this paper we present a new full-Newton step IIPM for $P_*(\kappa)$-LCP. In our analysis, a specific feasibility step is introduced to induce the search directions, which is different from that of the algorithm in [15]. By using some new analytic tools, we show that the algorithm is quadratically convergent and the iteration complexity coincides with the currently best known one for IIPMs.

We use the following notation throughout the article: $\mathbf{R}^n_+$ denotes the nonnegative orthant of $\mathbf{R}^n$. x_{min} is the smallest component of $x=(x_1,x_2,...,x_n)^T\in\mathbf{R}^n$. $X=diag(x)$ denotes the diagonal matrix whose diagonal entries are the components of x. The all-one vector of length n is denoted by e, and $I=\{1,2,...,n\}$ is the index set. As usual, $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the 1-norm, 2-norm and infinity norm, respectively. Furthermore, the feasible set of $P_*(\kappa)$-LCP is denoted by

$$\mathcal{F}:=\{(x,s)\in\mathbf{R}^{2n}_+ : s=Mx+q\}.$$

Finally, the solution set of $P_*(\kappa)$-LCP is expressed by

$$\mathcal{F}^*:=\{(x^*,s^*)\in\mathcal{F} : (x^*)^Ts^*=0\}.$$

Throughout this paper it will be assumed that $\mathcal{F}^*$ is not empty, i.e., $P_*(\kappa)$-LCP has at least one solution. The proofs and technical lemmas are presented in Appendix.

2. Preliminaries

In case of the IIPM, we call (x, s) an ε-solution of $P_*(\kappa)$-LCP if the value of duality gap $x^T s$ and the norm of the residual vector $\|s - Mx - q\|$ do not exceed ε. We are now ready to present an IIPM that generates an ε-solution for $P_*(\kappa)$-LCP, if it exists.

2.1 The perturbed problems

As usual, we choose arbitrarily $(x^0,\ s^0) > 0$ as the initial point such that $(x^0)^T s^0 = \mu^0 e$ for some (positive) number μ^0. The initial value of the residual is denoted as

$$r^0 = s^0 - Mx^0 - q.$$

For any ν with $0 < \nu \leq 1$, we consider the perturbed problem

$$(P_\nu) \qquad s - Mx - q = \nu r^0, \quad (x, s) \geq 0.$$

Note that if $\nu = 1$ then $(x, s) = (x^0, s^0)$ yields a strictly feasible solution of (P_ν). We may conclude that if $\nu = 1$ then the problem (P_ν) has a strictly feasible solution, which means that the perturbed problem (P_ν) satisfies the interior-point condition (IPC).

Lemma 1. *If the original problem (P) is feasible, then the perturbed problem (P_ν) satisfies the IPC.*

Assuming that (P) is feasible and $0 < \nu \leq 1$, then Lemma 1 shows that the problem (P_ν) satisfies the IPC, and its central path exists. This means that the following system

$$s - Mx - q = \nu r^0, \quad xs = \mu e, \quad x \geq 0, \quad s \geq 0, \tag{1}$$

has a unique solution for any $\mu > 0$, we donate this solution in the sequel as $(x(\mu, \nu), s(\mu, \nu))$. It is the μ-center of the perturbed problem (P_ν). In the following, we assume that the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$.

To measure the proximity of iterates (x, s) to the μ-center of the problem (P_ν), let us introduce a measures $\delta(x, s; \mu)$ as follows

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\|v - v^{-1}\|, \quad \text{where} \quad v := \sqrt{\frac{xs}{\mu}}. \tag{2}$$

Now, we describe the main iteration of our algorithm. Assume that for some $\mu \in (0, \mu^0)$, we have (x, s) satisfying the feasibility condition (1) with $\nu = \frac{\mu}{\mu_0}$, and $x^T s \leq (n + \delta^2)\mu$, $\delta(x, s; \mu) \leq \tau$. For given $\theta \in (0, 1)$, we can find new iterates (x^+, s^+) that satisfy the perturbed problem (P_{ν^+}), with $\mu^+ := (1 - \theta)\mu$, $\nu^+ := \frac{\mu^+}{\mu^0}$, such that $(x^+)^T s^+ \leq (n + \delta^2)\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

More specifically, each main iteration consists of a feasibility step, a μ-update and a few centering steps. The feasibility step aims to get the iterates (x^f, s^f), which is strictly feasible for (P_{ν^+}) and close enough to its μ-center $(x(\mu^+, \nu^+), s(\mu^+, \nu^+))$. In fact, the feasibility step is designed in such a way that $\delta(x^f, s^f; \mu^+) \le \frac{1}{\sqrt{2(1+4\kappa)}}$. Since (x^f, s^f) is strictly feasible for (P_{ν^+}), by performing a few centering steps starting at (x^f, s^f) and targeting at the μ^+-center of (P_{ν^+}), it is easy to get iterates (x^+, s^+) that are strictly feasible for (P_{ν^+}) such that $\delta(x^+, s^+; \mu^+) \le \tau$.

For the feasibility step, the system

$$M\Delta^f x - \Delta^f s = \theta\nu r^0, \quad s\Delta^f x + x\Delta^f s = \mu e - xs, \tag{3}$$

is used to define the search directions $\Delta^f x$ and $\Delta^f s$. After the feasibility step the iterates given by $x^f := x + \Delta^f x, s^f := s + \Delta^f s$. We conclude that after the feasibility step the iterates satisfy the system (1) with $\nu = \nu^+$. The hard part in the analysis will be to guarantee that (x^f, s^f) are positive and satisfy $\delta(x^f, s^f; \mu^+) \le \frac{1}{\sqrt{2(1+4\kappa)}}$.

In the centering steps, it starts at the iterates (x^f, s^f) and targets at the μ-centers of (P_{ν^+}). The search directions Δx and Δs are defined by the system $M\Delta x - \Delta s = 0, s\Delta x + x\Delta s = \mu e - xs$, which are the usual primal-dual Newton directions. After a centering step, the iterates are denoted as x^+ and s^+ which satisfy $(x^+)^T s^+ \le (n + \delta^2)\mu^+$ and $\delta(x^+, s^+; \mu^+) \le \tau$. More formal description of the algorithm is given in Fig. 1

Generic Full-Newton Step IIPM For $P_*(\kappa)$-LCP

Input:
  A threshold parameter $\tau \in (0, 1)$; an accuracy parameter $\varepsilon > 0$;
  barrier update parameter $\theta \in (0, 1)$;
  a starting point $(x^0, s^0) > 0$ and $\mu^0 > 0$ such that $x^0 s^0 = \mu^0 e$;
begin
  $x := x^0;\ s := s^0;\ \mu := \mu^0;$
  while $\max(\mu(n + \delta^2), ||r^0||) \ge \varepsilon$ do
  begin
    feasibility step: $(x, s) := (x, s) + (\Delta^f x, \Delta^f s);$
    update of μ : $\mu := (1 - \theta)\mu;$
    centering steps:
    while $\delta(x, s; \mu) \ge \tau$ do
     begin
      $(x, s) := (x, s) + (\Delta x, \Delta s);$
     end
  end
end

Fig. 1 Infeasible interior point algorithm

The required number of centering steps can be obtained by using Theorem A2 and Corollary A3. Indeed, supposing that $\delta = \delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, after k centering steps we will get iterates (x^+, s^+) that are still feasible for (P_{ν^+}) and satisfy $\delta(x^+, s^+; \mu^+) \leq (\frac{1}{\sqrt{2}})^{2^k} \frac{1}{1+4\kappa}$. From this one can deduce that $\delta(x^+, s^+; \mu^+) \leq \tau$ will be satisfied after at most $1 + \lceil \log_2(\log_2 \frac{1}{\tau(1+4\kappa)}) \rceil$ centering steps.

In particular, in the current paper we change the definition for feasibility step by the following system

$$M\Delta^f x - \Delta^f s = \theta\nu r^0, \quad s\Delta^f x + x\Delta^f s = \beta xs, \tag{4}$$

where we first require $\beta > -1$, and we will give further requirement on β.

3. Analysis of the algorithm

Let (x, s) denote the starting point of an iteration and assume $\delta(x, s; \mu) \leq \tau$. Recall that this is certainly true at the start of the first iteration, because $\delta(x, s; \mu) = 0$.

3.1 Effect of the feasibility step

We already knew that the feasibility step generates new iterates (x^f, s^f) that satisfy the feasibility condition for (P_{ν^+}), except possibly the nonnegativity conditions. The crucial element in the analysis is to show that after the feasibility step $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, which guarantees the new iterates are within the region where the Newton process targeting at the μ^+-center of (P_{ν^+}) is quadratically convergent.

Defining

$$d_x^f := \frac{\upsilon\Delta^f x}{x}, \qquad d_s^f := \frac{\upsilon\Delta^f s}{s}, \qquad \text{where } \upsilon = \sqrt{\frac{xs}{\mu}}. \tag{5}$$

Then, the system (4) can be rewritten as

$$DMDd_x^f - d_s^f = \theta\nu\upsilon s^{-1} r^0, \qquad d_x^f + d_s^f = \beta\upsilon, \tag{6}$$

where $D := (S^{-1}X)^{\frac{1}{2}}, S := diag(s), X := diag(x)$. By using (4) and (5), we have

$$\begin{aligned} x^f s^f &= xs + (s\Delta^f x + x\Delta^f s) + \Delta^f x\Delta^f s \\ &= (1+\beta)xs + \Delta^f x\Delta^f s = (1+\beta)\mu\upsilon^2 + \mu d_x^f d_s^f. \end{aligned} \tag{7}$$

Lemma 2 (Lemma 5 in [17]). *The iterates (x^f, s^f) are strictly feasible if and only if $(1+\beta)\upsilon^2 + d_x^f d_s^f \geq 0$.*

Lemma 3 (Lemma 2.2 in [15]). *The iterates* (x^f, s^f) *are certainly strictly feasible if* $\|d_x^f\| < \frac{1}{\rho(\delta)}$ *and* $\|d_s^f\| < \frac{1}{\rho(\delta)}$, *where* $\rho(\delta) := \delta + \sqrt{1+\delta^2}$.

The proof of Lemma 3 in [15] makes clear that the elements of the vector v satisfy

$$\frac{1}{\rho(\delta)} \leq v_i \leq \rho(\delta), \qquad i = 1, ..., n. \tag{8}$$

In the following, let us denote

$$\omega_i := \omega_i(v) := \frac{1}{2}\sqrt{|d_{xi}^f|^2 + |d_{si}^f|^2} \text{ and } \omega := \omega(v) := \|(\omega_1, ..., \omega_n)\|.$$

We can get

$$\begin{aligned} (d_x^f)^T d_s^f &\leq \|d_x^f\|\|d_s^f\| \leq \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) \leq 2\omega^2, \\ \|d_{xi}^f d_{si}^f\| &= \|d_{xi}^f\|\|d_{si}^f\| \leq \frac{1}{2}(\|d_{xi}^f\|^2 + \|d_{si}^f\|^2) \leq 2\omega_i^2 \leq 2\omega^2, \quad 1 \leq i \leq n. \end{aligned}$$

Assuming $(1+\beta)v^2 + d_x^f d_s^f \geq 0$, which according to Lemma 2 holds if and only if the iterates (x^f, s^f) are strictly feasible, we proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. According to definition (2) it holds that

$$\delta(x^f, s^f; \mu^+) = \frac{1}{2}\|v^f - \frac{e}{v^f}\|, \qquad \text{where} \quad v^f = \sqrt{\frac{x^f s^f}{\mu^+}}. \tag{9}$$

For the convenience of analysis, in the sequel $\delta(x^f, s^f; \mu^+)$ and $\delta(x, s; \mu)$ will be denoted shortly by $\delta(v^f)$ and $\delta(v)$, respectively.

Lemma 4 (Lemma 7 in [17]). *Assuming* $(1+\beta)v^2 + d_x^f d_s^f > 0$, *one has*

$$4\delta^2(v^f) \leq \frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\omega^2\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)\omega^2\big)}.$$

Because we need to have $\delta(v^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$, it follows from Lemma 4 that it suffices if

$$\frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\omega^2\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)\omega^2\big)} \leq \frac{2}{(1+4\kappa)^2}.$$

Now, let us define the function

$$\psi(t) = \frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2t}{1-\theta} + \frac{2(1-\theta)t\rho^4(\delta)}{(1+\beta)\big((1+\beta) - 2\rho^2(\delta)t\big)} - \frac{2}{(1+4\kappa)^2},$$

and the other two functions

$$\begin{aligned}\psi_1(t) &= \frac{4(1-\theta)\delta^2}{1+\beta} + \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)} + \frac{2t}{1-\theta} - \frac{3}{2(1+4\kappa)^2},\\ \psi_2(t) &= \frac{2(1-\theta)t\rho^4(\delta)}{(1+\beta)\big((1+\beta)-2\rho^2(\delta)t\big)} - \frac{1}{2(1+4\kappa)^2}.\end{aligned}$$

Note that $\psi(t)=\psi_1(t)+\psi_2(t)$, and both $\psi_1(t)$ and $\psi_2(t)$ are monotone increasing to t. By applying Lemma A5, the root t^* of $\psi(t)$ satisfies: $t^* \geq \min\{t_1^*, t_2^*\}$, where t_1^*, t_2^* are respectively the roots of $\psi_1(t)=0, \psi_2(t)=0$, i.e.,

$$\begin{aligned}t_1^* &= \Big(\frac{3}{2(1+4\kappa)^2} - \frac{4(1-\theta)\delta^2}{1+\beta} - \frac{(\theta+\beta)^2 n}{(1-\theta)(1+\beta)}\Big)\frac{(1-\theta)}{2},\\ t_2^* &= \frac{(1+\beta)^2}{4(1-\theta)(1+4\kappa)^2\rho^4(\delta)+2(1+\beta)\rho^2(\delta)}.\end{aligned}$$

At this stage, we decide to choose

$$0<\tau\leq\frac{1}{8(1+4\kappa)},\quad 0<\theta\leq\frac{\alpha}{2\sqrt{2n}(1+4\kappa)},\quad -\frac{1}{2}\leq\beta\leq 0,\quad \alpha\leq 1.$$

It is easy to check that $t_1^* \geq \frac{25}{64(1+4\kappa)^2}$ and $t_2^* \geq \frac{1}{57(1+4\kappa)^2}$. Furthermore, from the proof of Lemma A6, one can easily verify that if

$$\omega^2 \leq \min\{\frac{25}{64(1+4\kappa)^2}, \frac{1}{57(1+4\kappa)^2}, \frac{2}{13}\} = \frac{1}{57(1+4\kappa)^2}, \tag{10}$$

namely, $w < \frac{1}{\sqrt{57}(1+4\kappa)}$, then $\delta(v^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$.

3.2 Upper bound for $\|d_x^f\|^2 + \|d_s^f\|^2$

Now we are ready to derive an upper bound for $\|d_x^f\|^2 + \|d_s^f\|^2$. We first give the following Lemma.

Lemma 5. *Let $x>0$ and $s>0$ be two n-dimensional vectors, and let $M \in R^{n\times n}$ be a $P_*(\kappa)$-matrix. Then the solution (u,z) of the linear system*

$$DMDu - z = a, \quad u + z = b, \tag{11}$$

satisfies the following relations

$$\|u\| \leq \sqrt{1+2\kappa}\|a+b\|, \tag{12}$$

$$\|u\|^2 + \|z\|^2 \leq \|b\|^2 + 2\kappa\|a+b\|^2 + 2\sqrt{1+2\kappa}\|a+b\|\cdot\|a\|, \tag{13}$$

where $D=(S^{-1}X)^{\frac{1}{2}}$.

According to Lemma 5, by replacing u, z, a and b with $d_x^f, d_z^f, \theta\nu\upsilon s^{-1}r^0$ and $\beta\upsilon$, respectively, the inequality (13) becomes

$$\begin{aligned} \|d_x^f\|^2 + \|d_s^f\|^2 \;\leq\; & \|\beta\upsilon\|^2 + 2\kappa\|\theta\nu\upsilon s^{-1}r^0 + \beta\upsilon\|^2 \\ & + 2\sqrt{1+2\kappa}\|\theta\nu\upsilon s^{-1}r^0 + \beta\upsilon\|\|\theta\nu\upsilon s^{-1}r^0\|. \end{aligned} \tag{14}$$

To proceed, we have to specify our initial iterates (x^0, s^0). We assume that ρ_p and ρ_d satisfy

$$\|x^*\|_\infty \leq \rho_p, \quad \max\{\|s^*\|_\infty, \rho_p\|Me\|_\infty, \|q\|_\infty\} \leq \rho_d, \tag{15}$$

for some $(x^*, s^*) \in \mathcal{F}^*$. As usual, we initialize the algorithm with $x^0 = \rho_p e, s^0 = \rho_d e, \mu^0 = \rho_p \rho_d$. For such starting points, we have

$$\|\theta\nu\upsilon s^{-1}r^0\| \leq \frac{3\theta}{\rho_p \upsilon_{\min}}\|x\|_1. \tag{16}$$

By using (16) and (2) in (14), we get

$$\begin{aligned} \|d_x^f\|^2 + \|d_s^f\|^2 \;\leq\; & \beta^2 n(1+2\delta)^2 + 2\kappa\Big(\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1 + \beta\sqrt{n}(1+2\delta)\Big)^2 \\ & + 2\sqrt{1+2\kappa}\Big(\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1 + \beta\sqrt{n}(1+2\delta)\Big)\frac{3\theta\rho(\delta)}{\rho_p}\|x\|_1. \end{aligned} \tag{17}$$

To specify a bound for $\|d_x^f\|^2 + \|d_s^f\|^2$, we need an upper bound for $\|x\|_1$.

Lemma 6. (Lemma 4.5 in [19]) *Let (x, s) be feasible for the perturbed problem (P_ν), $\rho(\delta)$ as defined in Lemma 3. Then*

$$\|x\|_1 \leq (1+4\kappa)n\rho_p\big(2+\rho^2(\delta)\big), \quad \|s\|_1 \leq (1+4\kappa)n\rho_p\big(2+\rho^2(\delta)\big). \tag{18}$$

By substituting (18) into (17), we get

$$\begin{aligned} \|d_x^f\|^2 + \|d_s^f\|^2 \leq f(\delta) = {} & \beta^2 n(1+2\delta)^2 + 2\kappa\Big(3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big) \\ & + \beta\sqrt{n}(1+2\delta)\Big)^2 + 2\sqrt{1+2\kappa}\Big(3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big) \\ & + \beta\sqrt{n}(1+2\delta)\Big)3\theta\rho(\delta)(1+4\kappa)n\big(2+\rho(\delta)^2\big). \end{aligned} \tag{19}$$

We have found that $\delta(\upsilon^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$ holds if the inequality (10) is satisfied. Then by (19), inequality (10) holds if $f(\delta) \leq \frac{4}{57(1+4\kappa)^2}$ holds. To proceed, we give the values of θ and τ by the following Lemma.

Lemma 7. *Let $\delta = \delta(\upsilon)$ be given by (2), for $n \geq 2$, $\rho(\delta) = \delta + \sqrt{1+\delta^2}$ and $\delta \leq \tau$, if $\theta = \frac{1}{52n(1+4\kappa)^{\frac{5}{2}}}$, $\tau = \frac{1}{50(1+4\kappa)^{\frac{3}{2}}}$ and $-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$. Then $f(\delta) \leq \frac{4}{57(1+4\kappa)^2}$.*

3.3 Complexity analysis

In the previous sections, we have suggested that if at the start of an iteration the iterates satisfy $\delta(x, s; \mu) \leq \tau$, then after the feasibility step, the iterates satisfied $\delta(v^f) \leq \frac{1}{\sqrt{2}(1+4\kappa)}$. Hence, after the feasibility step, the proposed algorithm at most

$$1 + \lceil \log_2(\log_2 \frac{1}{\tau(1+4\kappa)})\rceil = 1 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}}))\rceil$$

centering steps are sufficient to get an iterate (x^+, s^+) that satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. Each main iteration consists of one feasibility step and at most

$$1 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}}))\rceil$$

centering steps. After the main iteration, both the norms of the residual vectors and the duality gap are reduced by the factor $1-\theta$. So the total number of main iterations is bounded above by $\frac{1}{\theta}\log\frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}$. If we take the value of θ as in Lemma 7, then the total number of inner iterations is bounded above by

$$52(2 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}}))\rceil)(1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}.$$

Theorem 2. *If problem (P) has optimal solution (x^*, s^*) such that $\|x^*\|_\infty \leq \rho_p$ and $\max\{\|s^*\|_\infty, \rho_p\|Me\|_\infty, \|q\|_\infty\} \leq \rho_d$, then after at most*

$$52(2 + \lceil \log_2(\log_2(50(1+4\kappa)^{\frac{1}{2}}))\rceil)(1+4\kappa)^{\frac{5}{2}} n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\varepsilon}$$

iterations the algorithm finds an ε-solution of $P_*(\kappa)$-LCP.

4. Numerical results

In this section, we present numerical results of the full-Newton step IIPM for $P_*(\kappa)$-LCP under the MatlabR2012a environment.

We consider the following problems. It should be pointed out that Problems 5.1-5.2 are $P_*(0)$-LCPs, which are important subclasses of $P_*(\kappa)$-LCP.

Problem 5.1 (Fathi's example in [7]).

$$M = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -3 \\ 1 \\ -1 \\ 5 \\ 4 \\ -1.5 \end{bmatrix}.$$

Problem 5.2 (Fathi's example in [7])

$$M = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}.$$

These problems are solved by applying the proposed algorithm. We take $\varepsilon = 10^{-5}$ as the accuracy parameter for all problems in the implementation. With an arbitrary positive point being initial point (x^0, s^0), the number of inner iterates (IIN) and main iterates (MIN) for combinations of the update parameters θ, value of β and the dimension of the problems 5.1-5.2 are shown in Tables 1-2, respectively. Here, we should notice that β lies in different interval for the problem with different dimension. In the following numerical results, we always choose β for three different values, which are the endpoint and midpoint of the interval corresponding to different dimension.

Table 1: The number of iterations for the problems 5.1.

x^0	s^0	DIM	β	θ	MIN	IIN
			-.0222	.0027	4176	8354
$(.514, \cdots, .514)$	$(.594, \cdots, .594)$	7	-.0111	.0027	4176	6266
			0	.0027	4176	4874

Table 2: The number of iterations for the problems 5.2.

x^0	s^0	DIM	β	θ	MIN	IIN
			-.0416	.0096	1031	2062
$(.29, .29)$	$(.729, .729)$	2	-.0208	.0096	1031	1374
			0	.0096	1031	1374
			-.0340	.0064	1549	3098
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	3	-.0170	.0064	1549	2065
			0	.0064	1549	1936
			-.0294	.0048	2067	4134
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	4	-.0147	.0048	2067	2756
			0	.0048	2067	2480
			-.0263	.0038	2584	5168
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	5	-.0132	.0038	2584	3876
			0	.0038	2584	3100
			-.0240	.0032	3102	6204
$(.29, \cdots, .29)$	$(.729, \cdots, .729)$	6	-.0120	.0032	3102	4653
			0	.0032	3102	3619

5. Concluding remarks

In this paper, we propose a full-Newton step IIPM based on new directions for $P_*(\kappa)$-LCP. The new directions are defined by equation $s\Delta^f x + x\Delta^f s = \beta xs$, where$-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$. The iteration bound coincides with the currently best known iteration bound of IIPMs for $P_*(\kappa)$-LCP.

For further research, this algorithm may be possible extended to the Cartesian $P_*(\kappa)$-LCP over symmetric cones.

Also, in this paper, the case $-\frac{1}{17\sqrt{n}(1+4\kappa)^{\frac{3}{2}}} \leq \beta \leq 0$ is discussed, a larger range of β will be explored.

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