Application of anti-control strategy based on a modified washout filter controller

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Abstract. The problem of anti-control of Hopf bifurcation for Lü system is considered in this paper. A modified washout filter-aided dynamic feedback control law is introduced for the problem. The necessary conditions are presented in the controlled system, so that a certain bifurcation is created at equilibria with preferred stability. In addition, we find the control law can be applied to control the stability of the original bifurcated solution. Anti-control of chaotic attractor is also given. The direction of bifurcation and the stability of the bifurcating limit cycle are determined by the normal form theory and the center manifold theorem. Finally, some computer simulations are provided to illustrate the efficiency of the anti-control approach.
Keywords: Hopf bifurcation, limit cycle, Lü system, anti-control strategy.

1. Introduction

Bifurcation and chaos are well known for their complexity, and have been intensively studied in various fields such as physics, chemistry, biology, engineering and information. In recent years, control and anti-control of bifurcation have attracted the interest of many scholars. Bifurcation control means that a controller is designed to change the bifurcation characteristics and then some satisfactory dynamic behaviors are obtained [1, 2, 3, 4, 5, 6, 7, 8, 9]. Different from bifurcation control, an ideal bifurcation is generated artificially in a suitable location, which is called anti-control of bifurcation [10, 11]. Anti-control of Hopf bifurcation is regarded as a way to produce limit cycles in nonlinear dynamic systems [12, 13, 14]. In many practical applications, artificial bifurcation is required when it is beneficial and useful. Washout filter-based control strategy has been

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fully recognized in various fields. These include chaotic synchronization control [15], bifurcation control [16, 17], and anti-control of Hopf bifurcation [18, 19]. The main advantage of this controller is that the equilibrium structure of the original system can be guaranteed.

Lü system is described by Ref. [20] in the following:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where x, y, z are the state variables and a, b, c are real parameters. When the parameters are fixed as $a = 36, b = 3, c = 20$, system (1) is chaotic. For Lü system, most of the research articles mainly focus on the bifurcation analysis [21], bifurcation and chaos control via various controllers [22, 23] and chaos synchronization control [24]. As far as we know, anti-control of Hopf bifurcation for Lü system has not been found before.

In this paper, applying anti-control strategy to produce bifurcation in desired location in Lü system is discussed. In fact, we indicate a modified washout filter control law for generating limit cycles and ideal dynamic state. The rest of this paper is arranged as follows. In Section 2, the local stability of equilibria of the system is analyzed. In Section 3, a modified washout filter-aided controller is used to the model, and then the conditions of creating Hopf bifurcation at the equilibrium point are given. In Section 4, based on the normal form theory and the center manifold theorem, the direction of bifurcation and the stability of the bifurcating periodic solutions are studied. In Section 5, the results of numerical simulation show that the anti-control strategy is effective. Finally, it makes a summary in Section 6.

2. Local stability of the equilibria

The Jacobian matrix of system (1) at the point (x_0, y_0, z_0) is given by

$$J = \begin{pmatrix} -a & a & 0 \\ -z_0 & c & -x_0 \\ y_0 & x_0 & -b \end{pmatrix}. \tag{2}$$

The characteristic equation of system (1) is

$$\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0, \tag{3}$$

where

$$\begin{aligned} k_1 &= a + b - c, \\ k_2 &= x_0^2 + az_0 + ab - ac - bc, \\ k_3 &= ax_0^2 + ax_0y_0 + abz_0 - abc. \end{aligned}$$

For any parameter values, system (1) has equilibrium $S_0(0,0,0)$. For $bc > 0$, it also has the symmetric equilibria $S_{\pm}(\pm\sqrt{bc}, \pm\sqrt{bc}, c)$. By the Routh-Hurwitz theorem, we get that:
(1) the equilibrium $S_0(0,0,0)$ is a saddle point, and is unstable for any $a, b, c \in (0, +\infty)$.
(2) for any $a + b > 3c$, the symmetric equilibria $S_{\pm}(\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ are stable.
(3) for any $a+b = 3c$, the characteristic equation at equilibria $S_{\pm}(\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ has a pair of pure imaginary eigenvalues $\lambda_{1,2} = \pm\mathrm{i}\sqrt{\frac{k_3}{k_1}}$.
(4) for any $a+b < 3c$, the symmetric equilibria $S_{\pm}(\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ are unstable.

3. Creation of Hopf bifurcations

In order to anti-control Hopf bifurcation effectively, the objective of this section is to design a control law to model (1), so that the system can create bifurcation at the location we need. A modified washout filters controller is added to the system, then the controlled system obtained in this paper is shown below:

$$\begin{cases} \dot{x} = a(y - x) + u, \\ \dot{y} = -xz + cy + u, \\ \dot{z} = xy - bz, \\ \dot{u} = m(x - y) + n(x - y)^3 - du, \end{cases} \tag{4}$$

where u is a control input, and the real parameters m,n,d are control gains. The first two equations are chosen to control. Noticed that the equilibrium points of the original system are preserved during the control process, which is similar to the traditional washout filters controller. It is worth mentioning that the controller designed in this paper is different from the published controllers.

Next, the conditions of Hopf bifurcation at equilibrium point in the controlled system are discussed. To facilitate further discussion, we assume that $a, b, c \in (0, +\infty)$.

3.1 Existence of Hopf bifurcation at the steady state S_0

It is pointed out that if $a, b, c \in (0, +\infty)$, S_0 is a saddle point, which do not occur Hopf bifurcation in the original system(1). The following will realize Hopf bifurcation creating from S_0. The characteristic equation of the controlled model (4) at S_0 is

$$\lambda^4 + l_1\lambda^3 + l_2\lambda^2 + l_3\lambda + l_4 = 0, \tag{5}$$

where

$$\begin{aligned} l_1 &= a + b - c + d, \\ l_2 &= ab - ac - bc + ad + bd - cd, \\ l_3 &= -abc + abd - acd - bcd + cm, \end{aligned}$$

$$l_4 = -abcd + bcm.$$

Computing the following determinants:

$$\Delta_1 = l_1,$$
$$\Delta_2 = \begin{vmatrix} l_1 & 1 \\ l_3 & l_2 \end{vmatrix} = l_1 l_2 - l_3,$$
$$\Delta_3 = \begin{vmatrix} l_1 & 1 & 0 \\ l_3 & l_2 & l_1 \\ 0 & l_4 & l_3 \end{vmatrix} = l_3(l_1 l_2 - l_3) - l_1^2 l_4,$$
$$\Delta_4 = \begin{vmatrix} l_1 & 1 & 0 & 0 \\ l_3 & l_2 & l_1 & 1 \\ 0 & l_4 & l_3 & l_2 \\ 0 & 0 & 0 & l_4 \end{vmatrix} = l_4 \Delta_3.$$

If $l_1 > 0, l_3 > 0, l_4 > 0$ and $l_3(l_1 l_2 - l_3) - l_1^2 l_4 > 0$, then $\Delta_i > 0 (i = 1, 2, 3, 4)$. Based on Routh-Hurwitz criteria, all roots of the characteristic equation have negative real parts. Thus, S_0 is locally asymptotically stable. If $l_3(l_1 l_2 - l_3) - l_1^2 l_4 \leq 0$, and $l_i > 0 (i = 1, 2, 3, 4)$, S_0 is unstable and non-hyperbolic. Choosing m as the bifurcation parameter. When

$$m = m_0 = \frac{-a^2 c + ac^2 + a^2 d - 2acd + c^2 d + ad^2 - cd^2}{c}, \tag{6}$$

the characteristic Eq.(5) has a pair of pure imaginary roots as follows:

$$\lambda_{1,2} = \pm \mathrm{i}\omega_0 = \pm \mathrm{i}\sqrt{ad - ac - cd} \quad (ad - ac - cd > 0). \tag{7}$$

The other two eigenvalues are

$$\lambda_3 = -b < 0, \tag{8}$$
$$\lambda_4 = -a + c - d < 0. \tag{9}$$

It is easy to verify that the following transversality condition holds:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda = \mathrm{i}\omega_0}) = \frac{c}{2(a^2 - 3ac + c^2 + 3ad - 3cd + d^2)} \neq 0. \tag{10}$$

So, according to Hopf bifurcation theory [25], the controlled model (4) occurs Hopf bifurcation at S_0.

3.2 Existence of Hopf bifurcation at the steady states S_+ and S_-

Since equilibrium S_+ and S_- are symmetric, we only consider the bifurcation at point S_+. The characteristic equation of the controlled model (4) at S_+ is

$$\lambda^4 + h_1 \lambda^3 + h_2 \lambda^2 + h_3 \lambda + h_4 = 0, \tag{11}$$

where

$$h_1 = a + b - c + d,$$
$$h_2 = ab + ad + bd - cd,$$
$$h_3 = 2abc + abd,$$
$$h_4 = 2abcd - 2bcm.$$

As discussed above, choosing m as the bifurcation parameter, when

$$m = m_0 = \frac{2abcdh_1^2 - h_1h_2h_3 + h_3^2}{2bch_1^2}, \tag{12}$$

the characteristic Eq.(11) has a pair of pure imaginary roots:

$$\lambda_{1,2} = \pm \mathrm{i}\omega_0 = \pm \mathrm{i}\sqrt{\frac{h_3}{h_1}}, \tag{13}$$

and other two negative roots. The following transversality condition is also satisfied:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda=\mathrm{i}\omega_0}) = \frac{2bc(h_3 - 3h_1\omega_0^2)}{(h_3 - 3h_1\omega_0^2)^2 + (2h_2\omega_0 - 4\omega_0^3)^2} \neq 0. \tag{14}$$

Therefore, according to Hopf bifurcation theory [25], the controlled model (4) occurs Hopf bifurcation at S_+ . Section 2 shows that system (1) will undergo Hopf bifurcation at S_+ when $c = \frac{a+b}{3}$. So our control scheme also enlarges the parameter region in which Hopf bifurcation might occur at S_+.

4. Stability of the bifurcating limit cycle

By the normal form theory and the center manifold theorem [25], the direction of bifurcation and the stability of created limit cycle are determined in this section. Equilibrium S_0 is chosen as an example to discuss, and other equilibria can be similar. By the linear transform $(x, y, z, u)^T = P(X, Y, Z, U)^T$, where

$$P = \begin{pmatrix} \omega_0 & a - c & 0 & \frac{d}{c-d} \\ \omega_0 & a & 0 & -1 \\ 0 & 0 & 1 & 0 \\ (a-c)\omega_0 & -ac - \omega_0^2 & 0 & a + d \end{pmatrix}, \tag{15}$$

system (4) is transformed into the normal form

$$\begin{cases} \dot{X} = -\omega_0 Y + F_1(X, Y, Z, U), \\ \dot{Y} = \omega_0 X + F_2(X, Y, Z, U), \\ \dot{Z} = -bZ + F_3(X, Y, Z, U), \\ \dot{U} = (-a + c - d)U + F_4(X, Y, Z, U), \end{cases} \tag{16}$$

where F_1, F_2, F_3, F_4 are high order nonlinear functions of X, Y, Z, U. Because the expressions are too complex, it is omitted here. With the aid of detailed calculation, we get:

$$C_1(0) = \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \tag{17}$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{\alpha'(0)}, \tag{18}$$

$$\tau_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(0)\}}{\omega_0}, \tag{19}$$

$$\beta_2 = 2Re\{C_1(0)\}. \tag{20}$$

The expressions of $g_{11}, g_{02}, g_{20}, g_{21}$ can be found in Ref. [25]. Then we can get the following conclusions for the controlled model in accordance with the Ref. [25].

The properties of Hopf bifurcation in the controlled system (4) is determined by the parameters μ_2, β_2 and τ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical, and it is subcritical if $\mu_2 < 0$. Parameter β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$, the bifurcating periodic solutions is stable, and it is unstable if $\beta_2 > 0$. Parameter τ_2 determines the period of the bifurcating periodic solution. If $\tau_2 > 0$, the period increases and the period decreases if $\tau_2 < 0$.

5. Numerical analysis

In this section, some numerical simulations are presented to verify the above theoretical analyses. For the controlled model (4), at the equilibrium S_0, we choose $a = 2, b = 2, c = 1, d = 4$, then the bifurcation critical value is $m_0 = 18$. Choosing $n = 2$, we have results: $\mu_2 = 71.74987, \tau_2 = -1.06250, \beta_2 = -2.6574$. There exits a stable periodic orbit near the equilibrium point S_0. The bifurcation figure is shown in Fig. 1.

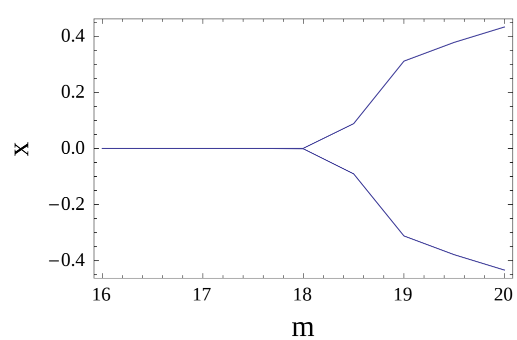

Figure 1: Bifurcation diagram of controlled model at the equilibrium S_0.

At the equilibrium S_+, as discussed in Section 2, S_+ is stable in the original system (1) when $c < \frac{a+b}{3}$. While in the controlled system (4), we choose $a = 1, b = 5, c = 1.5, d = 2$, the bifurcation critical value $m_0 = -0.60355$ is obtained. Then we have $\mu_2 = -0.02383, \tau_2 = 0.004163, \beta_2 = -0.01149$ when $n = 0.1$. This suggests that the controlled system (4) occur Hopf bifurcation at S_+ when $c < \frac{a+b}{3}$. The figure is shown in Fig. 2.

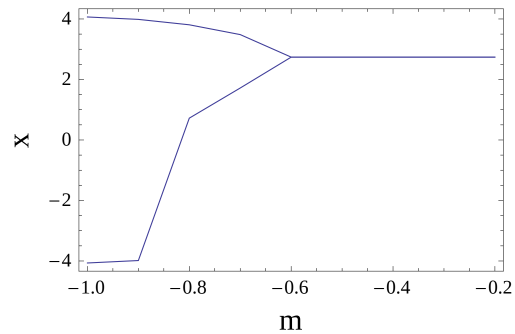

Figure 2: Bifurcation diagram of controlled model at the equilibrium S_+ when $c < \frac{a+b}{3}$.

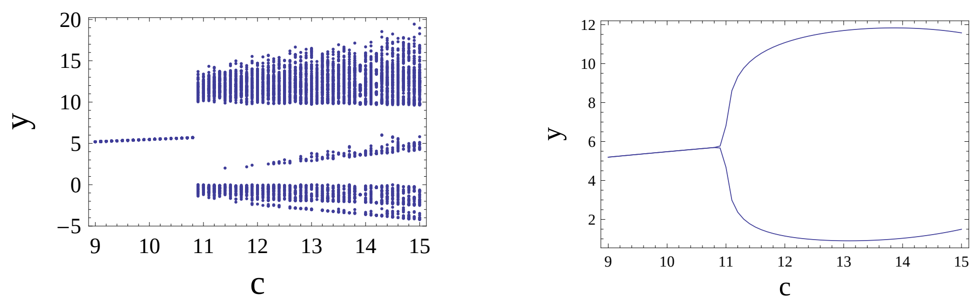

Figure 3: Original and controlled model bifurcate at the equilibrium S_+.

Section 2 also points out that for any $c = \frac{a+b}{3}$, the characteristic equation at equilibrium S_+ has a pair of pure imaginary eigenvalues $\lambda_{1,2} = \pm i\sqrt{\frac{k_3}{k_1}}$. At this point, c is chosen as the bifurcation parameter. We choose $a = 30, b = 3$ for example, the original system (1) can occur Hopf bifurcation at $c_0 = 11$. After some calculations, the stability index $\beta_2 = 0.475766$ is obtained, which shows the bifurcating periodic solution is unstable. Now we will show the controller can change the stability of the existing bifurcation. In the controlled system (4), setting $a = 30, b = 3, d = 1, m = 0.01$, the bifurcation critical value is still at $c_0 = 11$. When $n = 4$, the stability index $\beta_2 = -0.0011313$ shows the bifurcating periodic solutions is stable. So the anti-control strategy in this paper can not only enlarge the Hopf bifurcation parameter region, but also stabilize the existing bifurcation at S_+. The comparison of bifurcation figures are shown in Fig. 3.

Next, we are also interested in anti-control of the chaotic attractor. The original system is chaotic when $a = 36, b = 3, c = 20$. Fig. 4 shows the chaotic attractor and the limit cycle generated from the equilibrium when $d = 46, n = 0.1$ in the controlled system (4).

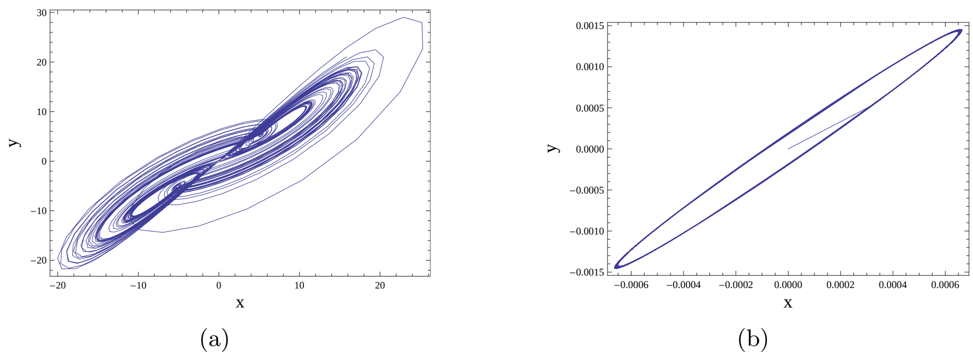

Figure 4: Anti-control of the chaotic attractor.

6. Conclusions

This paper studies the problem of anti-control of Hopf bifurcation for chaotic Lü system. A modified washout filter-aided dynamic feedback control law is designed for the creation of Hopf bifurcations. That is, we succeed in generating limit cycle at equilibria through appropriate control parameters. Not only that, but we also achieve to stabilize an unstable bifurcated solution of the original system. The direction of bifurcation and the stability of created limit cycle are analyzed based on the normal form theory. It is worth mentioning that this controller has realized the stability control of an unstable Hopf bifurcation of the original system, which is rare in the existing literature. Simulation results indicate the effectiveness of the proposed approach. The anti-control strategy can also be applied to other chaotic systems.

Acknowledgements

This work is supported by Natural Science Foundation of Fujian Province of China (Nos. 2017J05012, 2018J01550); and National Natural Science Foundation of China (No. 61603174); and the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Date Science and Statistics.

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Accepted: 23.07.2019