Simple endomorphism semirings of semilattices

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Abstract. In the paper, various types of simple endomorphism semirings of semilattices are investigated.

Keywords: semiring, semimodule, semilattice, endomorphism.

1. Introduction

1.1 Semilattices. An (algebraic) semilattice M $(= M(+))$ is a commutative and idempotent semigroup (i.e., $x + x = x$, $x + y = y + x$ and $x + (y + z) = (x + y) + z$ for all $x, y, z \in M$). The basic relation of order is given on M by $x \leq y$ if and only if $x + y = y$. We have $x + y = \sup(x, y)$ for all $x, y \in M$.

An element $w \in M$ is the smallest (greatest, resp.) element of the ordered set $M(\leq)$ exactly when $w = 0_M$ ($w = o_M$, resp.) is the uniquely determined (additively) neutral (absorbing, resp.) element of the semilattice $M(+)$. If such an element exists in M, we simply write $0_M \in M$ ($o_M \in M$, resp.). If not, then we write $0_M \notin M$ ($o_M \notin M$, resp.). In the sequel, we will frequently use the

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following notation: $N = M \setminus \{o_M\}$, $L = M \setminus \{0_M\}$ and $K = M \setminus \{0_M, o_M\} = N \cap L$.

A non-empty subset A of M is an *ideal* of the semilattice M if $M + A \subseteq A$ (then $M + A = A$). An ideal A is called *prime* if the set $M \setminus A$ is a subsemilattice of M.

1.2 Semirings. A semiring S ($= S(=, \cdot)$) is an algebraic structure equipped with two associative binary operations (usually denoted as addition and multiplication), where the addition os commutative and the multiplication distributes over the addition.

A non-empty subset I of S is a *left* (*right*) *ideal* of S if $SI \cup (I + I) \subseteq I$ ($IS \cup (I + I) \subseteq I$), I is an *ideal* if it is both left and right ideal, and I is a *bi-ideal* if $SI \cup IS \cup (S + I) \subseteq I$. In the latter case, the relation $(I + I) \cup \mathrm{id}_S$ is a congruence of the semiring S.

An element $a \in S$ is called *left (right) multiplicatively absorbing* if $ab = a$ ($ba = a$) for every $b \in S$. The set of all right multiplicatively absorbing elements will be denoted by $\underline{R}(S)$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is an ideal of S and $\underline{R}(S) \subseteq I$ for every right ideal of S.

The semiring S is called (*congruence-*)*simple* if S has precisely two congruence relations.

1.3 Endomorphism semirings. If M ($= M(+)$) is a commutative semigroup then the set ($\underline{E}(M) =$) $\underline{E}$ of all endomorphisms of M is a semiring via $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$. The semiring $\underline{E}$ is unitary, since the identity automorphism $\mathrm{id}_M = 1_{\underline{E}}$ is the (unique) multiplicatively neutral element of $\underline{E}$.

If M is a semilattice then the semiring $\underline{E}$ is additively idempotent. This means that the additive semigroup $\underline{E}(+)$ is a semilattice, too.

2. $0, 1$-preserving endomorphisms (a)

Let M be a semilattice such that $0_M, o_M \in M$ and $|M| \geq 4$. The set $\underline{E}_{0,1} = \{\, f \in \underline{E} \mid f(0_M) = 0_M, f(o_M) = o_M \,\}$ is a unitary subsemiring of $\underline{E}$. The set $\underline{D}_{0,1} = \{\, f \in \underline{E}_{0,1} \mid f(M) = \{0_M, o_M\} \,\}$ is an ideal of the semiring $\underline{E}_{0,1}$ and there is a one-to-one correspondence between prime ideals of the semilattice M and the endomorphisms from $\underline{D}_{0,1}$; if A is a prime ideal then the corresponding endomorphisms q_A is given by $q_A(A) = \{o_M\}$ and $q_A(M \setminus A) = \{0_M\}$. If $x \in N = M \setminus \{o_M\}$ then $A_x = \{\, y \in M \mid y \not\leq x \,\}$ is just the prime ideal determined by x and the corresponding endomorphism will be denoted by q_x. We put $\underline{B}_{01} = \{\, q_x \mid x \in N \,\}$ and $C_{0,1} = \{\, q_{x_1} + \cdots + q_{x_n} \mid n \geq 1, x_i \in N \,\}$. Then $\underline{C}_{0,1}$ is just the subsemiring of $\underline{D}_{0,1}$ generated by the set $\underline{B}_{0,1}$ and one sees readily that $\underline{C}_{0,1} = \underline{B}_{0,1}$ iff the semilattice M is a lattice.

2.1 The one-element set $\{o_M\}$ is a prime ideal iff $N + N = N$. The corresponding endomorphism will be denoted by $\underline{\xi}$. We have $\underline{\xi}(N) = \{0_M\}$, $\underline{\xi} + f = f$ for every

$f \in \underline{E}_{0,1}$, and if $o_N \in N$ then $\underline{\xi} = q_{o_N}$. If $o_N \notin N$ then $\underline{\xi} \notin \underline{B}_{0,1}$. Anyway, if E is a subsemiring of $\underline{E}_{0,1}$ such that $\underline{\xi} \in E$ then $\underline{\xi} = 0_E$.

2.2 We have $q_{0_M} + f = q_{0_M}$ for every $f \in \underline{E}_{0,1}$. In particular, if E is a subsemiring of $|E_{0,1}$ such that $q_{0_M} \in E$ then $q_{0_M} = o_E$. Clearly, $q_{0_M} = q_L$, $L = M \setminus \{0_M\}$.

2.3 If E is a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ then $\underline{C}_{0,1} \subseteq E$, $o_E = q_{0_M} \in E$ and $\underline{R}(E) = E \cap \underline{D}_{0,1}$.

2.4 Assume that the ordered set $K(\leq)$ (or $L(\leq)$) is downwards directed (here $L = M \setminus \{0_M\}$ and $K = M \setminus \{0_M, o_M\} = N \cap L$). Then the set ${}^1\underline{F}_{0,1} = \{\, f \in \underline{E}_{0,1} \,|\, 0_M \in f(K)\,\}$ is a subsemiring of $\underline{E}_{0,1}$. We have $\underline{E}_{0,1}\,{}^1\underline{F}_{0,1} \subseteq {}^1\underline{F}_{0,1}$ and ${}^1\underline{F}_{0,1}$ is a left ideal of the semiring $\underline{E}_{0,1}$. Moreover, $\underline{B}_{0,1} \setminus \{q_{0_M}\} \subseteq {}^1\underline{F}_{0,1}$. More generally, if A is a prime ideal, $A \neq L$, then $q_A \in {}^1\underline{F}_{0,1}$. The set ${}^1\underline{F}'_{0,1} = {}^1\underline{F}_{0,1} \cup \{q_{0_M}\}$ is a left ideal of $\underline{E}_{0,1}$ and $\underline{D}_{0,1} \subseteq {}^1\underline{F}'0,1$. Furthermore, ${}^2\underline{F}0,1$ and ${}^2\underline{F}'_{0,1} = {}^2\underline{F}_{0,1} \cup \{q_{0_M}\}$ are subsemirings of $\underline{E}_{0,1}$, where ${}^2\underline{F}_{0,1}$ is the set of the endomorphisms $f \in \underline{E}_{0,1}$ such that the set $f(K)$ is downwards cofinal in H (i.e., for every $v \in K$ there is $u \in K$ with $f(u) \leq v$). Now, ${}^1\underline{F}_{0,1}$ is an ideal of ${}^2\underline{F}_{0,1}$, ${}^1\underline{F}'_{0,1}$ is an ideal of ${}^2\underline{F}'_{0,1}$ and the semirings ${}^2\underline{F}_{0,1}$ and ${}^2\underline{F}'_{0,1}$ are unitary.

2.5 Assume that the ordered set $K(\leq)$ (or $N(\leq)$) is upwards directed (equivalently, $N = N + N$). Then the set ${}^1\underline{G}_{0,1} = \{\, f \in \underline{E}_{0,1} \,|\, o_M \in f(K)\,\}$ is a subsemiring of $\underline{E}_{0,1}$. We have $\underline{E}_{0,1}\,{}^1\underline{G}_{0,1} \subseteq {}^1\underline{G}_{0,1}$, $\underline{E}_{0,1} + {}^1\underline{G}_{0,1} \subseteq {}^1\underline{G}_{0,1}$ and ${}^1\underline{G}_{0,1}$ is a left ideal of the semiring $\underline{E}_{0,1}$. Moreover, $\underline{D}_{0,1} \setminus \{\underline{\xi}\} \subseteq {}^1\underline{G}_{0,1}$ and we put ${}^1\underline{G}'_{0,1} = {}^1\underline{G}_{0,1} \cup \{\underline{\xi}\{$, so that $\underline{D}_{0,1} \subseteq {}^1\underline{G}'_{0,1}$.

Let ${}^2\underline{G}_{0,1}$ be the set of the endomorphisms $f \in \underline{E}_{0,1}$ such that the set $f(K)$ is upwards cofinal in K (i.e., for every $v \in K$ there is $u \in K$ with $v \leq f(u)$. It is easy to see that ${}^2\underline{G}_{0,1}$ and ${}^2\underline{G}'_{0,1} = {}^2\underline{G}_{0,1} \cup \{\underline{\xi}\}$ are subsemirings of $\underline{E}_{0,1}$. Besides, ${}^1\underline{G}_{0,1}$ is a bi-ideal of the semiring ${}^2\underline{G}_{0,1}$ and ${}^1\underline{G}'_{0,1}$ is an ideal of ${}^2\underline{G}'_{0,1}$, The semirings ${}^2\underline{G}_{0,1}$ and ${}^2\underline{G}'_{0,1}$ are unitary.

2.6 Let $\alpha = (u, v) \in K \times K$. Define f_α by $f_\alpha(0_M) = 0_M$, $f_\alpha(y) = v$ for $0_M < y \leq u$ and $f_\alpha(z) = o_M$ for $z \not\leq u$. Then $f_\alpha \in \underline{E}_{0,1}$, $f_\alpha(u) = v$, $f_\alpha(M) = \{0_m, v, o_<\}$ and $q_u \leq f_\alpha$. If $u = o_N$ then $f_\alpha(K) = \{v\}$. If $u \neq o_N$ then $f_\alpha(K) = \{v, o_M\}$, and if $N = N + N$ then $f_\alpha \in {}^1\underline{G}_{0,1}$ (see 2.5).

Assume that $N = N + N$ and define f'_α by $f'_\alpha(x) = 0_M$ for $x \leq u$, $f'_\alpha(y) = v$ for $y \in N$, $y \not\leq u$, and $f'_\alpha(o_M) = o_M$. Again, $f'_\alpha \in \underline{E}_{0,1}$, and if $u \neq o_N$ then $f'_\alpha(M) = \{0_M, v, o_M\}$. If $u \neq 0_M, o_N$ then then $f'_\alpha(K) = \{0_M, v\}$ (if $K(\leq)$ is downwards directed then $f'_\alpha \in^1 \underline{F}_{0,1}$ - see 2.4).

2.7 Let $\beta = (u, v, w) \in N \times K \times K$, $u < v$. Define g_β by $g_\beta(x) = 0_M$ for $x \leq u$, $g_\beta(y) = w$ for $y \leq v$, $y \not\leq u$, and $g_\beta(z) = o_M$ for $z \not\leq v$. Then $g_\beta \in \underline{E}_{0,1}$, $g_\beta(u) = 0_M$, $g_\beta(v) = w$, $g_\beta(M) = \{0_M, w, o_M\}$ and $q_v \leq g_\beta$. If $u = 0_M$ then $g_\beta = f_\alpha$, $\alpha = (v, w)$ (see 2.6). If $u \neq 0_M$ then $\{0_M, w\} \subseteq g_\beta(K)$ and $g_\beta(K) = \{0_M, w, o_M\}$ iff $v \neq o_N$. If $v \neq o_N$ then $\{o_M, w\} \subseteq g_\beta(K)$ and $g_\beta(K) = \{0_M, w, o_M\}$ iff $u \neq o_M$.

If $K(\leq)$ is downwards directed and $u \neq 0_M$ then $g_\beta \in {}^1\underline{F}_{0,1}$ (see 2.4). If $N = N + N$ and $w \neq o_N$ then $g_\beta \in {}^1\underline{G}_{0,1}$ (see 2.5).

2.8 Put $\underline{E}'_{0,1} = \underline{D}_{0,1} + \underline{E}_{0,1}$. If $N+N = N$ then $\underline{\xi} \in \underline{D}_{0,1}$ and we get $\underline{E}'_{0,1} = \underline{E}_{0,1}$. Assume, therefore, that $N+N \neq N$. Then $\mathrm{id_M} \notin \underline{\mathrm{E}}'_{0,1}$, and hence $\underline{E}'_{0,1}$ is a proper bi-ideal of the semiring $\underline{E}_{0,1}$ (in particular, the semiring $\underline{E}_{0,1}$ is not simple).

Lemma 2.8.1. *Let $f \in \underline{E}_{0,1}$, $B = \{\, x \in M \mid f(x) = o + M \,\}$ and $C = M \setminus B$. Then:*
(i) *$f \in \underline{E}'_{0,1}$ iff there is a prime ideal A of M such that $A \subseteq B$.*
(ii) *If $f \in \underline{E}'_{0,1}$ and $A \neq \{o_M\}$ then $o_M \in f(K)$* (see 2.5)*.*
(iii) *If $o_C \in C$ then $f \in \underline{E}'_{0,1}$.*

Proof. It is easy. □

2.9 Put $\underline{E}''_{0,1} = \underline{B}_{0,1} + \underline{E}_{0,1}$. If $o_N \in N$ then $\underline{\xi} = q_{o_N} \in \underline{B}_{0,1}$ and we have $\underline{E}''_{0,1} = \underline{E}_{0,1}$. Assume, therefore, that $o_N \notin N$. Then $\mathrm{id}_M \notin \underline{E}''_{0,1} \subseteq \underline{E}'_{0,1}$.

Lemma 2.9.1. *Let $f \in \underline{E}_{0,1}$, $B = \{\, x \in M \mid f(x) = o_M \,\}$ and $C = M \setminus B$. Then:*
(i) *$f \in \underline{E}''_{0,1}$ iff there is $w \in N$ such that $\{\, z \in M \mid z \nleq w \,\} \subseteq B$.*
(ii) *If $f \in \underline{E}''_{0,1}$ and $q_u \leq f$ for $u \neq o_N$ then $o_M \in f(K)$* (see 2.5)*.*
(iii) *If $o_C \in C$ then $f \in \underline{E}''_{0,1}$.*

Proof. It is easy. □

Proposition 2.10. ([1, 3.8]) *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and E is simple. Then:*
(1) *For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $f \in E$ such that $f(w) \leq v$, $f(w) \nleq u$ and $f(z) \nleq v$ for every $z \nleq w$.*
[2] *If $0_E \in E$ (equivalently, $N + N = N$ and $\underline{\xi} \in E$) then for all $u \in N \setminus \{o_N\}$ and $w \in K$ there is at least one $f \in E$ such that $f(w) \in K$, $f(w) \nleq u$ and $f(z) = o_M$ for every $z \nleq w$.*
[3] *If $0_E \in E$ then for all $u \in N$ and $v \in K$, $u < v$, there is at least one $f \in E$ such that $f(K) \leq v$ and $f(z) \nleq u$ for at least one $z \in K$.*
[4] *If $0_E \in E$ then for every $u \in N \setminus \{o_N\}$ there is at least one $f \in E$ such that $f(K) \subseteq N$ and $f(z) \nleq u$ for at least one $z \in K$.*
[5] *$E \subseteq \underline{E}''_{0,1}$* (see 2.8)*.*

Theorem 2.11. *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and $f_\alpha \in E$ for every $\alpha \in K \times K$* (see 2.6)*. If $E \subseteq \underline{E}''_{0,1}$* (see 2.9) *then the semiring E is simple.*

Proof. Let $\varrho \neq \mathrm{id}_E$ be a congruence of the semiring E. Then there are $g, h \in E$ such that $g < h$, $(g,h) \in \varrho$, and there is $v \in K$ such that $g(v) < h(v)$. Let $u \in K$ and $\alpha = (u,v)$. Then $(q_u, q_{0_M}) = (q_{g(v)} g f_\alpha, q_{g(v)} h f_\alpha) \in \varrho$. It follows that $(q_x, q_{0_M}) \in \varrho$ for every $x \in N$. Finally, if $k, t \in E$ then there are $y, z \in N$ such that $q_y \leq k$ and $q_z \leq t$. Consequently, $(k, q_{0_M}) = (k + q_y, q_{0_M} + q_y) \in \varrho$, $t, q_{0_M}) = (t + q_z, q_{0_M} + q_z) \in \varrho$ and $(k,t) \in \varrho$. □

Theorem 2.12. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$, $f_\alpha \in E$ for every $\alpha = (o_N, v)$, $v \in K \setminus \{o_N\}$* (see 2.6), *and $k(w) = o_N$ for some $k \in E$ and $w \in K \setminus \{o_N\}$. Then the semiring E is simple.*

Proof. Let $\varrho \neq \mathrm{id}_E$ be a congruence of E. Then there are $g, h \in E$ such that $g < h$ and $g(v) < h(v)$ for some $v \in K \setminus \{o_N\}$ (consider the endomorphisms gk, hk and put $v = w$ when necessary). Now, $(q_{o_N}, q_{0_M}) = (q_{g(v)} g f_\alpha, q_{g(v)} h f_\alpha) \in \varrho$, where $\alpha = (o_N, v)$, and it follows easily that $\varrho = E \times E$. □

Lemma 2.13. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{),1}$ such that $o_N \notin E(M \setminus \{o_N\})$. Define a relation $\underline{\tau}$ on E by $(f,g) \in \underline{\tau}$ iff $f \mid M \setminus \{o_N\} = g \mid M \setminus \{o_N\}$. Then:*
(i) *$\underline{\tau}$ is a congruence of the semiring E.*
(ii) *If $q_{0_M} \in E$ then $(f, q_{0_M}) \notin \underline{\tau}$ for every $f \in E$, $f \neq q_{0_M}$.*
(iii) *If $\underline{\xi} = q_{o_N} \in E$ and $(f, \underline{\xi}) \in \underline{\tau}$ then $f(N \setminus \{o_N\}) = \{0_M\}$.*
(iv) *$\underline{\tau} = \mathrm{id}_E$ iff for all $f, g \in E$, $f \neq g$, there is $v \in K \setminus \{o_N\}$ with $f(v) \neq g(v)$.*

Proof. It is easy. □

Theorem 2.14. *Assume that $o_N \in N$. Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$, $f_\alpha \in E$ for every $\alpha = (o_N, v)$, $v \in K \setminus \{o_N\}$* (see 2.6) *and $o_N \notin E(M \setminus \{o_N\})$. Then the semiring E is simple if and only if for all $f, g \in E$, $f \neq g$, there is at least one $w \in K \setminus \{o_N\}$ with $f(w) \neq g(w)$.*

Proof. If E is simple then $\underline{\tau} = \mathrm{id}_E$ follows from 2.3(i), (ii) and hence 2.13 (iv) applies. Conversely, if $\underline{\tau} = \mathrm{id}_E$ then we proceed in a similar way as in the proof of 2.12 to show that E is simple. □

2.15 Let A be a prime ideal of the semilattice M. For every $v \in M$, define $f_{A,v}$ by $f_{A,v}(0_M) = 0_M$, $f_{A,v}(y) = v$ for $y \in L \setminus A$ and $f_{A,v}(A) = \{o_M\}$. Then $f_{A,v} \in \underline{E}_{0,1}$ and $f_{A,v} = \{0_M, v, o_M\}$, provided that $A \neq L = M \setminus \{0_M\}$. Of course, $q_A \leq f_{A,v}$, and so $f_{A,v} \in \underline{E}'_{0,1}$.

Put ${}^3\underline{F}_{0,1} = {}^2\underline{F}'_{0,1} \setminus {}^1\underline{F}_{0,1}$. Then ${}^3\underline{F}_{0,1}$ is a unitary subsemiring of ${}^2\underline{F}'_{0,1}$ and we have ${}^3\underline{F}_{0,1} + {}^2\underline{F}'_{0,1} = {}^3\underline{F}_{0,1}$. The relation $({}^3\underline{F}_{0,1} \times {}^3\underline{F}_{0,1}) \cup ({}^1\underline{F}_{0,1} \times {}^1\underline{F}_{0,1})$

is a congruence of the semiring ${}^2\underline{F}'_{0,1}$ (use 2.4), and hence this semiring os not simple.

Theorem 2.16. ([2, 3.2]) *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{B}_{0,1} \subseteq E$ and $f_{A,w} \in E$ for every prime ideal A of M and all $w \in K$* (see 2.15). *Then $\underline{D}_{0,1} \subseteq E$ and the semiring E is simple if and only if $E \subseteq \underline{E}'_{0,1}$* (see 2.8).

Proof. First, $q_w f_{A,w} = q_A$, and so $D_{0,1} \subseteq E$. The relation $\varsigma = (F \times F) \cup \mathrm{id}_E$, $F = E \cap \underline{E}'_{0,1}$, is a congruence of the semiring E (see 2.8) and $\varsigma \neq \mathrm{id}_E$, since $\underline{D}_{0,1} \subseteq F$. Thus $F = E$, provided that $\varsigma = E \times E$.

As concerns the converse implication, we proceed similarly as in the proof of 2.11 (just notice that $q_{g(v)} g f_{A,v} = q_A$ and $q_{g(v)} h f_{A,v} = q_{0_M}$). □

Corollary 2.17. ([2, 3.3]] *The semiring $\underline{E}_{0,1}$ is simple if and only if $N + N = N$.* □

Corollary 2.18. ([2, 3.3]) *The semiring $\underline{E}'_{0,1}$* (see 2.8) *is always simple.* □

Remark 2.19. (cf. 2.18) Let E be any subsemiring of $\underline{E}_{0,1}$ and $H = E \cap \underline{E}'_{0,1}$. If $H = \{f\}$ is a one-element set then f is a bi-absorbing element of the semiring E. If E is simple and $|H| \geq 2$ then $H = E$ and $E \subseteq \underline{E}'_{0,1}$.

Remark 2.20. Assume that $K(\leq)$ is downwards directed (see 2.4). It is easy to see that the relation $({}^1\underline{F}_{0,1} \times {}^1\underline{F}_{0,1}) \cup \mathrm{id}$ os a congruence of the semiring ${}^1\underline{F}'_{0,1}$. Consequently, this semiring is not simple (cf. 2.15).

Remark 2.21. Let $o_N \notin N = N + N$. Then $\underline{E}''_{0,1} \subseteq {}^1\underline{G}_{0,1}$. If ${}^1\underline{G}_{0,1} = \underline{E}''_{0,1}$ then the semiring ${}^1\underline{G}_{0,1}$ is simple (use 2.11).

Remark 2.22. Let $N = N + N$. We have $\underline{D}_{0,1} \subseteq {}^1\underline{G}_{0,1}$ and the relation $({}^1\underline{G}_{0,1} \times {}^1\underline{G}_{0,1}) \cup \mathrm{id}$ is a congruence of ${}^1\underline{G}'_{0,1}$ Therefore, the semiring ${}^1\underline{G}'_{0,1}$ is not simple. Similarly, the semiring ${}^2\underline{G}'_{0,1}$ is not simple.

3. 0, 1-preserving endomorphisms (b)

The preceding section is immediately continued. Here, we will assume, moreover, that every infinite strictly increasing sequence $x_1 < x_2 < \dots$ of elements from M os *upwards cofinal* in K (or M). The following result is obvious.

Proposition 3.1. (i) *The ordered set $M(\leq)$ is a lattice.*
(ii) $\underline{B}_{0,1} = \underline{C}_{0,1}$ *and $\underline{E}''_{0,1}$ is a subsemiring of $\underline{E}_{0,1}$.*
(iii) *If $N + N \neq N$ then $o_N \notin N$, there is no infinite strictly increasing sequence of elements in M, $\underline{B}_{0,1} = \underline{D}_{0,1}$ and $\underline{E}'_{0,1} = \underline{E}''_{0,1}$.*
(iv) *If $o_N \in N$ then $N + N = N$, there is no strictly increasing sequence of elements in M, $B_{0,1} = \underline{D}_{0,1}$ and $\underline{E}'_{0,1} = \underline{E}''_{0,1} = \underline{E}_{0,1}$.*

(iv) *If* $o_N \notin N = N + N$ *then the set* N *has no maximal element,* $\underline{\xi} \notin \underline{B}_{0,1}$, $\underline{D}_{0,1} = \underline{B}_{0,1} \cup \{\underline{\xi}\}$, $\underline{E}'_{0,1} = \underline{E}_{0,1}$ *and* $\mathrm{id}_M \notin \underline{E}''_{0,1}$. □

Corollary 3.2. $\underline{D}_{0,1} = \underline{B}_{0,1}$ *except for the case when* $o_N \notin N = N + N$. *Then* $\underline{\xi} \notin \underline{B}_{0,1}$ *and* $\underline{D}_{0,1} = \underline{B}_{0,1} \cup \{\underline{\xi}\}$. □

Proposition 3.3. ([1, 4.7]) *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$, E *is simple and* E *does not operate transitively on* K. *Then:*
(i) *Both* E *and* M *are infinite.*
(ii) $o_N \in N$, $M \setminus \{o_N\}$ *is a subsemilattice of* M *and* $E(M \setminus \{o_N\}) \subseteq M \setminus \{o_N\}$.
(iii) E *operates transitively on the set* $K \setminus \{o_N\}$.
(iv) $E9x) = m \setminus \{o_N\}$ *for every* $x \in K \setminus \{o_N\}$.
(v) $M \setminus \{o_N\} \subseteq E(o_N)$.

Proposition 3.4. (i) *If* $N + N \neq N$ *then* $\underline{E}'_{0,1} = \underline{E}''_{0,1}$ *is just the set of the endomorphisms* $f \in \underline{E}_{0,1}$ *such that* $f(x) = o_M$ *for a maximal element* $w \in N$ *and all* $x \nleq w$.
(ii) *If* $o_N \notin N = N + N$ *then* $\underline{E}''_{0,1} = {}^1\underline{G}_{0,1}$ (see 2.5) *and* $\underline{E}'_{0,1} = \underline{E}_{0,1}$.
(iii) *If* $o_N \in N$ *then* $\underline{E}'_{0,1} = \underline{E}''_{0,1} = \underline{E}_{0,1}$.

Proof. It is easy. □

Remark 3.5. Let $f \in \underline{E}_{0,1}$ be such that $f(M) = \{0_M, w, o_M\}$ is a three-element subsemilattice (we have $w \in K$). Put $A = \{\, x \in M \mid f(x) = 0_M \,\}$, $B = \{\, y \in M \mid f(y) = w \,\}$ and $C = \{\, z \in M \mid f(z) = o_M \,\}$. Then $A + A = A$, $B + (A \cup B) = B$ and $M + C = C$. Moreover, $u = o_A \in A$ and $A = \{\, x \in M \mid x \leq u \,\}$. If $v = o_{A\cup B} \in A \cup B$ then $v \in B$, $u < v$, $B = \{\, y \in M \mid y \leq v, y \nleq u \,\}$ and $C = \}\, z \in M \mid z \nleq v \,\}$. Thus $f = g_{(u,v,w)}$ in this case (see 2.7). On the other hand, if $o_{A\cup B} \notin A \cup B$ then $A \cup B = N$, $N + N = N$, $o_N \notin N$, $B = N \setminus A = \{\, y \in N \mid y \nleq u \,\}$ and $C = \{o_M\}$. Thus $f = f'_{(u,w)}$ (see 2.6).

In the remaining part of this section, assume that every infinite strictly decreasing sequence $x_1 > x_2 > \dots$ of elements from M is *downwards cofinal* in K (or L).

Theorem 3.6. ([1, 5.1]) *Assume that* $o_N \notin N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \notin E$.
(i) *If the set* $L = M \setminus \{0_M\}$ *has at least one minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{E}''_{0,1}$ (see 3.4(i),(ii)) *and the following condition is satisfied:*

(ii1) *For all* $u \in N$ *and* $v, w \in K$, $u < v$, *there is ar least one* $f \in E$ *such that* $f(w) \leq v$ *and* $f(w) \nleq u$.

(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{E}''_{0,1}$, *the condition* (i1) *is true and, besides, the following condition is satisfied:*

(ii1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq o_M$ *and* $f(x) \nleq v$ *for every* $x \in K$. □

Theorem 3.7. ([1, 5.2]) *Assume that* $o_N \in N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$.
(i) *If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 3.6(i1) *is satisfied.*
(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if the conditions* 3.6(i1) *and* 3.6(ii1) *are satisfied.* □

Theorem 3.8. ([1, 5.3]) *Assume that* $o_N \notin N = N+N$. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \in E$.
(i) *If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 3.6(i1) *is satisfied and, moreover, the following condition is satisfied as well:*

(i1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq 0_M$ *and* $f(x) \leq v$ *for every* $x \in K$.

(ii) *If the set* L *has no minimal element then the semiring* E *is simple if and only if the conditions* (i1), 3.6(i1) *and* 3.6(ii1) *are satisfied.* □

Corollary 3.9. *Let* E, F *be subsemirings of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \subseteq F$ *and* E *is simple.*
(i) *If* $o_N \in N$ *then* F *is simple.*
(ii) *If* $o_N \notin N = N + N$ *and* $\underline{\xi} \in E$ *then* F *is simple.*
(iii) *If* $o_N \notin N$ *and* $\underline{\xi} \notin F$ *then* F *is simple if and only if* $F \subseteq \underline{E}''_{0,1}$.
(iv) *If* $o_N \notin N = N+N$ *and* $\underline{\xi} \notin F$ *then* F *is simple if and only if the condition* 3.8(i1) *is true for* F. □

Proposition 3.10. *Assume that* $o_N \notin N = N + N$.
(i) $\underline{B}_{0,1} \subseteq {}^1\underline{G}_{0,1} = \underline{E}''0, 1$ *and the semiring* ${}^1\underline{G}_{0,1}$ *is simple.*
(ii) *If* E *is a simple subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* $\underline{\xi} \notin E$ *then* $E \subseteq {}^1\underline{G}_{0,1}$.
(iii) *If* E *is a subsemiring such that* ${}^1\underline{G}_{0,1} \subsetneqq E \subseteq {}^2\underline{G}'_{0,1}$ *then* E *is not simple.*
(iv) *If* E *is a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \nsubseteq {}^2\underline{G}'_{0,1}$ *and* $\underline{\xi} \in E$ *then* E *satisfies* 3.8(i1).

Proof. (i) See 3.4(i) and 2.2.1.
(ii) Use 3.6.
(iii) If E is simple then $\underline{\xi} \in E$. Now, by 3.8, E satisfies 3.8(i1), and consequently $E \nsubseteq {}^2\underline{G}'_{0,1}$.
(iv) There is $\in E \setminus {}^2\underline{G}''_{0,1}$. Then there is $v \in K$ such that $v \nleq f(x)$ for every $x \in N$ and it follows that $w = o_A \in A = f(N) \subseteq N$. Thus $f(N) \leq w$. □

Proposition 3.11. *Assume that* $o_N \in N$. *Then:*
(i) ${}^1\underline{G}_{0,1} = \{ f \in \underline{E}_{0,1} \,|\, f(o_N) = o_N \}$ *and* ${}^2\underline{G}_{0,1} = \{ f \in \underline{E}_{0,1} \,|\, f(o_N) \in$

$\{o_N, o_M\}\}$.
(ii) $\underline{B}_{0,1} \subseteq {}^1\underline{G}'_{0,1} \subseteq {}^2\underline{G}'_{0,1}$ *and these semirings are not simple.*
(iii) *If* E *is a subsemiring such that* $\underline{B}_{0,1} \subseteq E \subseteq {}^2\underline{G}'_{0,1}$ *then* E *is not simple.*
(iv) *If* E *is a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E \nsubseteq {}^2\underline{G}'_{0,1}$ *then* E *satisfies* 3.8(i1).

Proof. (i) See 2.5.
(ii) and (iii). If E is simple then E satisfies 3.6(i1) (use 3.7) and there is $f \in E$ such that $0_M \neq f(o_N) < o_N$. Then $f \notin {}^2\underline{G}'_{0,1}$.
(iv) We proceed similarly as in the proof of 4.10(iv). □

Proposition 3.12. *The set* $\underline{A} = \underline{B}_{0,1} \cup \{f_\alpha \mid \alpha \in K \times K\}$ *is a subsemiring of* $\underline{E}_{0,1}$.

Proof. Easy to check directly. □

In the remaining part of this section, assume that the set $L = M \setminus \{0_M\}$ has at least one minimal element and denote by P the set of minimal elements.

Proposition 3.13. *The set* $\underline{A}' = \{\underline{B}_{0,1} \cup \{f(\alpha) \mid \alpha \in P \times K\}$ *is a subsemiring of* $\underline{A}$ (see 3.12).

Proof. Easy to check directly. □

Proposition 3.14. *Let* E *be a simple subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$. *If either* $o_N \notin N$ *or* M *is finite or* $o_N \in E(M \setminus \{o_N\})$ *then* E *operates transitively on* K.

Proof. See 3.3. □

Proposition 3.15. *Let* E *be a subsemiring of* $\underline{E}_{0,1}$ *such that* $\underline{B}_{0,1} \subseteq E$ *and* E *operates transitively on* K. *Then:*
(i) E *is simple, provided that either* $o_N \in N$, *or* $o_N \notin N$ *and* $\underline{\xi} \notin E \subseteq \underline{E}''_{0,1}$.
(ii) $\underline{A}' \subseteq E$.
(iii) *For all* $u, v \in K$ *there is* $u_1 \in N$ *such that* $u_1 < u$ *and* $g_{(u_1,u,v)} \in E$ (see 2.7 and 3.5).

Proof. (i) Clearly, 3.6(i1) is true and it remains to use 3.6(i) or 3.7(i).
(ii) We have $f_{(u,v)} = f + q_u$, where $u \in P$, $v \in K$, $f \in E$ and $f(u) = v$.
(iii) Choose $w \in P$ and find $g \in E$ with $g(u) = w$. Then there is $u_1 \in N$ such that $u_1 < u$ and $g_{(u_1,u,v)} = f_{(w,v)}g + q_u$. □

Lemma 3.16. *Let* E *be a subsemiring of* $\underline{A}$ *such that* $\underline{B}_{0,1} \subseteq E$. *Then* E *operates transitively on* K *if and only if* $E = \underline{A}$.

Proof. This is easy (see 3.12). □

Proposition 3.17. (i) *The semiring $\underline{A}$ is simple.*
(ii) *If E is a subsemiring of $\underline{A}$ such that $\underline{\ }B_{0,1} \subseteq E$ and E is simple then $o_N \in N$, $o_N \notin E(M \setminus \{o_N\})$, M is infinite and E operates transitively on $K \setminus \{o_N\}$.*

Proof. Combine 3.14, 3.15, 3.16 and 3.3. □

Remark 3.18. Let E be a proper subsemiring of $\underline{A}$ such that $\underline{B}_{0,1} \subseteq E$ and E is simple (see 3.3 and 3.17). Then $o_N \in N$, $E = \underline{B}_{0,1} \cup \{ f_{(u,v)} \mid u, v \in K, v \neq o_N \}$, $o_{N_1} \in N_1 = M \setminus \{o_M, o_N\}$ and M is infinite. If E_1 is a simple subsemiring of E such that $\underline{B}_{0,1} \subseteq E_1$ then $E_1 = E$.

Proposition 3.19. *Let E be a subsemiring of $\underline{E}_{0,1}$ such that $\underline{A}' \subseteq E$. Let η denote the congruence of E generated by the pairs (q_u, q_{0_M}), $u \in P$. If $\varrho \neq \mathrm{id}_E$ is a congruence of E then $\eta \subseteq \varrho$. IN particular, the semiring E is simple if and only if $\eta = E \times E$.*

Proof. First, if $(f, g) \in \varrho$ and $f(w) < g(w)$ for some $w \in P$ then $q_u = q_{f(w)} f f_\alpha$, $q_{0_M} = q_{f(w)} g f_\alpha$, $\alpha = (u, w)$, $u \in P$, and hence $(q_u, q_{0_M}) \in \varrho$ and $\eta \subseteq \varrho$. On the other hand, if $f(w) = g(w)$ for every pair $(f, g) \in \varrho$ then $f(wv) = f f_\beta(w) = g f_\beta(w) = g(v)$ for all $v \in K$, $\beta = (w, v)$, and hence $f = g$ and $\varrho = \mathrm{id}_E$. □

4. 0-preserving endomorphisms

Let M be a semilattice such that $0_M \in M$ and $|M| \geq 3$. The set $\underline{E}_0 = \{ f \in \underline{E} \mid f(0_M) = 0_M \}$ is a unitary subsemiring of $\underline{E}$ and the constant endomorphism $\underline{\zeta}$, $\underline{\zeta}(M) = 0_M$, is the zero element of the semiring $\underline{E}_0$ (i.e., $\underline{\zeta}$ is additively neutral and multiplicatively absorbing). For every $f \in \underline{E}_0$, $f \neq \underline{\zeta}$, the set $A_f = \{ x \in M \mid f(x) \neq 0_M \}$ is a prime ideal of the semilattice M. For every prime ideal A of M and all $v \in M$, we have $q_{A,v} \in \underline{E}_0$, where $q_{A,v}(A) = \{v\}$ and $q_{A,v}(M \setminus A) = \{0_M\}$. Now, $q_{A,0_M} = \underline{\zeta}$, $q_{A_1,v_1} q_{A_2,v_2} = q_{A_2,v_1}$ if $v_2 \in A$, $q_{A_1,v_1} q_{A_2.v_2} = \underline{\xi}$ otherwise and the set $\underline{D}_0 = \{ q_{A_1,v_1} + \dots + q_{A_n,v_n} \mid n \geq 1 \}$ is an ideal of the semiring $\underline{E}_0$.

For $u, v \in M$, $u \neq o_M$, put $q_{u,v} = q_{A,v}$, where $A = A_u = \{ x \in M \mid x \not\leq u \}$, and $q_{o_M,v} = \underline{\xi}$. Let $\underline{B}_0 = \{ q_{u,v} \mid u, v \in M \}$ and $\underline{C}_0 = \{q_{u_1,v_1} + \dots + q_{u_n,v_n} \mid n \geq 1 \}$. Then $\underline{B}_0 \subseteq \underline{C}_0 \subseteq \underline{D}_0$ and $\underline{C}_0$ is a left ideal of $\underline{E}_0$.

Theorem 4.1. *Let E be a subsemiring of $\underline{E}_0$ such that $\underline{B}_0 \subseteq E$. Then the semiring E is simple if and only if for every $f \in E$ there is an element $w \in M$ such that $f(M) \leq w$.*

Proof. The result follows immediately from [5, 2.6] (but see also [3, 3.6]). □

Corollary 4.2. ([3], [6]) *Put* $\underline{F}_0 = \{ f \in \underline{E}_0 \mid f(M) \leq w$ for some $w \in M \}$. *Then:*
(i) *$\underline{F}_0$ is an ideal of $\underline{E}_0$ and $\underline{D}_0 \subseteq \underline{F}_0$.*
(ii) *If E is a subsemiring of $\underline{E}_0$ such that $\underline{B}_0 \subseteq E$ then the semiring E is simple if and only if ($\underline{C}_0 \subseteq$) $E \subseteq \underline{F}_0$.*

(iii) *The semirings* $\underline{C}_0$ *and* $\underline{F}_0$ *are simple.*
(iv) $\underline{F}_0 = \underline{E}_0$ *if and only if* $o_M \in M$ *(or, equivalently,* $\mathrm{id}_M \in \underline{F}_0$*).*
(v) $\underline{C}_0 = \underline{E}_0$*) if and only if* M *is a finite distributive lattice.* □

Remark 4.3. Let E be a subsemiring such that $\underline{B}_0 \subseteq E \subseteq \underline{F}_0$. If $e \in F$ is (left, right) multiplicatively neutral then $e = \mathrm{id}_M = 1_E$., and hence $o_M \in M$ and $\underline{F}_0 = \underline{E}_0$.

Remark 4.4. $\underline{D}_0 = \underline{E}_0$ if and only if M is a finite distributive lattice.

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