Partially ordered objects in the topos of M set

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Abstract. In this paper first we introduce the concept of partially ordered objects and monotone morphisms in an arbitrary topos. Partially ordered objects are counterparts of partially ordered sets, commonly known as posets, in an arbitraty topos. We then study partially ordered objects in the topos M set. We will prove that partially ordered objects in M set is equivalent to the category M pos, of posets with compatible actions of a monoid M on them.

Keywords: partial order, monotone, topos, monoid, action.

1. Introduction

Partially ordered sets are among the most important mathematical structures. They have a wide range of applications in mathematics. In what follows we mention some of the most important usage of partially ordered sets.

Perhaps the best known use of the language of partially ordered sets is Zorn's Lemma. Proved independantly by Kazimierz Kuratowski in 1922 and Max Zorn in 1935, it states that every partially ordered set in which every chain has an upper bound has a maximal element [3].

Zorn's Lemma has many important and crucial results and consequences such as the existence of maximal ideals in rings, existence of basis for vector spaces and strong completeness of first-order logic.

So, according to what we mentioned above, it is evident that the study of posets is very important for mathematicians.

Introduced by Lawvere and Tierney, topos theory is a branch of mathematics which studies categories behaving similar to the category of sets and functions.

Each topos defines a frame work for practicing mathematics. Classical mathematics is the mathematics studied in the frame work defined by the category of sets and functions.

Every topos has an underlying logic which is a super intuitionistic one. The formulas valid in every topos are exactly intuitionistic valid formulas. Therefore, from the prespective of logic, practicing mathematics in a topos is the same as studying intuitionistic or constructivist mathematics. Constructivisim in mathematics has gained much attention recently. Besides its philosphical arguments it is important because of its applications in computer science, for example see [8].

One line of research in topos theory is to recreat classical mathematics in an arbitrary topos. Acording to what we said previously, partially ordered sets have all sorts of applications in mathematics. Therefore it seems important to study these mathematical structures in a topos.

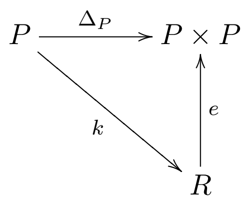
In the first section of this paper we introduce the concept of a partially ordered object which is actually due to MacLane and Moerdijk [10]. Next we define what we mean by a monotone morphism. It is easy to see that these form a subcategory of the underlying topos. In the other sections we study partially ordered objects in the topos $M\mathbf{Set}$, of sets with an action of the monoid M on them. We prove that a category of partially ordered objects and monotone morphisms in this topos is finitely complete and cartesian closed. Besides these properties we mention some other interesting properties, as well.

2. Partially ordered objects and monotone morphisms in a topos

Our aim in this section is to define partially ordered object (po-object) in an arbirary topos. Having in mind what a poset (partially ordered set) $(P, \leq)$ is, we give the following definitions. For more information regarding this definition see [10] page 199.

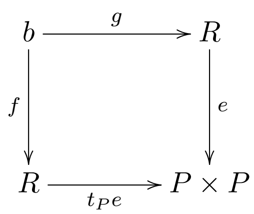
Definition 2.1. *Let P be an object of a topos $\mathcal{E}$. By a binary relation e on P we mean a subobject $e : R \rightarrowtail P \times P$ of $P \times P$. Now we have the following definitions:*

1. The binary relation $e : R \rightarrowtail P \times P$ on P is said to be reflexive whenever $\Delta_P : P \to P \times P$ factors through $e : R \rightarrowtail P \times P$. That is there exists a morphism $k : P \to R$ such that the



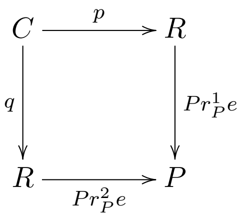
commutes. In other words, e is reflexive whenever $\Delta_P \leq e$.

2. We say that $e : R \to P \times P$ is anti-symmetric whenever $e \cap t_P e \leq \Delta_P$, where t_P is the twisting morphism which is $(Pr_P^2, Pr_P^1) : P \times P \to P \times P$ and Pr denotes the projection morphisms. That is, considering the pullback square



there exists a morphism $m : b \to P$ such that $\Delta_P m = eg = t_P ef$.

3. We say that $e : R \to P \times P$ is transitive whenever $(Pr^1_P eq, Pr^2_P ep) : C \to P \times P$ factors through e, where C is defined by the pullback square



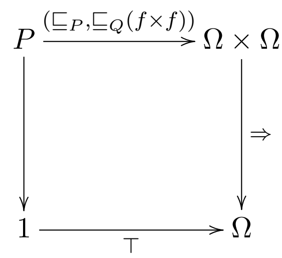
4. A partially ordered object (po-object) in a topos $\mathcal{E}$ is a pair $(P, e :\leq_P \to P \times P)$, where P is an $\mathcal{E}$-object and e is a binary relation on P in the topos, which is reflexive, anti-symmetric, and transitive.

Here we fix some notations. For a po-object P we denote its order by $e_P :\leq_P \to P \times P$, and by $\sqsubseteq_P$ we mean the characterstic morphism of the relation on P.

Note that a monotone morphism $f : P \to Q$ between posets is a function satisfying

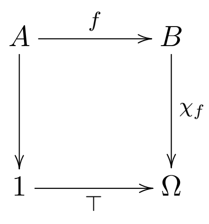
$$x \leq_P y \Rightarrow f(x) \leq_Q f(y).$$

Definition 2.2. *Let P and Q be two po-objests in a topos $\mathcal{E}$. An $\mathcal{E}$-morphism $f : P \to Q$ is said to be monotone whenever $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. That is,*



is commutative.

Remark 1. Ω-*axiom.* The Ω-axiom states that for each monomorphism $f : A \to B$ in a topos with subobject clasifier $\top : 1 \to \Omega$ there exists a unique morphism $\chi_f : B \to \Omega$ such that



is a pullback square.

Theorem 2.1. *Assume P and Q are po-objects. A morphism $f : P \to Q$ is monotone if and only if there exists a morphism $h :\leq_P \to \leq_Q$ such that*

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_p} & P \times P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times f} \\ \leq_q & \xrightarrow[e_Q]{} & Q \times Q \end{array}$$

is commutative.

Proof. Assume $f : P \to Q$ is monotone. That is, $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. Consider the diagram

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P, \sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \wedge} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

Since $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$, the outer square is a pullback. The lower square is the pullback defining $\wedge$. So, by the pullback lemma,

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P, \sqsubseteq_Q(f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \end{array}$$

is also a pullback. Therefore, $\sqsubseteq_Q (f \times f)e_P = \sqsubseteq_P e_P = \top_{\leq_P}$. Now, consider the pullback square

$$\begin{array}{ccc} \leq_Q & \xrightarrow{e_Q} & Q \times Q \\ \downarrow & & \downarrow{\scriptstyle \sqsubseteq_Q} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

Since $\sqsubseteq_Q (f \times f)e_P = \top_{\leq_P}$, there exists a morphism $h :\leq_P \to \leq_Q$ such that $e_Q h = (f \times f)e_P$. For the converse, assume there exists a morphism $h :\leq_P \to \leq_Q$

such that

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times f} \\ \leq_Q & \xrightarrow{e_Q} & Q \times Q \end{array}$$

is commutative. Consider the square

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P, \sqsubseteq_Q (f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \end{array}$$

One can see that the above is pullback. Now, consider the diagram

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow & & \downarrow{\scriptstyle (\sqsubseteq_P, \sqsubseteq_Q (f\times f))} \\ 1 & \xrightarrow[(\top,\top)]{} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \wedge} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

Since the upper and lower squares are pullbacks, the whole square is pullback. Therefore, by Ω-axiom, we have $\sqsubseteq_P \wedge \sqsubseteq_Q (f \times f) = \sqsubseteq_P$. □

From now on we use this equivalent form of monotonicity. We have defined partially ordered objects and monotone morphisms in a topos. In the following sections we study the above definitions in the topos M**Set**, of sets with actions of the monoid M on them (for M**Set** see for example [6]).

Before we go on any further note that an M-set, where M is monoid is a set X equipped with a function $\mu : M \times X \to X$ that satisfies (for the sake of simplicity $\mu(s, x)$ is denoted sx)

$$ex = x, \quad s(tx) = (st)x.$$

An action-preserving function between two M-sets X and Y is a function $f : X \to Y$ such that

$$f(sx) = sf(x).$$

3. Partially ordered objects in MSet

Now we study the partially orderd objects in the topos M**Set**. We will show that the category of po-objects in M**Set** is equivalent to the catgeory of M-posets. The category of M-posets has been the subject of major studies (for example see [1]).

Definition 3.1. *An M-poset is a poset with an action of the monoid M such that for $x, y \in P$ and $s \in M$ we have*

$$x \leq y \Rightarrow sx \leq sy.$$

An action-preserving function f between M-posets P and Q is said to be monotone whenever for each $x, y \in P$ we have

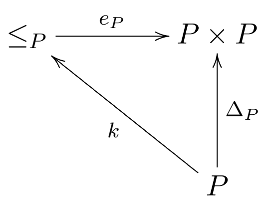
$$x \leq y \Rightarrow f(x) \leq f(y).$$

Let us fix a notation wich we use from now on. If $f : A \to B$ is a function and $C \subseteq A$ then by $f[C]$we mean the image of f restricted to the set C.

Theorem 3.1. *Let P be a po-object* (*as in Definition* 2.1) *in the topos M***Set** *then* $(P, e_P[\leq_P])$ *is an M-poset.*

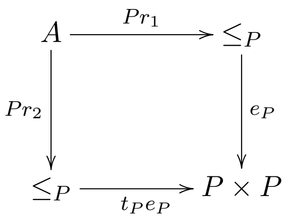
Proof. First of all it is easy to see that $e_P[\leq_P]$ is a sub M-set of $P \times P$.

To see that $e_P[\leq_P]$ is reflexive note that there exists an action-presereving map $k : P \to \leq_P$ such that



is commutative. Let $x \in P$, we want to show that $(x, x) \in e_P[\leq_P]$. We have $e_P k(x) \in e_P[\leq_P]$. That is $\Delta_P(x) \in e_P[\leq_P]$. So, $(x, x) \in e_P[\leq_P]$.

To see that $e_P[\leq_P]$ is anti-symmetric let $(x, y) \in e_P[\leq_P]$ and $(y, x) \in e_P[\leq_P]$. We want to prove that $x = y$. We have $(x, y) \in e_P[\leq_P] \cap t_P e_P[\leq_P]$. Let



be a pullback square in M-**Set**. Note that $A = \{(a, b) \in \leq_P \times \leq_P : e_P(a) = t_P e_P(b)\}$ and Pr_1, Pr_2 are the usuall projection maps Now, by anti-symmetry of e_P, there exists a morphism $h : A \to P$ such that

$$\begin{array}{ccc} A & \xrightarrow{Pr_1} & \leq_P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle e_P} \\ P & \xrightarrow[\Delta_P]{} & P \times P \end{array}$$

is commutative. Since $(x, y), (y, x) \in e_P[\leq_P]$, there exist $a, b \in \leq_P$ such that $e_P(a) = (x, y)$ and $e_P(b) = (y, x)$. So, $t_P e_P(b) = e_P(a)$. Hence $(a, b) \in A$. Therefore, $e_P Pr_1(a, b) = \Delta_P h(a, b)$. That is, $(x, y) = \Delta_P h(a, b)$. So, $x = y$.

The last step is to prove that $e_P[\leq_P]$ is transitive. Let $(x, y), (y, z) \in e_P[\leq_P]$. That is, there exist $a, b \in \leq_P$, such that $e_P(a) = (x, y)$ and $e_P(b) = (y, z)$. Thus $Pr^2_P e_P(a) = Pr^1_P e_P(b)$. Let

$$\begin{array}{ccc} C & \xrightarrow{\pi_2} & \leq_P \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle Pr^1_P e_P} \\ \leq_P & \xrightarrow[Pr^2_P e_P]{} & P \end{array}$$

be a pullback in M-**Set**. We can consider C to be $\{(r, s) \in \leq_P \times \leq_P : Pr^1_P e_P(s) = Pr^2_P e_P(r)\}$. Then, $(a, b) \in C$. Since e_P is transitive, there exists a morphism $l : C \to \leq_P$ such that

$$\begin{array}{ccc} C & \xrightarrow{(Pr^1_P e_P \pi_1, Pr^2_P e_P \pi_2)} & P \times P \\ & {\scriptstyle l}\searrow & \uparrow{\scriptstyle e_P} \\ & & \leq_P \end{array}$$

is commutative. As a result $e_P l(a, b) = (x, z)$, and so $(x, z) \in e_P[\leq_P]$. □

Theorem 3.2. *Assume that P and Q are po-objects in M**Set** and $f : P \to Q$ is monotone. Then $f : (P, e_P[\leq_P]) \to (Q, e_Q[\leq_Q])$ is a monotone action-preserving function between M-posets $(P, e_P[\leq_P])$ and $(Q, e_Q[\leq_Q])$.*

Proof. Since $f : P \to Q$ is monotone as a morphism between po-objects in M**Set**, there exists a morphism $h : \leq_P \to \leq_Q$ such that

$$\begin{array}{ccc} \leq_P & \xrightarrow{e_P} & P \times P \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f\times f} \\ \leq_Q & \xrightarrow[e_Q]{} & Q \times Q \end{array}$$

is commutative. We have to prove that

$$(x, y) \in e_P[\leq_P] \Rightarrow (f(x), f(y)) \in e_Q[\leq_Q].$$

So, let $(x, y) \in e_P[\leq_P]$. We have $(f(x), f(y)) = (f \times f)(x, y)$. Since $(x, y) \in e_P[\leq_P]$, there exists $a \in \leq_P$ such that $e_P(a) = (x, y)$. So, $(f(x), f(y)) = (f \times f)e_P(a)$. By the commutativity of the previous diagram, $(f(x), f(y)) = e_Q h(a) \in e_Q[\leq_Q]$. □

Let us denote the category of poset object and monotone maps of M-**Set** by **Po**M**Set**. By the above theorems, one can see that we have a functor F : **Po**M**Set** $\to M$**Pos** which sends each po-object P to the M-poset $(P, e_P[\leq_P])$ and each monotone morphism $f : P \to Q$ between po-objects to itself as a monotone morphism between M-posets $(P, e_P[\leq_P])$ and $(Q, e_Q[\leq_Q])$.

Now if $(P, \leq_P)$ is an M-poset then one can easily see that $(P, i_{\leq_P})$ is a po-object in the topos M-**Set**, where $i_{\leq_P}$ is the inclusion function from $\leq_P$ to $P \times P$. It is also easy to see that an action-preserving monotone funtion between M-posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is monotone as a morphism between po-objects $(P, i_{\leq_P})$ and $(Q, i_{\leq_Q})$. So, we have a functor $G : M$**Pos** $\to$ **Po**M**Set** such that it sends each M-poset $(P, \leq_P)$ to the po-object $(P, i_{\leq_P})$, and each monotone function $f : (P, \leq_P) \to (Q, \leq_Q)$ between M-posets to itself considered as a morphism between po-objects (P, i_P) and (Q, i_Q).

Theorem 3.3. *The functors F and G constitute an equivalence between the categories* **Po**M**Set** *and* M**Pos**.

Proof. Let (P, e_P) be an object of **Po**M**Set**. One can see that

$$GF(P, e_P) = (P, i_{e_P[\leq_P]}).$$

One can see that $id_P : (P, e_P) \to (P, i_{e_P[\leq_P]})$ is a natural isomorphism.

On the other hand if $(P, \leq_P)$ is an M-poset then

$$FG(P, \leq_P) = (P, i_{\leq_p}[\leq_P]) = (P, \leq_P).$$

□

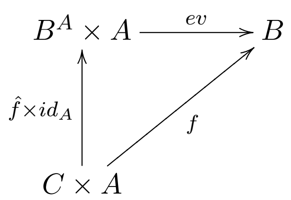
Now that we have proved **Po**M**Set** is equivalent to M**Pos**, we can study the categorical properties of **Po**M**Set** by investigating M**Pos**.

M**Pos** has been the subject of major studies, for example see [1], [2], [13]. However, we state some theorems which capture the categorical properties of M**Pos** completely.

Theorem 3.4. *The category* M**Pos** *is isomorphic to* $\mathbf{Pos}^M$.

Proof. For the proof of this theorem see [6]. □

Remark 2. Let $\mathcal{C}$ be a category and A and B objects of this category. The exponential object of A and B denoted by B^A is an object of $\mathcal{C}$ together with a morphism $ev : B^A \times A \to B$ such that for every morphism $f : C \times A \to B$ in $\mathcal{C}$ there exists a unique morphism $\hat{f} : C \to B^A$ making



commutative.

A category with finite products is said to be Cartesian closed whenever all exponential for each pair of objects exist.

Remark 3. Let P and Q be two M-sets then Q^P is the following set

$$\{f : M \times P \to Q : f \text{ is action-preserving}\}$$

equipped with an action such that for $s \in M$ and $f \in Q^P$ we have

$$sf : M \times P \to Q, \quad (m, x) \mapsto f(ms, x).$$

The M-set Q^P is the exponentiation of Q and P in M**Set**. If f is a member of Q^P and $s \in M$ then by f_s we mean the following function

$$f_s : P \to Q, \quad x \mapsto f(s, x).$$

Assuming that P and Q are two objects of M**Pos**, one can see that their product object is the M-set $P \times Q$ equipped with the point-wise order. Also, it can be seen that M**Pos** is a catesian closed category. The exponential object $[P \to Q]$ is the subobject of Q^P such that $f \in [P \to Q]$ if and only if for each $s \in M$ the function f_s is monotone.

4. Partial monotone morphism classifier

In this section we first describe the partial morphism classifier of the topos of M-sets using the language introduced in [5] and the process of [9]. After this we use the partiall morphism classifier to construct the partial monotone morphims classifier.

Remark 4. (a) Let A be an M-set, $X \in Sub(M \times A)$, and $s \in M$. We equip $Sub(M \times A)$ with the action $sX = \{(t, a) \in M \times A : (ts, a) \in X\}$. One can see that $Sub(M \times A)$ with this action is indeed an M-set.

(b) It is well-known that Ω^A is isomorphic to $Sub(M \times A)$. In fact, the isomorphism $\Lambda : \Omega^A \to Sub(M \times A)$ is given by $\Lambda(f) = f^{-1}(\{M\})$.

(c)If $X \in Sub(M \times A)$ then we denote it as $(X_m)_{m\in M}$, where $X_m = \{x \in A : (m, x) \in X\}$.

Now, we will use Kock and Wraith's construction to obtain the partial morphism classifier [9]. After this we equip this object with a partial order. We will prove that the resulting object can classify partial monotone morphisms with up M-sets as their domains. Note that an up M-subset of an M-poset is an upset of it which is also a sub M-set.

We define $\{\cdot\}_A : A \to Sub(M \times A)$ to be the exponential adjoint of $\delta_A : A \times A \to \Omega$, which is the characterstic morphism of $\Delta_A : A \to A \times A$. That is, $\delta_A(x, y) = \{s \in M : sx = sy\}$. Hence we have

$$\{\cdot\}_A : A \to Sub(M \times A), \quad x \to \{x\}_A,$$

where $\{x\}_A = \{(m, mx) : m \in M\}$. Equivalently we have $\{x\}_A = (\{mx\})_{m\in M}$. Thus, we have a monomorphism

$$(\{\cdot\}, id_A) : A \to Sub(M \times A) \times A, \quad x \mapsto (\{mx\}_{m\in M}, x).$$

Let us denote the characterstic morphism of this monomorphism by h. Therefore $h : Sub(M \times A) \times A \to \Omega$ is defined by

$$((X_m)_{m\in M}, x) \mapsto \{s \in M : \forall m \in M \quad X_{ms} = \{msx\}\}.$$

As a result, one can see that

$$h((X_m), x) = M \Leftrightarrow \forall m \in M \quad X_m = \{mx\}.$$

The morphism h has the exponential adjoint

$$\hat{h} : Sub(M \times A) \to Sub(M \times A),$$

which is given by

$$\hat{h}((X_t)_{t\in M}) = (\{x \in A : h(m(X_t)_{t\in M}, x) = M\})_{m\in M},$$

which is equal to

$$(\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\})_{m\in M}.$$

Definition 4.1. *We define $\tilde{A}$ to be the equalizer of $\hat{h}$ and $id_{Sub(M\times A)}$. One can see that $\tilde{A}$ is the set of all $(X_t)_{t\in M} \in Sub(M \times A)$ such that*

$$\forall m \in M, \quad \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} = X_m.$$

Theorem 4.1. *We have $(X_m)_{m\in M} \in \Omega^A$ is a member of $\tilde{A}$ if and only if for each $m \in M$, X_M is either empty or has exactly one element.*

Proof. Assume for $m \in M$, $X_m \neq \emptyset$. Thus

$$\exists x \in A \quad \forall t \in M \quad X_{tm} = \{tx\}.$$

So, for $t = e$ we have $X_m = \{x\}$, for some $x \in A$. Hence X_m has exactly one element.

To prove that $(X_m)_{m\in M}$ is a member of $\tilde{A}$, it is enough to show that for each $m \in M$, we have

$$X_m = \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\}.$$

We prove that if $X_m = \emptyset$, then

$$\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} = \emptyset.$$

Assume $X_m = \emptyset$. If

$$\exists x \in A \quad \forall t \in M \quad X_{tm} = \{tx\},$$

then, replacing t by e, we have $X_m = \{x\}$, for some $x \in A$, which is a contradiction. So, $\{x \in A : \forall t \in M \quad X_{tm} = \{tx\} = \emptyset\}$. If $X_m \neq \emptyset$, then by the hypothesis, there exists $x \in A$ such that $X_m = \{x\}$. Note that, since $(X_m)_{m\in M}$ is a member of Ω^A, for each $t, m \in M$ we have $tX_m = X_{tm}$. So, for each $t \in M$ we have $X_{tm} = tX_m = t\{x\} = \{tx\}$. Hence, x satisfies

$$\forall t \in M \quad X_{tm} = \{tx\}.$$

So,

$$X_m \subseteq \{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\}.$$

Now assume a satisfies the property

$$\forall t \in M, \quad X_{tm} = \{ta\}.$$

Replacing t by e, we get $X_m = \{a\}$. So, $\{a\} = \{x\}$. That is $a \in X_m$. Therefore,

$$\{x \in A : \forall t \in M \quad X_{tm} = \{tx\}\} \subseteq X_m.$$

□

Theorem 4.2. *Let P be an M-poset. We equip $\tilde{P}$ with the relation*

$$(X_t)_{t\in M} \quad \tilde{\leq} \quad (Y_t)_{t\in M} \Leftrightarrow \forall t \in M \quad \downarrow X_t \subseteq \downarrow Y_t.$$

Then $\tilde{P}$ is an M-poset

Proof. Obviously $\tilde{P}$ is a poset. So, we show that $\tilde{\leq}$ is compatible with the action of M. Let $(X_t)_{t\in M} \ \tilde{\leq}\ (Y_t)_{t\in M}$ and $m \in M$ then, we have

$$\forall t \in M \quad \downarrow X_t \subseteq \downarrow Y_t.$$

So,

$$\forall t \in M \quad \downarrow X_{tm} \subseteq \downarrow Y_{tm}.$$

Therefore, $(X_{tm})_{t\in M} \ \tilde{\leq}\ (Y_{tm})_{t\in M}$. That is $m(X_t)_{t\in M} \ \tilde{\leq}\ m(Y_t)_{t\in M}$. □

Theorem 4.3. *Let P be an M-poset. Then the function*

$$\eta_P : P \to \tilde{P}, \quad x \mapsto (\{mx\})_{m\in M}$$

is an action-preserving, monotone function.

Proof. To see that η_P is well-defined, let $m, t \in M$. We have $t\{mx\} = \{tmx\} \subseteq \{tmx\}$. To see that it is action-preserving, let $s \in M$. We have $s \cdot \eta_P(x) = (\{msx\})_{m\in M}$. On the other hand $\eta_P(sx) = (\{msx\})_{m\in M}$.

Next, we prove that η_P is monotone. Let $x \leq y$. Then $\downarrow \{mx\} \subseteq \downarrow \{my\}$, because $x \leq y$ implies $mx \leq my$. This gives that $(\{mx\})_{m\in M} \tilde{\leq} (\{my\})_{m\in M}$. □

Definition 4.2. *Let P be an M-poset. By an M-upset of P we mean a sub M-set F of P which is an upset with respect to the order of P. By a partial monotone morphism from an M-posets P to an M-poset Q we mean a morphism from an M-upset F of P to Q.*

Theorem 4.4. *Let P be an M-poset and $f : P \rightharpoonup Q$ be a partial monotone morphism such that $dom(f) = F$. Then $\hat{f} : P \to Q$ is monotone, where $\hat{f}$ is the unique morphism making the following square a pullback one in M-**Set**:*

$$\begin{array}{ccc} F & \xrightarrow{i_F} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ Q & \xrightarrow[\eta_Q]{} & \tilde{Q} \end{array}$$

Proof. First of all one can see that $\tilde{f}$ is defined by

$$\tilde{f} : P \to \tilde{Q}, \quad x \mapsto (X_m)_{m\in M},$$

such that for each $m \in M$ we have

$$X_m = \begin{cases} \{f(mx)\}, & \text{if } mx \in F \\ \emptyset, & \text{otherwise.} \end{cases}$$

To show that this morphism is monotone, let $x \leq y$ in P. We should prove that $\tilde{f}(x) \;\tilde{\leq}\; \tilde{f}(y)$. Let

$$\tilde{f}(x) = (X_m)_{m\in M}, \quad \tilde{f}(y) = (Y_m)_{m\in M}.$$

Then, we have to show that

$$\forall m \in M \quad \downarrow X_m \subseteq \downarrow Y_m.$$

If $X_m = \emptyset$, then obviously $\downarrow X_m \subseteq \downarrow Y_m$. If X_m is non-empty, then we have

$$X_m = \{f(mx)\}, \quad mx \in F.$$

But, since $mx \leq my$ and F is an upset, we have $my \in F$. So, $Y_m = \{f(my)\}$. Now, since f is monotone, we have $f(mx) \leq f(my)$. Hence $\downarrow X_m \subseteq \downarrow Y_m$. □

Note that the above diagram is obviously a pullback in M**Pos** and $\hat{f}$ is the only monotone morphism making into a pullback square.

5. Some adjoint situations

Let X be an M-set. Obviously $(X, =)$ is an M-poset. If $f : X \to Y$ is an action-preserving function, we have $f : (X, =) \to (Y, =)$ is monotone. So, we have a functor $F : M\mathbf{Set} \to M\mathbf{Pos}$ which sends each M-set X to $(X, =)$ and sends each action-preserving funtion to itself.

Let $(P, \leq)$ be an M-poset. We have a functor $G : M\mathbf{Pos} \to M\mathbf{Set}$ which forgets the order of each M-poset.

Theorem 5.1. *The functor F is the left adjoint of the functor G.*

Proof. Note that for an M-set X we have $GFX = X$. One can see that the natural transformation $\eta : id_{M\mathbf{Set}} \to GF$, with the X'th component being id_X, serves as the unit of this adjunction. □

Now we consider the obvious forgetful functor $R : M\mathbf{Pos} \to \mathbf{Pos}$. Define the functor $L : \mathbf{Pos} \to M\mathbf{Pos}$ such that for each poset P we have $L(P) = M \times P$, where the action on $M \times P$ is the product action, where P is considered as an M-set with the trivial action, and take the order on $M \times P$ to be the pointwise order, where M is considered as a poset with the equality order.

Theorem 5.2. *The functor L is the left adjoint of the functor R.*

Proof. The natural transformation $\eta : id_{\mathbf{Pos}} \to RF$ with the P'th component being

$$\eta_P \to M \times P, \quad x \mapsto (e, x),$$

is the unit of this adjunction. For $f : P \to RQ$ a monotone function between a poset P and the underlying poset of an M-poset Q the adjoint morphism $\hat{f} : LP \to Q$ sends (s, p) to $f(sp)$. □

Next is the zero functor $Z : M\mathbf{Pos} \to \mathbf{Pos}$ which sends each M-poset P to the poset of zero elements of P. For an action-preserving monotone map $f : P \to Q$, Zf is the restriction of f to the set ZP.

This functor has a left adjoint $T : \mathbf{Pos} \to M\mathbf{Pos}$, which sends a poset P to the M-poset P, where the action on P is trivial. Also T sends monotone maps to themselves considered as action-preserving monotone functions.

Theorem 5.3. *The functor Z is the right adjoint of the functor T.*

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