Sumudu transform for solving some classes of fractional differential equations

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Abstract. Many authors recently, have demonstrated the usefulness of fractional calculus especially in the derivation of solutions for linear partial differential equations (PDEs) and ordinary differential equations (ODEs). In this paper, some properties of fractional Sumudu transform for solving fractional differential equations (FrDEs) have introduced. The approximated solutions of some classes of FrDEs using Sumudu transform method have studied. The objective of this work is to show the advantages of application of Sumudu transform method and the expansion of the coefficients of a binomial series for solving fractional differential equations.
Keywords: Sumudu transform, Laplace transform, fractional differential equations (FrDEs), binomial series, models, class.

1. Introduction

DEs are useful tools in mathematical models of life problems and applied mathematics. DEs have played very important role in different applications of mathematics for a long time and with the development of the computer. Thus, the investigation and the analysis of DEs had increase in applications leding to several mathematical problems; therefore, there are a lot of different techniques of transformations have proposed by authors in order to solve different types of DEs like: Laplace, Fourier, Mellin, Hankel transformations and Sumudu transform which is little known and not widely used in solving DEs yet. The single Sumudu transform (or Sumudu transform) was originally proposed by [1]

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for that functions of exponential order for solving DEs and control engineering problems and proved many interesting properties in t-domain and u-domain, also the properties and applications of Sumudu transform to ODEs have described by him. Among the integral transforms, Sumudu transform has units preserving properties and thus may be used to solve problems without resorting them to the frequency domain and this is one of many strength points of this new transform especially with the applications of problems with physical dimensions. The nonlinear PDEs that appears in many branches of physics, engineering and applied mathematics can not be described successfully, thus they turn out to describe them by models from fractional calculus. [2] derived formulas for the single Sumudu transform of partial derivatives and applied them in solving initial value problems (IVP) also, studied the applications of Sumudu transform in solving PDEs. The Sumudu transform of partial derivatives is derived by [2] and its applicability demonstrated using three different PDEs. [3] extended Sumudu transform to that functions of two variables. Using this extended definition, a function of two variables with emphasis on solutions to PDEs such as $f(x, y)$ is transformed to a function such as $F(u, v)$. [4] introduced the analytical investigations of the Sumudu transform and applications to integral production equations. Laplace transform is dual to Sumudu transform in solving mathematical problems. However, Sumudu transform may be used to solve mathematical problems without resorting to a new frequency domain. In the past two decades the subject of fractional calculus was widely investigated and has remarkably gained importance and popularity. [5] and [6] have studied the approximated solutions of FrDEs of Lane-emden type by method of collocation and least square method respectively. [7] extended the theory and the applications of Sumudu transform and used it to solve the FrDEs by direct integration methods. [8] studied and proved some of Sumudu transform properties and Laplace-Sumudu transforms duality and the complex inversion formula while [9] has developed an analytical methods for solving FrPDEs and extended Sumudu transform iterative method to solve a variety of time and space FrPDEs as well as systems of them. They demonstrated the utility of the method by finding the exact solutions to a large number of FrPDEs. [10] used Sumudu transform techniques to obtain the solutions of a cauchy problems for DEs with the Caputo fractional derivative and the solution of fractional Diffusion-Wave equation while [11] used Sumudu transform and variational iteration method (VIM) to approximate the solutions of FrDEs related to entropy, wavelets etc. [18] derived the approximated solutions to some homogeneous FrPDEs by applying the Laplace of the fractional derivative and the expansion of the coefficients of a binomial series. Lastly, several researchers studied fractional Sumudu transform in [12, 13, 14, 15, 16, 17]. In this paper, we derived a standard formulas for finding the approximated solutions to some classes of homogeneous FrPDEs by using the Sumudu fractional derivative and the expansion of the coefficients of binomial series.we solve the same equations solved in

[18]using Laplace transform but we use Sumudu transform instead and we get the same results.

2. Preliminary

In this section, we introduce preliminaries concepts and some definitions for this study.

Definition 2.1. *The Fractional Derivative of a Casual Function*
The fractional derivative of a casual function $f(t)$ is defined by [18] as follows:

$$\frac{d^\alpha}{dt^\alpha}f(t)=\begin{cases} f^{(n)}(t), & \text{if } \alpha=n\in N, \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, & \text{if } n-1<\alpha<n, \end{cases} \tag{1}$$

where the Euler gamma function $\Gamma(.)$ is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, (R(z)>0).$$

Definition 2.2 (The Sumudu Transform of a Function)**.** *Let $f(t)$ be a real function defined on the domain $(0,\infty)$ the Sumudu transform of $f(t)$ is defined by [18] as follows:*

$$G(u)=S[f(t)]=\int_0^\infty e^{-t}f(ut)\,dt, u\in C$$

provided the integral exists for some u.

Definition 2.3 (The Mittag-Leffler Function)**.** *The Mittag-Leffler function [18] is defined by*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+\beta)}, \quad (z,\alpha,\beta\in C, R(\alpha>0).$$

Definition 2.4 (The Simplest Wright Function)**.** *The simplest Wright function is defined by [18] as follows:*

$$\o(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\beta)}.\frac{z^k}{k!}, (z,\alpha,\beta\in C).$$

Definition 2.5 (The Riemann-Liouville Fractional Derivatives)**.** *The Riemann-Liouville fractional derivatives $D^\alpha_{a+}y$ and $D^\alpha_{a-}y$ of order $\alpha\in C(R(\alpha)\geqq 0)$ are defined by [18] as follows:*

$$(D^\alpha_{a+}y)(x)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x\frac{y(t)dt}{(x-t)^{\alpha-n+1}}; \qquad n=[R(\alpha)]+1; x>a$$

and

$$(D^{\alpha}_{b-}y)(x) = \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dx})^n \int_a^x \frac{y(t)dt}{(t-x)^{\alpha-n+1}}; \qquad n = [R(\alpha)] + 1; x > b$$

respectively, where $[R(\alpha)]$ *means the integral part of* $R(\alpha)$.

Definition 2.6 (The Pochhammer Symbol)**.** *The Pochhammer symbol (or the shifted factorial, since* $(1)_n = n!$ *of* $n \in N_0 = \{0, 1, 2, ...\}$*) given by [18] as following:*

$$(\lambda)_n = \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\dots(\lambda+n-1), & (n \in N_0/\{0\}). \end{cases}$$

Definition 2.7 (The Binomial Coefficients)**.** *The binomial coefficients are defined by [18] as follows:*

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!},$$

where λ *and* n *are integers.*

Observe that 0!=1, then,

$$\binom{\lambda}{0} = \binom{\lambda}{\lambda} = 1$$

and

$$(1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda+r-1}{r} z^r.$$

2.1 Fractional Sumudu transform

Using the definition of fractional Sumudu transform, we can easily obtain the following operational formulas:

1. $S_{\alpha}\{f(at)\} = G_{\alpha}(au)$,
2. $S_{\alpha}\{f(t-b)\} = E_{\alpha}(-b^{\alpha}) = G_{\alpha}(u)$,
3. $S_{\alpha}\{E_{\alpha}(-c^{\alpha}t^{\alpha})\} = \frac{1}{(1+cu)^{\alpha}} G_{\alpha}(\frac{u}{1+cu})$,
4. $S_{\alpha}\{\int_0^t f(t)(dt)^{\alpha}\} = u^{\alpha}\Gamma(1+\alpha)G_{\alpha}(u)$,
5. $S_{\alpha}\{f^{\alpha}(t)\} = \frac{G_{\alpha}(u) - \Gamma(1+\alpha)f(0)}{u^{\alpha}}$,
6. $S^2_{\alpha}\{f(at, bx)\} = G_{\alpha}(au)H_{\alpha}(bv)$,
7. $S^2_{\alpha}\{f(at)g(bx)\} = G^2_{\alpha}(au, bv)$,
8. $S^2_{\alpha}\{f(t-a, x-b)\} = E_{\alpha}(-(A+B)^{\alpha})G^2_{\alpha}(au, bv)$,

9. $S^2_\alpha\{\partial^\alpha_t f(t,x)\} = \frac{G^2_\alpha(u,v)-\Gamma(1+\alpha)f(0,x)}{u^\alpha}$,

10. If one defines the convolution of order of the two function $f(t)$ and $g(t)$ by the expression

$$(f(x) * g(x))_\alpha = \int_0^x f(x-v)g(v)(dv)^\alpha,$$

then $S_\alpha\{(f(t) * g(t))_\alpha\} = u^\alpha G_\alpha(u)H_\alpha(u)$, where $G_\alpha(u) = S_\alpha\{f(t)\}$ and $H_\alpha(u) = S_\alpha\{g(t)\}$.

and for the detailed proof of the above properties.

Theorem 2.1. *Let $f(t)$ be a function, then*

$$S[D^\alpha f(t)](u) = u^{-\alpha}S[f(t)] - \sum_{k=0}^{n} u^{\alpha-k}f^{(k-1)}(0).$$

Proof. By the technique of integral transform and the Definition 2.2 of Sumudu transform, we have the following:

$$\begin{aligned}
S[D^\alpha f(t)](u) &= \int_0^\infty e^{-t}[D^\alpha f(ut)]dt, \\
&= \int_0^\infty e^{-t}\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(x)}{(ut-x)^{\alpha-n+1}}dxdt, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty\int_x^\infty e^{-t}\frac{f^{(n)}(x)}{(ut-x)^{\alpha-n+1}}dxdt, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty\int_x^\infty e^{-\frac{z+x}{u}}f^{(n)}(x)z^{n-\alpha+1}\frac{1}{u}dzdx, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}\int_x^\infty e^{-\frac{z}{u}}z^{n-\alpha+1}dzdx, \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}u^{n-\alpha}\Gamma(n-\alpha)dx, \\
&= u^{n-\alpha}\int_0^\infty \frac{f^{(n)}(x)}{u}e^{-\frac{x}{u}}dx, \\
&= u^{n-\alpha}[\frac{Sf(t)}{u^n} - \frac{f(0)}{u^n} - \frac{f'(0)}{u^{n-1}}........\frac{f^{(n-1)}(0)}{u^{n-\alpha}}],
\end{aligned}$$

Hence

$$S[D^\alpha f(t)](u) = u^{-\alpha}S[f(t)] - \sum_{k=0}^{n} u^{\alpha-k}f^{(k-1)}(0).$$

□

3. Method of solution of fractional differential equations

Throughout this section, consider $y(t)$ such that for some value of the parameter u, the Sumudu transform $G(u) = S(y(t))$ converges.

Theorem 3.1. *Let $o < \alpha < 1$ and $b \in R$. Then, the FrDE*

$$y^{\alpha}(t) - by(t) = 0, \quad t \geq 0. \tag{2}$$

with the initial condition $y(0) = c_0$ the solution is

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bu^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

Proof. Applying Sumudu transform for Equation (2)

$$\frac{G(u) - G(0)}{u^{\alpha}} - bG(u) = 0.$$

Then

$$G(u) = \frac{c_0}{1 - bu^{\alpha}}.$$

Since

$$\frac{1}{1 - bu^{\alpha}} = \sum_{k=0}^{\infty} (bu^{\alpha})^k.$$

So,

$$G(u) = c_0 \sum_{k=0}^{\infty} (bu^{\alpha})^k. \tag{3}$$

Taking inverse Sumudu transform for Equation (3)

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bu^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

□

Theorem 3.2. *Let $1 < \alpha < 2$ and $a, b \in R$. Then, the following FrDE*

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0. \tag{4}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r + \alpha k + 1) r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r + \alpha k + 2) r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{k+\alpha-1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r + \alpha k + \alpha) r!}. \end{aligned}$$

Proof. Applying Sumudu transform for Equation (4)

$$\frac{G(u)-G(0)}{u^{-\alpha}}-\frac{G'(0)}{u^{\alpha-1}}+\frac{a(G(u)-G(0))}{u}+bG(u)=0.$$

That is,

$$G(u)(u^{-\alpha}+au^{-1}+b)=(c_0u^{-\alpha}+c_1u^{1-\alpha}+ac_0u^{-1}),$$
$$G(u)=\frac{c_0u^{-\alpha}+c_1u^{1-\alpha}+ac_0u^{-1}}{u^{-\alpha}+au^{-1}+b}.$$

Since

$$\begin{aligned}
\frac{1}{u^{-\alpha}+au^{-1}+b} &= \frac{u}{u^{1-\alpha}+a+bu},\\
&= \frac{u}{(u^{1-\alpha}+a)(1+\frac{bu}{u^{1-\alpha}+a})},\\
&= \frac{u}{u^{1-\alpha}+a}\sum_{k=0}^{\infty}(\frac{-bu}{u^{1-\alpha}}+a)^k,\\
&= \sum_{k=0}^{\infty}\frac{(-b)^k u^{\alpha k+\alpha}}{(1+au^{\alpha-1})^{k+1}},\\
&= \sum_{k=0}^{\infty}(-b)^k u^{\alpha k+\alpha}\sum_{r=o}^{\infty}(-au^{\alpha-1})^r\binom{k+r}{r},\\
&= \sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha}.
\end{aligned}$$

Then,

$$\begin{aligned}
G(u) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k}\\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+1}\\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha-1}.
\end{aligned} \tag{5}$$

After taking inverse Sumudu transform for Equation (5), we obtain the solution

$$\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+1)r!}\\
&+ c_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+2)r!}\\
&+ ac_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+\alpha-1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+\alpha)r!}
\end{aligned}$$

□

Example 3.1. If we let $\alpha = \frac{3}{2}, a = -1$ and $b = -2$ in Theorem 3.2, then, the following FrDE

$$y^{(\frac{3}{2})}(t) - y'(t) - 2y(t) = 0, \quad t \geq 0. \tag{6}$$

has a solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + 1) r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + 2) r!} \\ &- c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + \frac{3}{2}) r!}. \end{aligned} \tag{7}$$

[18] solved Equation (6) using Laplace transform with the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + 1) r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + 2) r!} \\ &- c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2} + \frac{3}{2}k + \frac{3}{2}) r!}. \end{aligned} \tag{8}$$

Hence, the two solutions Equation (3.6) and (3.7) are identical

3.1 Remark

Let $1 < \alpha < 2$ and a ,b $\in R$. Then, the FrDE

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0.$$

If $a = 0$ then, the FrDE

$$y^{(\alpha)}(t) + by(t) = 0$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k + 1)} + c_1 t \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k + 2)}.$$

Theorem 3.3. *Let $1 < \alpha < 2$ and a ,b $\in R$. Then, the FrDE*

$$y^{(\alpha)}(t) + ay''(t) + by(t) = 0, \quad t \geq 0. \tag{9}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2)r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3)r!} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4)r!}. \end{aligned}$$

Proof. Taking Sumudu transform for Equation (9)

$$\frac{G(u)-G(u)}{u^2} - \frac{G'(0)}{u} - \frac{aG(u)-aG(0)}{u^\alpha} - \frac{G'(0)}{u^{\alpha-2}} + bG(u)$$

$$G(u)(u^{-2} + au^{-\alpha} + b) = c_0 u^{-2} + c_1 u^{-1} + ac_0 u^{-\alpha} + ac_1 u^{2-\alpha}.$$

Since,

$$\begin{aligned} \frac{1}{u^{-2}+au^{-\alpha}+b} &= \frac{u^\alpha}{u^{\alpha-2}+a+bu^\alpha}, \\ &= \frac{u^\alpha}{(u^{\alpha-2}+a)(1+\frac{bu^\alpha}{u^{\alpha-a}})}, \\ &= \frac{u^\alpha}{(u^{\alpha-2}+a)} \sum_{k=0}^{\infty} (\frac{-bu^\alpha}{u^{\alpha-2}+a})^k, \\ &= \sum_{k=0}^{\infty} \frac{(-b)^k u^{k\alpha+\alpha}}{(u^{\alpha-2}+a)^{k+1}}, \\ &= \sum_{k=0}^{\infty} \frac{(-b)^k u^{2k+2}}{(1+au^{2-\alpha})^{k+1}}, \\ &= \sum_{k=0}^{\infty} (-b)^k u^{2k+2} \sum_{r=0}^{\infty} (-au^{2-\alpha})^r \binom{k+r}{r}, \\ &= \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2}. \end{aligned}$$

Then

$$
\begin{aligned}
G(u) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k} \\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k+1} \\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+2} \\
&+ ac_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+3}. \qquad (10)
\end{aligned}
$$

Taking inverse Sumudu transform for Equation (10)

$$
\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k}}{((2-\alpha)r+2k)!} \\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k+1}}{((2-\alpha)r+2k+1)!} \\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k-\alpha+2}}{((2-\alpha)r+2k-\alpha+2)!} \\
&+ ac_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k-\alpha+3}}{((2-\alpha)r+2k-\alpha+3)!}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1)r!} \\
&+ c_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2)r!} \\
&+ ac_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k-\alpha+2}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3)r!} \\
&+ ac_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k-\alpha+3}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4)r!}.
\end{aligned}
$$

□

Example 3.2. If we let $\alpha = \frac{3}{2}, a = \sqrt{3}$ and $b = 8$ in Theorem 3.2, then, the FrDE

$$y''(t) + ay^{(\alpha)} + by(t) = 0, \quad t \geq 0. \qquad (11)$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{12}$$

[18] solved Equation (11) using Laplace transform with the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{13}$$

Hence, the two solutions in equations (12) and (13) are identical

4. Conclusion

In this paper, Fractional Sumudu transform has been studied. The operational formulas of fractional Sumudu transform have been derived. Fractional Sumudu transform and the expansion of the coefficients of a binomial series used to solve some classes of FrDEs, which led to many interesting consequences. The approximated solutions of the problems using he new method agrees very well with the analytical solutions. As such, this method is more cost effective in terms of computation steps than other existing transform methods. Hence, we can conclude that the new method is computationally very efficient in solving some classes of FrDEs.

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