Chain continuity for Zadeh's extension

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Abstract. This paper aims to study chain continuity for Zadeh's extension. More specially, the relations between (finite) chain continuity of discrete dynamical system and its induced fuzzy system have been discussed.

Keywords: Zadeh's extension, (finite) chain continuity, fuzzy, discrete dynamical system.

1. Introduction

Throughout this paper, let (X, d) be a compact metric space with metric d and $f : X \to X$ be continuous. A discrete dynamical system is a pair (X, f). For other notions and notations mentioned in this section, we refer to Section 2. A discrete dynamical system (X, f) can naturally induce its fuzzified counterpart and set-valued counterpart, i.e., fuzzy dynamical system $(\mathcal{F}(X), d_\infty)$ and set-valued dynamical system $(\mathcal{K}(X), d_H)$, respectively. The dynamics of set-valued dynamical system have been extensively studied and many elegant results have been obtained [1, 2, 3, 4, 5, and the references therein]. An initial work on the relations between dynamical properties of fuzzified and original system has been conducted[6]. Later the open question raised in the same article has been solved by Kupka[7]. From then on a systematic study on dynamics of induced fuzzy system has been developed [8, 9, 10, 11]. Meanwhile, the connections between original system and its induced fuzzy system attract lots of attention and quite a few inspiring results have been obtained [12, 13, 14, and the references therein].

On the other hand, the dynamics of the iteration of a continuous function can be very complicated. Hence it is natural to study the pseudo-orbits for a better understanding of true orbits. Along this line, the concept of chain continuity has been separately introduced in [15, 16]. Chain continuity is a pointwise notion and is generic, i.e., for certain spaces, most functions are chain continuous at most points. In this paper, the chain continuity for Zadeh's extension has been investigated. Below, basic notions are introduced in Section 2. Main results are presented in Section 3.

2. Preliminaries

In this section, we complete notations and recall some known definitions.

2.1 Basic concepts of dynamical systems

Let X be a compact metric space and $f : X \to X$ be continuous.

For $\delta > 0$, a *δ-pseudo-orbit* for f is a sequence in X such that $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{N}$.

A *δ-chain* is a finite *δ-pseudo-orbit* for f.

A point x is called a *chain continuity point* of f, or f is *chain continuous* at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that all δ-pseudo-orbits beginning δ close to x remains ϵ close to the points of the orbits of x, i.e., $d(x, x_0) < \delta$ and $d(f(x_i), x_{i+1}) < \delta$ imply $d(f^i(x), x_i) < \epsilon$ for $i = 1, 2, \cdots$.

A point x is called a *finite chain continuity point* of f, or f is *finite chain continuous* at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\{x_0, x_1, \cdots, x_m\}$ a δ-chain with $d(x, x_0) < \delta$ implies $d(f^i(x), x_i) < \delta$.

2.2 Metric space of fuzzy sets

Let $\mathcal{K}(X)$ be the class of all non-empty and compact subset of X. Define the ε-neighborhood of a nonempty subset A in X to be the set

$$U(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathcal{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 | \ A \subseteq U(B, \varepsilon)\},$$

The Hausdorff metric on $\mathcal{K}(X)$ is defined by letting

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

For a compact X, the topology generated by d_H coincides with the finite topology. In this case, the set of all finite subsets of X is dense in $\mathcal{K}(X)$.

Define $\mathcal{F}(X)$ as the class of all upper semicontinuous fuzzy sets $u : X \to [0, 1]$ such that $[u]_\alpha \in \mathcal{K}(X)$, where *$\alpha$-cuts* and the *support* of u are defined by

$$[u]_\alpha = \{x \in X | u(x) \geq \alpha\}, \alpha \in [0, 1]$$

and

$$supp(u) = \overline{\{x \in X | u(x) > 0\}},$$

respectively.

Moreover, for each $x \in X$, we denote $\hat{x}$ the characteristic function of x, it is clear that $\hat{x} \in \mathcal{F}(X)$ for all $x \in X$. Denote $\emptyset_X$ the *empty fuzzy set* ($\emptyset_X(x) = 0$ for all $x \in X$).

A *levelwise metric* d_∞ on $\mathcal{F}(X)$ is defined by

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha)$$

for all $u, v \in \mathcal{F}(X)$. It is well known that if (X, d) is complete, then $(\mathcal{F}(X), d_\infty)$ is also complete but is not compact and is not separable.

Lemma 2.1 ([6])**.** *Let $f : X \to X$ be continuous and $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ be its Zadeh's extension, then $[\hat{f}(u)]_\alpha = f([u]_\alpha)$.*

A fuzzy set u is *piecewise constant* if there exists a strictly decreasing sequence of closed subsets $\{C_1, C_2, \cdots, C_k\}$ of X and a strictly increasing sequence of real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_k\} \subseteq (0, 1]$ such that $[u]_\alpha = C_{i+1}$, where $\alpha \in (\alpha_i, \alpha_{i+1}]$.

Lemma 2.2 ([7])**.** *For any $v \in \mathcal{F}(X)$ and $\varepsilon > 0$ there exists a piecewise constant $u \in \mathcal{F}(X)$ such that $d_\infty(u, v) < \varepsilon$, i.e., the set of piecewise constant fuzzy sets is dense in $\mathcal{F}(X)$.*

Denote by $\mathcal{SF}(X)$ the set of piecewise constant fuzzy sets.

3. Main results

Denote by $\mathcal{K}_F(X)$ the set of all finite subsets of X. Recall that for a compact X, the topology generated by the Hausdorff metric coincides with the finite topology. The following lemma illustrates the relation between finite chain continuity of (X, f) and its induced set-valued system.

Lemma 3.1. *Let (X, f) be a dynamical system and $(\mathcal{K}(X), \overline{f})$ be its induced set-valued system. Assume that $\mathcal{K}_F(X)$ is the set of all finite subsets of X. Then f is finite chain continuous if and only if $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$.*

Proof. Assume that f is finite chain continuous. To prove that $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$, let $\Omega = \{A_0, A_1, \cdots, A_k\}$ be a δ-chain in $\mathcal{K}_F(X)$ and $|A_i| = n_i$ for $i = 0, 1, \cdots, k$. Thus

$$d_H(\overline{f}(A_{i-1}), A_i) < \delta.$$

For each i take $A_i = \{a_{ij}\}$, where $j = 1, \cdots, n_k$. Fix any $j \in \{1, \cdots, n_k\}$. As $\{A_0, A_1, \cdots, A_k\}$ is a δ-chain, there exist $a_{0j} \in A_0$ and $a_{1j} \in A_1$ such that

$$d(f(a_{0j}), a_{1j}) < \delta.$$

Again, there is $a_{2j} \in A_2$ such that

$$d(f(a_{1j}), a_{2j}) < \delta.$$

By continuing this process, we obtain a sequence of δ-chains in X, say $\Omega_j = \{a_{0j}, a_{1j}, \cdots, a_{kj}\}$ for each $j \in \{1, \cdots, n_k\}$. On the other hand, since X is compact, there exists a finite subset $B = \{b_0, b_1, \cdots, b_k\}$ with $\bigcup_{b_i \in B} U(b_i, \delta) = X$. Thus, there is $B_i = \{b_{ij}\}_{j=1}^{n_k} \subseteq B$ such that

$$A_i \subset \bigcup_{j=1}^{n_k} U(b_{ij}, \delta) \quad \text{and} \quad A_i \cap U(b_{ij}, \delta) \neq \emptyset.$$

Hence, for each $a_{0j} \in \Omega_j$, there exists $b_{0j} \in B_0$ such that $d(a_{0j}, b_{0j}) < \delta$. Since f is finite chain continuous, we have $d(f^i(b_{0j}), a_{ij}) < \epsilon$.

Set $W = \{b_{0j}\}_{j=1}^{n_k}$. Thus $\{A_0, A_1, \cdots, A_k\}$ is a δ-chain in $\mathcal{K}_F(X)$ with $H(W, A_0) < \delta$ satisfying $d_H(\overline{f}^i(W), A_i) < \epsilon$. It follows that $\overline{f}$ is finite chain continuous.

Conversely, fin any $a \in X$. Let $\Lambda = \{x_0, x_1, \cdots, x_m\}$ be any δ-chain with $d(x_0, a) < \delta$ in X. Thus $\overline{\Lambda} = \{\{x_0\}, \{x_1\}, \cdots, \{x_m\}\}$ is a δ-chain in $\mathcal{K}_F(X)$ with $d_H(\{x_0\}, \{a\}) < \delta$. Since $\overline{f}$ is finite chain continuous on $\mathcal{K}_F(X)$, $d_H(\overline{f}^i\{a\}, \{x_i\}) < \epsilon$ holds, which yields $d(f^i(a), x_i) < \epsilon$. This completes the proof. □

Theorem 3.1. *Let $f : X \to X$ be continuous and $\hat{f}$ be the Zadeh's extension of f. Suppose that $\mathcal{SF}(X)$ is the set of piecewise constant fuzzy sets. If f is finite chain continuous then $\hat{f}$ is finite chain continuous on $\mathcal{SF}(X)$.*

Proof. Let $\delta > 0$ be given by finite chain continuity for f. Fix any $\omega \in \mathcal{SF}(X)$. Let $\widehat{\Lambda} = \{u_i \mid u_i \in \mathcal{SF}(X)\}_{i=0}^m$ be a δ-chain with $d_\infty(\omega, u_0) < \delta$ in $\mathcal{SF}(X)$. According to the definition of piecewise constant fuzzy set, for ω and each u_i, there exist decreasing sequences of closed subsets $\{A_j\}_{j=1}^m$, $\{B_j^i\}_{j=1}^k$, and increasing sequences of reals $\{\alpha_j\}_{j=1}^m \subseteq (0,1]$, $\{\beta_j^i\}_{j=1}^k \subseteq (0,1]$ such that

$$[\omega]_\alpha = A_{j+1}, \quad \text{where} \quad \alpha \in (\alpha_j, \alpha_{j+1}],$$

$$[u]_\beta = B_{j+1}^i, \quad \text{where} \quad \beta \in (\beta_j^i, \beta_{j+1}^i],$$

respectively. Note that $\{\alpha_j\}_{j=1}^m$ and $\{\beta_j^i\}_{j=1}^k$ are both increasing, thus there exists an increasing sequence $\{\lambda_j\}_{j=1}^s \subseteq (0,1]$ which obtained by rearranging $\{\alpha_j\}_{j=1}^m$ and $\{\beta_j^i\}_{j=1}^k$, where $1 \le s \le m+k$. Therefore, one can find decreasing sequences of closed subsets $\{C_j\}_{j=1}^m$, $\{D_j^i\}_{j=1}^k$ with

$$[\omega]_\lambda = C_{j+1} \quad \text{and} \quad [u]_\lambda = D_{j+1}^i,$$

where $\lambda \in (\lambda_j, \lambda_{j+1}]$.

Since $\widehat{\Lambda} = \{u_i \mid u_i \in \mathcal{SF}(X)\}_{i=0}^m$ is a δ-chain with $d_\infty(\omega, u_0) < \delta$, it can be verified that $\{D_j^i\}_{j=1}^k$ is a δ-chain in $\mathcal{K}_F(X)$ satisfying $d_H(C_j, D_j^0) < \delta$. Combining the finite chain continuity of f with Lemma 3.1, $\overline{f}$ is finite chain continuous, i.e., for any $\epsilon > 0$, $d_H(\overline{f}^i(C_j), D_j^i) < \epsilon$ holds. Consequently, it follows that

$$\begin{aligned} d_\infty(\hat{f}^i(\omega), u_i) &= \sup_{\alpha\in[0,1]} d_H([\hat{f}^i(\omega)]_\alpha, [u_i]_\alpha) \\ &= \sup_{\alpha\in[0,1]} d_H(f^i[\omega]_\alpha, [u_i]_\alpha) \\ &= d_H(\overline{f}^i(C_j), D_j^i) < \epsilon, \end{aligned}$$

which implies that $\hat{f}$ is finite chain continuous on $\mathcal{SF}(X)$. □

Theorem 3.2. *Let (X, f) be a dynamical system and $(\mathcal{F}(X), \hat{f})$ be its induced fuzzy system. If $\hat{f}$ is chain continuous then f is chain continuous.*

Proof. Fix any $x^* \in X$. For $\epsilon > 0$, take a δ-pseudo-orbit $\Omega = \{x_0, x_1, \cdots\}$ with $d(x^*, x_0) < \delta$ in X. Thus $\widehat{\Omega} = \{\hat{x}_0, \hat{x}_1, \cdots\}$ is a δ-pseudo-orbit with $d_\infty(\hat{x}^*, \hat{x}_0) < \delta$ in $\mathcal{F}(X)$. Since $\hat{f}$ is chain continuous, we have $d_\infty(\hat{f}^i(\hat{x}^*), \hat{x}_i) < \epsilon$ for $i = 0, 1, \cdots$. Hence

$$\begin{aligned} d_\infty(\hat{f}^i(\hat{x}^*), \hat{x}_i) &= \sup_{\alpha\in[0,1]} d_H([\hat{f}^i(\hat{x}^*)]_\alpha, [\hat{x}_i]_\alpha) \\ &= \sup_{\alpha\in[0,1]} d_H(f^i[(\hat{x}^*)]_\alpha, [\hat{x}_i]_\alpha) \\ &= d_H(\{f^i(x^*)\}, \{x_i\}) < \epsilon. \end{aligned}$$

It follows that $d(f^i(x^*), x_i) < \epsilon$ and consequently f is chain continuous. □

According to the Theorem 3.2, the following corollary is straightforward:

Corollary 3.1. *Let (X, f) be a dynamical system and $(\mathcal{F}(X), \hat{f})$ be its induced fuzzy system. If $\hat{f}$ is finite chain continuous then f is finite chain continuous.*

Remark 3.1. Since $\mathcal{F}(X)$ is not compact (see [8]) and it is not true that chain continuity is equivalent to the chain continuity on a dense subset (see [17]), finite chain continuity of $\hat{f}$ can not imply its own chain continuity. However, the completeness of $\mathcal{F}(X)$ lead us to the following theorem which is involved with a concept of contraction.

Definition 3.1. *Let $(\mathcal{F}(X), d_\infty)$ be a complete metric space. A function $\hat{f}$: $\mathcal{F}(X) \to \mathcal{F}(X)$ is called a contraction if there exists $\lambda \in (0, 1)$ such that $d_\infty(\hat{f}(u), \hat{f}(v)) \leq \lambda d(x, y)$ holds for any $u, v \in \mathcal{F}(X)$.*

Theorem 3.3. *Let $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ be a contraction, then $\hat{f}$ is chain continuous.*

Proof. Fix $\epsilon > 0$ and take $\delta = \frac{1-\lambda}{4}\epsilon$. For $v \in \mathcal{F}(X)$, let $\{u_0, u_1, \cdots\}$ be a δ-pseudo-orbit with $d_\infty(v, u_0) < \delta$ in $\mathcal{F}(X)$. Define $U_n = U(u_n, \frac{\epsilon}{4})$ for $n \geq 1$.

Thus for any $\omega_n \in U_n$, it follows that

$$\begin{aligned}
&d_\infty(\hat{f}(\omega_{n-1}), u_n) \leq d_\infty(\hat{f}(\omega_{n-1}), \hat{f}(u_{n-1})) \\
&+d_\infty(\hat{f}(u_{n-1}), u_n) < \lambda d_\infty(\omega_{n-1}, u_{n-1}) + \delta \\
&< \frac{\lambda}{4}\epsilon + \frac{1-\lambda}{4}\epsilon = \frac{\epsilon}{4},
\end{aligned}$$

which implies $\hat{f}(U_{n-1}) \subset U_n$, and therefore $\hat{f}^n(U_0) \subset U_n$. Choose any $\omega_0 \in U_0$, $d_\infty(\hat{f}^n(\omega_0), u_n) < \frac{\epsilon}{4}$ holds.

On the other hand, note that

$$\begin{aligned}
d_\infty(v, \omega_0) \quad &\leq d_\infty(v, u_0) + d_\infty(u_0, \omega_0) \\
&< \delta + \frac{\epsilon}{4} = \frac{1-\lambda}{4}\epsilon + \frac{\epsilon}{4} \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Hence

$$d_\infty(\hat{f}^n(v), \hat{f}^n(\omega_0)) < \lambda^n d_\infty(v, \omega_0) < \lambda^n \frac{\epsilon}{2}.$$

Consequently,

$$\begin{aligned}
&d_\infty(\hat{f}^n(v), u_n) \leq d_\infty(\hat{f}^n(v), \hat{f}^n(\omega_0)) \\
&+d_\infty(\hat{f}^n(\omega_0), u_n) < \lambda^n \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon,
\end{aligned}$$

which proves the chain continuity of $\hat{f}$. □

4. Conclusions and discussions

There is a difference between chain continuity and shadowing. Actually, chain continuity is a pointwise notion and shadowing is a global one. Both of the concepts are important tools for understanding the true orbits in the dynamical systems. In this paper, we investigate the relations between (finite) chain continuity of discrete dynamical system and its induced fuzzy system.

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