A semi-partial isometries in Banach spaces

El Moctar Ould Beiba
Department of Mathematics and Computer Sciences
Faculty of Sciences and Techniques
University of Nouakchott Al Aasriya
P.O. Box 5026, Nouakchott
Mauritania
elbeiba@yahoo.fr

Abstract. The purpose of this paper is to introduce and study some basic properties of the class of A-Semi partial isometries on Banach spaces.

Keywords: isometry, partial isometry, semi-partial isometry, semi-partial (m,p)-isometry.

1. Introduction

Let $\mathcal{X}$ denote a complex Banach space and $\mathcal{H}$ denote a complex Hilbert space. $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{B}(\mathcal{H})$) is the algebra of all bounded linear operators on $\mathcal{X}$ (resp. on $\mathcal{H}$). For every $T \in \mathcal{B}(\mathcal{X})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and its adjoint by T^*. A subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T (or T-invariant) if $T\mathcal{M} \subset \mathcal{M}$.

Partial isometries on Hilbert spaces $\mathcal{H}$ form an attractive and important class of operators. The definition is simple: An operator $T \in \mathcal{B}(\mathcal{H})$ is a partial isometry if and only if $TT^*T = T$ or equivalently, an operator T whose restriction to the orthogonal complement of its null-space is an isometry ([7]) i.e., $\|Tx\| = \|x\|$ for every $x \in \mathcal{N}(T)^\perp$, where $\mathcal{N}(T)^\perp$ is the orthogonal complement of $\mathcal{N}(T)$.

Partial isometries play a vital role in operator theory; they enter, for instance, in the theory of the polar decomposition of arbitrary operators. The concept of partial isometry can be defined in other equivalent ways.

The notion of A-partial isometries means bounded linear operators defined on a Hilbert space that behave (in some sense) as partial isometries when the semi-inner product induced by a positive (semidefinite) operator is considered. This concept was first introduced in [3] and studied also in [4, 14] where several results concerning projections and metrical properties were developed.

The theory of partial isometries on Banach spaces has been studied by several authors. For more details, see [10] and [11] where authors characterize the class of partial isometries on Banach spaces. A bounded linear operator T on a

Banach space $\mathcal{X}$ is called a semi-partial isometry if

$$\|Tx\| = d(x, \mathcal{N}(T)), \quad \text{for all } \ x \in X.$$

This class of operators, which is a natural generalization of partial isometries from Hilbert to general Banach spaces, contains in particular the class of partial isometries introduced by M. Mbekhta [10].

In this paper, we introduce the class of A-semi-partial isometries, which is a natural generalization of partial isometries from Hilbert and Banach spaces. This class of operators contains, among others, A-isometries,A- co-isometries, partial isometries (in the sense of M. Mbekhta). Our main goal here is to investigate and to classify this class of operators.

The minimum modulus of an operator $T \in \mathcal{B}(\mathcal{X})$ is defined by

$$\gamma(T) := \inf\{\|Tx\| : \quad x \in \mathcal{X}, \ \|x\| = 1\}.$$

Note that $\gamma(T) > 0$ if and only if T is injective and has closed range. (For more detail see [6]. The outline of the paper is as follows: Section 2 contains basic results on A-isometric and A-partial isometric operators on Banach spaces. In section 3 we study the concept of A-semi-partial isometries on Banach spaces and we investigate various structural properties of this class of operators. In the final section of the paper, we give some algebraic and analytical stability properties for the class of A-semi partial isometry.

2. A-isometric and A-partial isometric operators

An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be an isometry if $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$. In the sequel, we extend this term in a general way as follows : $T \in \mathcal{B}(\mathcal{X})$ is said to be isometry on a subspace $\mathcal{Y}$ of $\mathcal{X}$ if $\|Ty\| = \|y\|$ for all $y \in \mathcal{Y}$. $T \in \mathcal{B}(\mathcal{X})$ is said to be co-isometry if $T^* \in \mathcal{B}(\mathcal{X}^*)$ is an isometry and unitary if both T and T^* are isometries (T is both isometry and co-isometry).

Definition 2.1. *Let T and $A \in \mathcal{B}(\mathcal{X})$. T is said to be an A-isometry operator if*

$$\|TAx\| = \|Ax\|, \qquad \forall\, x \in \mathcal{X}.$$

Remark 2.1. (i) If $A = I_{\mathcal{X}}$ then an A-isometry is an isometry.

(ii) Every isometry is an A-isometry.

(iii) If $T(\mathcal{R}(A)) \subset \mathcal{R}(A)$ then, T is an A-isometry if and only if $T_{/\overline{\mathcal{R}(A)}}$ is an isometry.

(iv) If A is surjective then T is an A-isometry if and only if T is an isometry.

Proposition 2.1. *Let $T, S, A \in \mathcal{B}(\mathcal{X})$, the following statements hold:*

(1) *If T is invertible A-isometry such that $TA = AT$, then T^{-1} is an A-isometry.*

(2) *If T is an A-isometry and S is an A-isometry such that $AS = SA$, then TS is an A-isometry.*

(3) *If T is an isometry and S is an A-isometry, then TS is an A-isometry.*

Proof. (1) Since $\|TAx\| = \|Ax\|$, for all $x \in \mathcal{X}$ and $TA = AT$ it follows that

$$\|T^{-1}Ax\| = \|Ax\|, \quad x \in \mathcal{X}.$$

(2) By the assumptions we have, for all $x \in \mathcal{X}$

$$\|TSAx\| = \|TASx\| = \|ASx\| = \|SAx\| = \|Ax\|.$$

The statement (3) is obvious. □

The following example shows that the condition $TA = AT$ in the statement (1) is necessary.

Example 2.1. Let $T = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u,v)\|^2 := |u|^2 + |v|^2$. A simple computation shows that T is invertible A-isometry such that $TA \neq AT$ and moreover T^{-1} is not A-isometry.

Definition 2.2. *Let $\mathcal{X}$ be a Hilbert space and $A, T \in \mathcal{B}(\mathcal{H})$. T is said to be A-partial isometry if*

$$\|TAx\| = \|Ax\|, \qquad \forall\, x \in A^{-1}(\mathcal{N}(T)^{\perp}),$$

or, equivalently

$$\|Tz\| = \|z\|, \quad \forall \;\; z \in \mathcal{R}(A) \cap \mathcal{N}(T)^{\perp}.$$

3. A-semi partial isometries

This section is devoted to study of the class of A-semi partial isometries.

Definition 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$. T is said to be an A-semi -partial isometry if T satisfies*

$$\|TAx\| = d(Ax, \mathcal{N}(T)) \;, \forall \;\; x \in \mathcal{X},$$

or, equivalently, if

$$\|Ty\| = d(y, \mathcal{N}(T)) \;, \forall \;\; y \in \overline{\mathcal{R}(A)}.$$

Remark 3.1. (i) Let $T \in \mathcal{B}(\mathcal{X})$ then, T is a semi partial isometry if and only if T is $I_{\mathcal{X}}$-semi partial isometry.

(ii) If T is a semi partial isometry then, T is A-semi partial isometry for all A in $\mathcal{B}(\mathcal{X})$.

(iii) If $\mathcal{R}(A) \subset \mathcal{N}(T)$ then, T is A-semi partial isometry.

In the following example, we give an operator T that is A-semi-partial isometry but it is not a semi-partial isometry.

Example 3.1. If $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u,v)\|^2 := |u|^2 + |v|^2$, then T is an A-semi-partial isometry.

Indeed, we have $TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$\|TA \begin{pmatrix} u \\ v \end{pmatrix} \|^2 = \| \begin{pmatrix} u+v \\ u+v \end{pmatrix} \|^2 = 2|u+v|^2$. On the other hand

$$d(A \begin{pmatrix} u \\ v \end{pmatrix}, \mathcal{N}(T))^2 = d(\begin{pmatrix} 0 \\ \sqrt{2}(u+v) \end{pmatrix}, \mathcal{N}(T))^2 = 2|u+v|^2.$$

Hence $\|TA \begin{pmatrix} u \\ v \end{pmatrix} \| = d(A \begin{pmatrix} u \\ v \end{pmatrix}, \mathcal{N}(T))$ and so that T is an A-partial isometry as required.

Note that T is not a semi-partial isometry. Indeed, $\|Te_1\| = \frac{1}{\sqrt{2}} \neq \|e_1\| = d(e_1, \mathcal{N}(T))$ where $e_1 = (1, 0)$.

Proposition 3.1. *Let T, $A \in \mathcal{B}(\mathcal{X})$. The following statements hold:*

(i) *if $\overline{\mathcal{R}(A)}$ is invariant by T and if T is an A-semi partial isometry then, T is a semi partial isometry on $\overline{\mathcal{R}(A)}$. In particular, if $\mathcal{R}(A)$ is dense in $\mathcal{X}$ and if T is an A-semi partial isometry then, T is a semi partial isometry;*

(ii) *if T is injective A-semi partial isometry and $\overline{\mathcal{R}(A)}$ is T-invariant, then T is an isometry on $\overline{\mathcal{R}(A)}$;*

(iii) *if T is injective A-semi partial isometry and A is an isometry such that $TA = AT$, then T is an isometry.*

Proof. (i) The restriction of T to $\overline{\mathcal{R}(A)}$ is a bounded operator on this space. Since T is an A-semi partial isometry then, by the second variant of the Definition 3.1, we have $\|Ty\| = d(y, \mathcal{N}(T)), \forall y \in \overline{\mathcal{R}(A)}$. Hence T is a semi partial isometry on $\overline{\mathcal{R}(A)}$.

If $\mathcal{R}(A)$ is dense in $\mathcal{X}$ then, $\overline{\mathcal{R}(A)}$ is the whole space $\mathcal{X}$ and hence the conclusion follows by applying the previous statement. (ii) Since T in injective A-semi partial isometry , it follows that $\|TAx\| = d(Ax, \mathcal{N}(T)) = \|Ax\|$, $x \in \mathcal{X}$ and hence $\|Ty\| = \|y\|$ for all $y \in \overline{\mathcal{R}(A)}$. Thus T is an isometry on $\overline{\mathcal{R}(A)}$. (iii) It is easy to check that $\|Tx\| = \|ATx\| = \|TAx\| = \|Ax\| = \|x\|$, $x \in \mathcal{X}$. □

Proposition 3.2. *Let $A, T \in \mathcal{B}(\mathcal{X})$ such that A is an isometry and $A(\mathcal{N}(TA)) = \mathcal{N}(T)$. Then the following statements are equivalent:*

(i) *T is an A-semi partial isometry;*

(ii) *TA is a semi partial isometry.*

Proof. Let x be in $\mathcal{X}$ we have $d(Ax, \mathcal{N}(T)) = \inf\{\|Ax - y\|, \, y \in \mathcal{N}(T)\}$. Since $A(\mathcal{N}(TA)) = \mathcal{N}(T)$, we get $d(Ax, \mathcal{N}(T)) = \inf\{\|Ax - Az\|, \, z \in \mathcal{N}(TA)\}$. Using the assumption that A is an isometry, the last equality becomes

$$d(Ax, \mathcal{N}(T)) = \inf\{\|x - z\|, \; z \in \mathcal{N}(TA)\} = d(x, \mathcal{N}(TA)).$$

Thus, we have

$$d(Ax, \mathcal{N}(T)) = d(x, \mathcal{N}(TA)), \; \forall x \in \mathcal{X}. \tag{3.1}$$

(3.1) gives us the desired equivalence. □

As a consequences of the Proposition 3.2, we get some corollaries:

Corollary 3.1. *Let A and T as in the Proposition 3.2 then, the following assertions are equivalent:*

(i) *T is A-semi partial isometry;*

(ii) $\gamma(TA) = \|TA\| = 1$;

(iii) *TA is a contraction and* $\gamma(TA) \geq 1$;

(iv) *The operator* $\widetilde{TA} : \mathcal{X}/\mathcal{N}(TA) \to \mathcal{X}$, *given by* $\widetilde{TA}\dot{x} = TAx$ *for all* $x \in \mathcal{X}$, *is an isometry.*

Proof. The assertion (i) is, by the Proposition 3.2, equivalent to that TA is semi partial isometry. Hence, we obtain the equivalence between (i)-(iv) by applying the Theorem 2.1 ([13]) to the operator TA. □

Corollary 3.2. *Let $A, T \in \mathcal{B}(\mathcal{X})$. If A is an invertible isometry, then T is A-semi partial isometry if and only if TA is a semi partial isometry.*

Proof. It is easy to check that A and T satisfy the conditions in the Proposition 3.2 and hence the conclusion follows by applying this proposition. □

Corollary 3.3. *Let $A, T \in \mathcal{B}(\mathcal{X})$. If A is an invertible isometry then, T is a semi-partial isometry if and only if TA^n is a semi-partial isometry for all $n \in \mathbb{N}$.*

Proof. By induction : To start, for $n = 1$, the result is an easy consequence of the statement (i) of the Proposition 3.1 combined with the Corollary 3.2. □

We give now some sufficient conditions for semi-partial isometry.

Proposition 3.3. *Let $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{N}(T) \subset \mathcal{R}(T)$. If T is T-semi partial isometry then, T is a semi partial isometry.*

Proof. It is obvious that $T(\mathcal{N}(T^2)) \subset \mathcal{N}(T)$. By the assumption $\mathcal{N}(T) \subset \mathcal{R}(T)$, we obtain the reverse inclusion and hence we get the equality $T(\mathcal{N}(T^2)) = \mathcal{N}(T)$. Now taking $A = T$, the Proposition 3.2 yields the desired result. □

Proposition 3.4. *Let $T \in \mathcal{B}(\mathcal{X})$. Then, T is a semi-partial isometry if and only if there is $A \in \mathcal{B}(\mathcal{X})$ the range of A is dense and T is A semi partial isometry.*

Proof. For the direct implication, Take $A = I_{\mathcal{X}}$. For the converse implication, use the second variant of the Definition 3.1 to conclude that T is a semi partial isometry □

The following proposition shows that the A-semi partial isometry is preserved when the parameter A runs over some specific class.

Proposition 3.5. *Let T, A_1 and A_2 be in $\mathcal{B}(\mathcal{X})$ such that assume that $TA_1 = TA_2$. Then T is A_1-semi partial isometry if and only if it is A_2-semi partial isometry.*

Proof. Let x be in $\mathcal{X}$. By the assumption $TA_1 = TA_2$ we have $\mathcal{R}(A_1 - A_2) \subset \mathcal{N}(T)$ and hence $A_1x - \mathcal{N}(T) = A_2x - \mathcal{N}(T)$. This implies $d(A_1x, \mathcal{N}(T)) = d(A_2x, \mathcal{N}(T))$. Consequently, $\|TA_1\| = d(A_1x, \mathcal{N}(T))$ if and only if $\|TA_2\| = d(A_2x, \mathcal{N}(T))$. This achieves the proof. □

Remark 3.2. The converse of the Proposition 3.5 is not true. Otherwise, by (ii) in the Remark 3.1applying to the identity, all operators will agree, which is possible only in the trivial case when $\mathcal{X} = \{0\}$.

If $\mathcal{H}$ is a Hilbert space, it is well know that $T \in \mathcal{B}(\mathcal{H})$ is a semi-partial isometry if and only if T is a partial isometry i.e; $\|Tx\| = \|x\| \;\; \forall\, x \in \mathcal{N}(T)^{\perp}$. The following theorem gives, under prerequisite condition, a necessary and sufficient condition for operator to be A-semi partial isometry in Hilbert space.

Theorem 3.1. *Let $T, A \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Assume that*

$$\mathcal{H} = A^{-1}(\mathcal{N}(T)) + A^{-1}(\mathcal{N}(T)^{\perp}). \tag{3.2}$$

The following statements are equivalent:
(i) *T is an A-semi-partial isometry;*
(ii) $\|TAx\| = \|Ax\|, \forall\, x \in A^{-1}\big(\mathcal{N}(T)^{\perp}\big)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Let $x \in \mathcal{X}, x = x_1 + x_2$ where $x_1 \in A^{-1}\big(\mathcal{N}(T)\big)(\Leftrightarrow Ax_1 \in \mathcal{N}(T))$ and $x_2 \in A^{-1}(\mathcal{N}(T)^{\perp})(\Leftrightarrow Ax_2 \in \mathcal{N}(T)^{\perp})$ and it follows that $\|TAx\| = \|Ax_2\| = d(Ax_2, \mathcal{N}(T))$. It is easy to see that for any $x \in \mathcal{X}$, $d(Ax, \mathcal{N}(T)) = d(Ax_2, \mathcal{N}(T))$.

We conclude that $\|TAx\| = d(Ax, \mathcal{N}(T)), \forall x \in \mathcal{X}$ and the proof of this implication is over. □

Remark 3.3. If A is invertible on a Hilbert space $\mathcal{H}$, the condition (3.2) is satisfied and hence T is A-semi partial isometry if and only if T is partial isometry.

It was observed in [13] that an operator $T \in \mathcal{B}(\mathcal{X})$ is a semi-partial isometry if and only if $T^* \in \mathcal{B}(\mathcal{X}^*)$ is a semi-partial isometry. This result fail to be true in general for A-semi-partial isometry as shown by the following example.

Example 3.2. If $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ on $X = \mathbb{C}^2$ equipped with the norm $\|(u,v)\|^2 := |u|^2 + |v|^2$.

We know that T is an A-semi-partial isometry,therefore T^* is not A^*-semi-partial isometry. Indeed, a simple computation shows that $T^*A^* = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$.

$$\|T^*A^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}\|^2 = 5 \quad \text{and} \quad d(A^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathcal{N}(T))^2 = 4.$$

Clearly, T^* is not A^*-semi-partial isometry.

4. Stability properties

We give in this section some algebraic and analytical stability properties for the A-semi partial isometry.

Proposition 4.1. *If T is A_1-semi partial isometry then, T is A_1A_2-semi partial isometry for any operator A_2.*

Proof. Applying the condition of A_1-semi partial isometry on the vector A_2x for an arbitrary vector$x$, we get $\|TA_1A_2x\| = d(A_1A_2x, \mathcal{N}(T))$. Thus T is A_1A_2-semi partial isometry. □

As a consequence of the Proposition 4.1 we obtain the following dynamic result:

Corollary 4.1. *If T is an A-semi partial isometry then, T is an A^n-semi-partial isometry for any integer $n \geq 1$.*

Proof. By induction, using Proposition 4.1. □

The following corollary is an immediate consequence of Corollary 4.1.

Corollary 4.2. *Let T be an A-semi-partial isometry. If A is idempotent ($A^2 = I_{\mathcal{X}}$) then, T is a semi-partial isometry.*

Proposition 4.2. *Let (A_n) be a sequence in $\mathcal{B}(\mathcal{X})$ which converges to A in the operator norm. If T is A_n-semi partial isometry for every n then, T is A-semi partial isometry.*

Proof. Let $x \in \mathcal{X}$. We have

$$\|TA_nx\| = d(A_nx, \mathcal{N}(T)), \quad \forall n \in \mathbb{N} \tag{4.1}$$

Since the left side of (4.1) tends to $\|TAx\|$ and the right side has as limit $d(Ax, \mathcal{N}(T))$, we obtain

$$\|TA\| = d\big(Ax, \mathcal{N}(T)\big).$$

Hence T is A-semi-partial isometry. □

Theorem 4.1. *Let $(T_n)_n$ be a sequence of semi partial isometry in $\mathcal{B}(\mathcal{H})$ such that $T_n \neq 0$ and T_n converges to $T(\neq 0)$ in norm (as $n \to \infty$). Suppose that there exists a positive integer m such that $\mathcal{N}(T) \subset \mathcal{N}(T_n)$ for every $n \geq m$, then T is a semi-partial isometry.*

Proof. Since $T_n \neq 0$ and T_n is a semi-partial isometry it follows by [13, Thoerem 2.1] that $\gamma(T_n) = \|T_n\| = 1$. As $T_n \to T$ in norm, we have $\|T_n\| \to \|T\|$ (as $n \to \infty$) and we get $\|T\| = 1$. Since by our assumption, $\mathcal{N}(T) \subset \mathcal{N}(T_n)$, for each $n \geq m$, it therefore follows from [9, Theorem 1.3] that $\gamma(T_n) \to \gamma(T)$ as $n \to \infty$ and so that $\gamma(T) = 1$. □

Theorem 4.2. *Let $(T_n)_n$ be a sequence of A-semi partial isometry in $\mathcal{B}(\mathcal{H})$ such that $T_n \neq 0$ and T_n converges to $T(\neq 0)$ in norm (as $n \to \infty$). Suppose that the following conditions are hold:*

(a) there exists a positive integer m such that $\mathcal{N}(TA) \subset \mathcal{N}(T_nA)$ for every $n \geq m$;

(b) $A(\mathcal{N}(T_nA)) = \mathcal{N}(T_n)$;

(c) $A(\mathcal{N}(TA)) = \mathcal{N}(T)$.

Then T is an A-semi-partial isometry.

Proof. Obviously, $T_nA \to TA$ as $n \to \infty$. By the condition (b) we know that T_nA is a semi-partial isometry (by Corollary 3.1). Now applying Theorem 4.1 we have that TA is a semi-partial isometry. The desired result follows from condition (c). □

Definition 4.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a linear subspace of $\mathcal{X}$. Let $\mathcal{Y}^\star$ be defined by*

$$\mathcal{Y}^\star = \{x \in \mathcal{X}, d(x, \mathcal{Y}) = \|x\|\}.$$

We say that $\mathcal{Y}$ is a $\star$-subspace if the following conditions hold:

(i)$_\star$ *$\mathcal{Y}^\star$ is linear subspace of $\mathcal{X}$;*

(ii)$_\star$ *$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\star$.*

Example 4.1. If $\mathcal{M}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$ then, $\mathcal{M}^\star = \mathcal{M}^\perp$. Hence $\mathcal{M}$ is $\star$-subspace.

Definition 4.2. *A Banach space $\mathcal{X}$ is said to be $\star$-Banach space, if any closed linear subspace of $\mathcal{X}$ is a $\star$-subspace.*

Example 4.2. Referring to the Example (0.1), any Hilbert space is a $\star$-Banach space.

Proposition 4.3. *Let $\mathcal{X}$ be a $\star$-Banach space, $\mathcal{Y}$ a linear closed subspace of $\mathcal{X}$ and $x \in \mathcal{X}$. Then, there exists a unique $y \in \mathcal{Y}$ such that $d(x, \mathcal{Y}) = \|x - y\|$, y is the unique vector in $\mathcal{Y}$ such that $x - y$ belongs to $\mathcal{Y}^\star$. y is called the $\star$-projection of x on $\mathcal{Y}$.*

Proof. The Proposition is an immediate consequence of the condition (ii). □

The following theorem gives a characterization of the semi partial isometry in the case of $\star$-Banach space.

Theorem 4.3. *Let $\mathcal{X}$ be a $\star$-Banach space and $T \in \mathcal{B}(\mathcal{X})$. Then, T is a semi partial isometry if and only if $\|Tx\| = \|x\|$, $\forall x \in \mathcal{N}(T)^\star$.*

Proof. For the direct implication, let $x \in \mathcal{N}(T)^\star$. Since T is a semi partial isometry, we have $\|Tx\| = d(x, \mathcal{N}(T))$. On the other hand, x satisfies

$d(x,\mathcal{N}(T)) = \|x\|$. Finally, we have $\|Tx\| = \|x\|$. Conversely, let $x \in \mathcal{X}$. By the Proposition 4.3, there is $y \in \mathcal{N}(T)$ such that $x - y \in \mathcal{N}(T)^\star$ and

$$d(x,\mathcal{N}(T)) = \|x - y\|. \tag{4.2}$$

By assumption, we have $\|T(x-y)\| = \|x-y\|$. Since $y \in \mathcal{N}(T)$, the last equality becomes

$$\|Tx\| = \|x - y\| \tag{4.3}$$

Taking (4.3) in (4.2), we get $\|Tx\| = d(x,\mathcal{N}(T))$. We can conclude that T is a semi partial isometry. $\square$

Theorem 4.4. *Let $\mathcal{H}$ be a Hilbert space, S and T are two partial isometries on $\mathcal{H}$ such that $S(\mathcal{N}(TS)^\perp) \subset \mathcal{N}(T)^\perp$. Then, TS is a partial isometry.*

Proof. Let $x \in \mathcal{N}(TS)^\perp$. We have $Sx \in \mathcal{N}(T)^\perp$. Since T is a semi partial isometry, we get

$$\|TSx\| = \|Sx\|,$$

Since S is a partial isometry then it is a semi partial isometry and hence we have

$$\|TSx\| = d(x,\mathcal{N}(S)). \tag{4.4}$$

Hence, we obtain

$$\|TSx\| \leq \|x\|. \tag{4.5}$$

By the inclusion $\mathcal{N}(S) \subset \mathcal{N}(TS)$, we have $d(x,\mathcal{N}(S)) \geq d(x,\mathcal{N}(TS)$. By the assumption $x \in \mathcal{N}(TS)^\perp$, we have $d(x,\mathcal{N}(TS) = \|x\|$. Thus $d(x,\mathcal{N}(S)) \geq \|x\|$. Which gives, from (4.4),

$$\|TSx\| \geq \|x\|. \tag{4.6}$$

(4.5) and (4.6) show that $\|TSx\| = \|x\|$. Thus, we have proved that $\|TSx\| = \|x\|,\ \forall x \in \mathcal{N}(TS)^\perp$. Hence, TS is a partial isometry. $\square$

In [2], A. Alahmari et al. introduced the concepts of semi-partial (m,p)-isometries an a generalization of (m,p)-isometries (see [1, 5, 8]. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be a semi partial (m,p)-isometry for some positive integer m and a real $p \geq 1$, if T satisfies

$$\sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = (-1)^{m+1} d(x,\mathcal{N}(T))^p, \qquad \forall\, x \in \mathcal{X}.$$

Clearly, T a semi-partial (m,p)-isometry if and only if T is an (m,p)-isometry. Also, a semi-partial isometry corresponds here to a semi- partial $(1,2)$-isometry. In the following definition, we extend this concept to the concept of A-semi-partial-(m,p)-isometry.

Definition 4.3. *Let $A, T \in \mathcal{B}(\mathcal{X})$ and let m be a positive integer, $p \in [1, \infty)$. T is said to be A-semi-partial-(m, p)-isometry if*

$$\sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|T^k Ax\|^p = (-1)^{m+1} d(Ax, \mathcal{N}(T))^p, \qquad \forall\, x \in \mathcal{X}.$$

Remark 4.1. We make the following observations:

(1) If $A = I_{\mathcal{X}}$, then every A-semi-partial-(m, p)-isometry is a semi-partial-(m, p)-isometry.

(2) Every A-semi-partial-$(1, 2)$-isometry is an A-semi-partial isometry (Definition 3.1).

(3) T is an A-semi partial-$(2, p)$- isometry if and only if

$$\|T^2 Ax\|^p - 2\|TAx\|^2 + d(Ax, \mathcal{N}(T))^p = 0, \qquad \forall\, x \in \mathcal{X}.$$

Proposition 4.4. Let $A, T \in \mathcal{B}(\mathcal{X})$ and let m be a positive integer, $p \in [1, \infty)$. Suppose that T is A-semi-partial-(m, p)-isometry. If V is isometry on $\mathcal{R}(T)$ with $VT = TV$ then VT is A-semi-partial-(m, p)-isometry.

Proof. Since V is isometry on $\mathcal{R}(T)$, we get

$$\|VTx\| = \|Tx\|, \ \forall\, x \in \mathcal{X} \ \text{ and } \ \mathcal{N}(VT) = \mathcal{N}(T).$$

This yields, by little calculation, taking in consideration that V commutes with T,

$$\|(VT)^k Ax\| = \|T^k Ax\|, \ 1 \leq k \leq m, \ x \in \mathcal{X}. \tag{4.7}$$

Now, since T is A-semi-partial-(m, p)-isometry, (4.7) implies

$$\begin{aligned} &\sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|(VT)^k Ax\|^p \\ &= (-1)^{m+1} d(Ax, \mathcal{N}(T))^p = (-1)^{m+1} d(Ax, \mathcal{N}(VT))^p, \qquad \forall\, x \in \mathcal{X}. \end{aligned}$$

Thus VT is A-semi-partial-(m, p)-isometry. □

It seems to be natural to enquire a detail study for the class of A-semi-partial-(m, p)-isometry. This study will be addressed in a forthcoming paper.

References

[1] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space I*, Integral Equations and Operator Theory, 21 (1995), 383-429.

[2] A. Alahmari, M. Mabrouk, M. A. Taoudi, *Discussions on partial isometries in Banach spaces and Banach algebra*, Bull. Korean Math. Soc., 54 (2017), 485-495.

[3] M.L. Arias, G. Corach, M.C. Gonzalez, *Partial isometries in semi-Hilbertian spaces*, Linear Algebra Appl., 428 (2008), 1460-1475.

[4] M. L. Arias, G. Corach, M. C. Gonzalez, *Metric properties of projections in semi-Hilbertian spaces*, Integral Equations and Operator Theory, 62 (2008), 11-28.

[5] F. Bayart, *m-isometries on Banach spaces*, Math. Nachr., 284 (2011), 2141-2147.

[6] A. Bourhim, M. Burgos, V. S. Shulman, *Linear maps preserving the minimum and reduced minimum moduli*, Journal of Functional Analysis, 258 (2010), 50-66.

[7] P. R. Halmos, J. E. Mclaughlin, *Partial isometries*, Pacific Journal of Mathematics, 13 (1963).

[8] P. Hoffman, M. Mackey, M. Ó Searcóid, *On the second parameter of an* $(m;p)$*-isometry*, Integral Equations and Operator Theory, 71 (2011), 389-405.

[9] I. S. Hwang, W. Y. Lee, *The reduced minimum modulus of operators*, Journal of Mathematical Analysis and Applications, 267 (2002), 679-694.

[10] M. Mbekhta, *Partial isometries and generalised inverses*, Acta Sci. Math. (Szeged), 70 (2004), 767-781.

[11] Ch. Schmoeger, *Spectal properties of generalized inverses*, Demonstratio Math., 37 (2004), 137-148.

[12] Ch. Schmoeger, *On a question of Mbekhta*, Extracta Mathematicae, 20 (2005), 281-290.

[13] M. A. Taoudi, *On a generalization of partial isometries in a Banach spaces*, Georgian Mathematical Journal, 15 (2008), 177-188.

[14] L. Suciu, *Quasi-isometries in semi-Hilbertian spaces*, Linear Algebra Appl., 430 (2009), 2474-2487.

Accepted: 25.07.2019