Generalized fractional integral inequalities for product of two convex functions

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Abstract. The aim of this paper is to generalize the results proved in [4] using generalized fractional integral. Some special cases are deduced from main results. Applying the techniques of our results, new results may be obtained during a similar manner for various operators.
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1. Introduction

If $\mathcal{F}_1 : \mathbf{I} \to \mathbb{R}$ be a convex mapping on the interval $\mathbf{I}$ of real numbers and $\wp_1, \wp_2 \in \mathbf{I}$ with $\wp_1 \neq \wp_2$ we have the following double inequality well known in the literature:

$$\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)dx \leq \frac{\mathcal{F}_1(\wp_1)+\mathcal{F}_1(\wp_2)}{2}, \tag{1}$$

double inequality (1) also known as trapezium inequality.

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There are several refinements of the Hermite-Hadamard inequality on convex functions which are extensively investigated by variety of authors (e.g., [1, 2, 3, 5, 6, 7, 8, 11]).

In [4], Chen established two new Hermite-Hadamard inequalities via Riemann-Liouville fractional integral for product of convex functions as follows:

Theorem 1.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for Riemann-Liouville fractional integral holds:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^{\alpha}}\left[J^{\alpha}_{\wp_1}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+J^{\alpha}_{\wp_2}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ \leq\ & \left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)\mathbf{M}(\wp_1,\wp_2) \\ & +\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathbf{N}(\wp_1,\wp_2), \end{aligned} \tag{2}$$

$$\begin{aligned} & 2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^{\alpha}}\left[I^{\alpha}_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ & +\mathbf{M}(\wp_1,\wp_2)\frac{\alpha}{(\alpha+1)(\alpha+2)} \\ & +\mathbf{N}(\wp_1,\wp_2)\left[\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}-\frac{1}{2}\right], \end{aligned} \tag{3}$$

where $\mathbf{M}(\wp_1,\wp_2)=\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)$, *and* $\mathbf{N}(\wp_1,\wp_2)=\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1)$.

Some new integral inequalities involving two nonnegative and integrable functions that are associated with the Hermite Hadamard type also are obtained by several authors. In [12], B.G. Pachpatte projected some Hermite-Hadamard kind inequalities involving two *log*-convex functions. An identical result for s-convex functions is established by Kirmaci et al. in [11]. In [15], Sarikaya conferred some integral inequalities for two h-convex functions. For recent results and generalizations regarding Hermite-Hadamard type inequality for product of two functions see [16] and therefore the references given in that.

It is outstanding that Sarikaya and Ertuğral in [14] defined the following generalized fractional integral:

$$_{\wp_1}I_{\varphi}\mathcal{F}_1(x)=\int_{\wp_1}^{x}\frac{\varphi(x-\jmath)}{x-\jmath}\mathcal{F}_1(\jmath)d\jmath,\quad x>\wp_1, \tag{4}$$

$$_{\wp_2}I_{\varphi}\mathcal{F}_1(x)=\int_{x}^{\wp_2}\frac{\varphi(\jmath-x)}{\jmath-x}\mathcal{F}_1(\jmath)d\jmath,\quad x<\wp_2 \tag{5}$$

where the function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(\jmath)}{\jmath} d\jmath < \infty.$$

Remark 1.1. If we take $\varphi(\jmath) = \jmath$, $\varphi(\jmath) = \frac{\jmath^\alpha}{\Gamma(\alpha)}$, $\varphi(\jmath) = \frac{1}{k\Gamma_k(\alpha)}\jmath^{\frac{\alpha}{k}}$ and $\varphi(\jmath) = \jmath(x - \jmath)^{\alpha-1}$ in the operators (4) and (5), then we have the Riemann integral, Riemann-Liouville fractional integral [10], k-Riemann-Liouville fractional integral given by Mubeen and Habibullah in [13] and conformable fractional operators given by Khalil et al. in [9], respectively.

By applying generalized fractional integral, Sarıkaya and Ertuğral established following fascinating Hermite-Hadamard type integral inequalities in the same paper:

Theorem 1.2. *Let $\mathcal{F}_1 : [\wp_1, \wp_2] \to \mathbb{R}$ is convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$\mathcal{F}_1\left(\frac{\wp_1 + \wp_2}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_\varphi \mathcal{F}_1(\wp_2) + {}_{\wp_2-}I_\varphi \mathcal{F}_1(\wp_1)\right] \leq \frac{\mathcal{F}_1(\wp_1) + \mathcal{F}_1(\wp_2)}{2}, \tag{6}$$

where $\Lambda(1) = \int_0^1 \frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath} d\jmath$.

The main objective of our this paper is to generalize the proved results in [4] with help of generalized fractional integrals. We shall also discuss some of its special cases.

2. Main results

Theorem 2.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$\left[{}_{\wp_1+}I_\varphi^{\wp_2} \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + {}_{\wp_2-}I_\varphi^{\wp_1} \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \leq J_1 \mathbf{M}(\wp_1, \wp_2) + J_2 \mathbf{N}(\wp_1, \wp_2) \tag{7}$$

where

$$\begin{aligned} J_1 &= \int_0^1 \frac{\varphi((\wp_2 - \wp_1)\jmath)}{\jmath}(2\jmath^2 - 2\jmath + 1)d\jmath, \\ J_2 &= \int_0^1 \frac{\varphi((\wp_2 - \wp_1)\jmath)}{\jmath}(2\jmath - 2\jmath^2)d\jmath, \\ \mathbf{M}(\wp_1, \wp_2) &= \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2), \\ \mathbf{N}(\wp_1, \wp_2) &= \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1). \end{aligned}$$

Proof. Since $\mathcal{F}_1, \mathcal{F}_2$ are convex functions, then according to the definition of convex function we have

$$\begin{aligned} &\mathcal{F}_1(\wp_1 + \jmath(\wp_2 - \wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2 - \wp_1)) \\ &\leq ((1 - \jmath)\mathcal{F}_1(\wp_1) + \jmath\mathcal{F}_1(\wp_2))\left((1 - \jmath)\mathcal{F}_2(\wp_1) + \jmath\mathcal{F}_2(\wp_2)\right), \end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\mathcal{F}_1(\wp_2 + \jmath(\wp_1 - \wp_2))\mathcal{F}_2(\wp_2 + \jmath(\wp_1 - \wp_2)) \\
&\leq (\jmath\mathcal{F}_1(\wp_1) + (1-\jmath)\mathcal{F}_1(\wp_2))\left(\jmath\mathcal{F}_2(\wp_1) + (1-\jmath)\mathcal{F}_2(\wp_2)\right). \tag{9}
\end{aligned}$$

Adding (8) and (9), we have

$$\begin{aligned}
&\mathcal{F}_1(\wp_1 + \jmath(\wp_2 - \wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2 - \wp_1)) \\
&+ \mathcal{F}_1(\wp_2 + \jmath(\wp_1 - \wp_2))\mathcal{F}_2(\wp_2 + \jmath(\wp_1 - \wp_2)) \\
&\leq ((1-\jmath)\mathcal{F}_1(\wp_1) + \jmath\mathcal{F}_1(\wp_2))\left((1-\jmath)\mathcal{F}_2(\wp_1) + \jmath\mathcal{F}_2(\wp_2)\right) \\
&+ (\jmath\mathcal{F}_1(\wp_1) + (1-\jmath)\mathcal{F}_1(\wp_2))\left(\jmath\mathcal{F}_2(\wp_1) + (1-\jmath)\mathcal{F}_2(\wp_2)\right).
\end{aligned}$$

So, we get

$$\begin{aligned}
&\mathcal{F}_1(\wp_1 + \jmath(\wp_2 - \wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2 - \wp_1)) \\
&+ \mathcal{F}_1(\wp_2 + \jmath(\wp_1 - \wp_2))\mathcal{F}_2(\wp_2 + \jmath(\wp_1 - \wp_2)) \\
&\leq (2\jmath^2 - 2\jmath + 1)\mathbf{M}(\wp_1, \wp_2) + 2\jmath(1-\jmath)\mathbf{N}(\wp_1, \wp_2), \tag{10}
\end{aligned}$$

Multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of inequality (10), then integrating the obtained inequality with respect to '$\jmath$' on $[0,1]$, we get our desired inequality (7). □

Remark 2.1. If we choose $\mathcal{F}_2(x) = 1$ for all $x \in [\wp_1, \wp_2]$ in Theorem 2.1, then we obtain the second inequality of (6).

Remark 2.2. If we use $\varphi(\jmath) = \jmath$ in Theorem 2.1, we have a well known inequality for the product of two convex function, given by B. G. Pachpatte in [12], i.e.,

$$\frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2} \mathcal{F}_1(x)\mathcal{F}_2(x)dx \leq \frac{1}{3}\mathbf{M}(\wp_1, \wp_2) + \frac{1}{6}\mathbf{N}(\wp_1, \wp_2),$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Remark 2.3. If we suppose $\varphi(\jmath) = \frac{\jmath^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, we have an inequality for the product of convex functions via Riemann-Liouville fractional integral given by Chen, F. in [4], i.e.,

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\wp_2 - \wp_1)^\alpha}\left[J^\alpha_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + J^\alpha_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\
&\leq \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)\mathbf{M}(\wp_1, \wp_2) \\
&+ \frac{\alpha}{(\alpha+1)(\alpha+2)}\mathbf{N}(\wp_1, \wp_2),
\end{aligned}$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Corollary 2.1. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for k-Riemann-Liouville fractional integral holds:*

$$
\begin{aligned}
&\frac{\Gamma_k(\alpha)}{(\wp_2-\wp_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\wp_1+,k}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-,k}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&\le \frac{k(\alpha^2+\alpha k+2k^2)}{\alpha(k+\alpha)(2k+\alpha)}\mathbf{M}(\wp_1,\wp_2)\\
&+\frac{2k^2}{(k+\alpha)(2k+\alpha)}\mathbf{N}(\wp_1,\wp_2), \qquad (11)
\end{aligned}
$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, we have our desired inequality (11). □

Theorem 2.2. . *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for generalized fractional integral holds:*

$$
\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\le \frac{1}{4\Lambda(1)}\left[{}_{\wp_1+}I^{\wp_2}_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+{}_{\wp_2-}I^{\wp_1}_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&+\frac{1}{4\Lambda(1)}\left\{J_2\mathbf{M}(\wp_1,\wp_2)+J_1\mathbf{N}(\wp_1,\wp_2)\right\}, \qquad (12)
\end{aligned}
$$

where J_1, J_2, $\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are same as used in Theorem 2.1.

Proof. Consider the left hand side of inequality (12) that can be written as

$$
\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&=\mathcal{F}_1\left(\frac{\jmath\wp_1+(1-\jmath)\wp_2}{2}+\frac{(1-\jmath)\wp_1+\jmath\wp_2}{2}\right) \qquad (13)\\
&\cdot\mathcal{F}_2\left(\frac{\jmath\wp_1+(1-\jmath)\wp_2}{2}+\frac{(1-\jmath)\wp_1+\jmath\wp_2}{2}\right)\\
&=\frac{1}{4}\{(\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2))\\
&\times(\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)+)+\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2))\}\\
&=\frac{1}{4}\{\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\\
&+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\\
&+\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\\
&+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\}. \qquad (14)
\end{aligned}
$$

Since $\mathcal{F}_1,\ \mathcal{F}_2$ are convex functions, so (14) can be written as

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{1}{4}\{\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\\
&+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\\
&+(\jmath\mathcal{F}_1(\wp_1)+(1-\jmath)\mathcal{F}_1(\wp_2))((1-\jmath)\mathcal{F}_2(\wp_1)+\jmath\mathcal{F}_2(\wp_2))\\
&+((1-\jmath)\mathcal{F}_1(\wp_1)+\jmath\mathcal{F}_1(\wp_2))(\jmath\mathcal{F}_2(\wp_1)+(1-\jmath)\mathcal{F}_2(\wp_2))\}\\
&=\frac{1}{4}\{\mathcal{F}_1(\jmath\wp_1+(1-\jmath)\wp_2)\mathcal{F}_2(\jmath\wp_1+(1-\jmath)\wp_2)\\
&+\mathcal{F}_1((1-\jmath)\wp_1+\jmath\wp_2)\mathcal{F}_2((1-\jmath)\wp_1+\jmath\wp_2)\}\\
&+\frac{1}{4}\{2\jmath(1-\jmath)\left[\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)(\wp_2)\right]\\
&+(2\jmath^2-2\jmath+1)\left[\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_2)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_1)\right]\}. \qquad (15)
\end{aligned}$$

Multiplying (15) with $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides and integrating the obtain inequality with respect to '$\jmath$' over $[0,1]$,then we have our required inequality (12). □

Remark 2.4. If we use $\varphi(\jmath)=\jmath$ in Theorem 2.2, we have a well known inequality for the product of two convex function, given by B. G. Pachpatte in [12], i.e.,

$$\begin{aligned}
2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \leq\ & \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}_1(x)\mathcal{F}_2(x)dx\\
&+\frac{1}{6}\mathbf{M}(\wp_1,\wp_2)+\frac{1}{3}\mathbf{N}(\wp_1,\wp_2),
\end{aligned}$$

where $\mathbf{M}(\wp_1,\wp_2)$ and $\mathbf{N}(\wp_1,\wp_2)$ are same as used in Theorem 2.1.

Remark 2.5. If we suppose $\varphi(\jmath)=\frac{\jmath^\alpha}{\Gamma(\alpha)}$ in Theorem 2.2, we have an inequality for the product of convex functions via Riemann-Liouville fractional integral given by Chen, F. in [4], i.e.,

$$\begin{aligned}
&2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^\alpha}\left[I^\alpha_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^\alpha_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&+\mathbf{M}(\wp_1,\wp_2)\frac{\alpha}{(\alpha+1)(\alpha+2)}\\
&+\mathbf{N}(\wp_1,\wp_2)\left[\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}-\frac{1}{2}\right].
\end{aligned}$$

Corollary 2.2. *If we choose $\mathcal{F}_2(x)=1$ for all $x\in[\wp_1,\wp_2]$ in Theorem 2.2, then we obtain the following inequality*

$$2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_\varphi^{\wp_2}\mathcal{F}_1(\wp_2)+{}_{\wp_2-}I_\varphi^{\wp_1}\mathcal{F}_1(\wp_1)\right]+\frac{\mathcal{F}_1(\wp_1)+\mathcal{F}_1(\wp_2)}{2}.$$

Corollary 2.3. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequality for k-Reimann-Liouville fractinal integral holds:*

$$\begin{aligned} &\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\leq \frac{\alpha\Gamma_k(\alpha)}{4(\wp_2-\wp_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\wp_1+,k}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-,k}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ &+\frac{\alpha}{4}\left[\frac{2k^2}{(k+\alpha)(2k+\alpha)}\mathbf{M}(\wp_1,\wp_2)+\frac{k(\alpha^2+\alpha k+2k^2)}{\alpha(k+\alpha)(2k+\alpha)}\mathbf{N}(\wp_1,\wp_2)\right], \end{aligned} \tag{16}$$

$\mathbf{M}(\wp_1, \wp_2)$ and $\mathbf{N}(\wp_1, \wp_2)$ are defined in Theorem 2.1.

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.2, we have our desired inequality (16). □

Theorem 2.3. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be real-valued, nonnegative and convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for generalized fractional integral holds:*

$$\begin{aligned} &\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+{}_{\wp_2-}I_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ &\leq \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \end{aligned} \tag{17}$$

Proof. Since $\mathcal{F}_1$, $\mathcal{F}_2$ are convex functions, then by using the definition of convex function we have

$$\begin{aligned} &2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &=2\mathcal{F}_1\left(\frac{x+y}{2}\right)\mathcal{F}_2\left(\frac{x+y}{2}\right)\leq \mathcal{F}_1(x)\mathcal{F}_2(x)+\mathcal{F}_1(y)\mathcal{F}_2(y), \end{aligned} \tag{18}$$

by setting $x = \wp_2 + \jmath(\wp_1 - \wp_2)$ and $y = \wp_1 + \jmath(\wp_2 - \wp_1)$ in18, we obtain

$$\begin{aligned} &2\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\leq \mathcal{F}_1(\wp_2+\jmath(\wp_1-\wp_2))\mathcal{F}_2(\wp_2+\jmath(\wp_1-\wp_2)) \\ &+\mathcal{F}_1(\wp_1+\jmath(\wp_2-\wp_1))\mathcal{F}_2(\wp_1+\jmath(\wp_2-\wp_1)). \end{aligned} \tag{19}$$

Now, multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of (19) and integrating resultant inequality with respect to '$\jmath$' over $[0, 1]$, we have

$$\begin{aligned} &\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\leq \frac{1}{2\Lambda(1)}\left[{}_{\wp_1+}I_{\varphi}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+{}_{\wp_2-}I_{\varphi}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]. \end{aligned} \tag{20}$$

Now, we have to prove right part of our inequality (19), according to given condition our $\mathcal{F}_1$, $\mathcal{F}_2$ are convex so we have

$$\begin{aligned} &\mathcal{F}_1(\wp_2 + \jmath(\wp_1 - \wp_2))\mathcal{F}_2(\wp_2 + \jmath(\wp_1 - \wp_2)) \\ &+ \mathcal{F}_1(\wp_1 + \jmath(\wp_2 - \wp_1))\mathcal{F}_2(\wp_1 + \jmath(\wp_2 - \wp_1)) \\ &\le \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2). \end{aligned} \tag{21}$$

Now, multiplying $\frac{\varphi((\wp_2-\wp_1)\jmath)}{\jmath}$ on both sides of (20) and integrating resultant inequality with respect to '$\jmath$' over $[0,1]$, we have

$$\begin{aligned} &[_{\wp_1+}I_\varphi \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) +_{\wp_2-} I_\varphi \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)] \\ &\le \mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2), \end{aligned} \tag{22}$$

by relating (20) and (22), we have our desired inequality (17). □

Remark 2.6. If we choose $\mathcal{F}_2(x) = 1$ for all $x \in [\wp_1, \wp_2]$ in Theorem 2.3, then we obtain the inequality (6).

Corollary 2.4. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following integral inequalities holds:*

$$\begin{aligned} \mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) &\le \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2} \mathcal{F}_1(x)\mathcal{F}_2(x)dx \\ &\le \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \end{aligned} \tag{23}$$

Proof. By using the idea $\varphi(\jmath) = \jmath$ in Theorem 2.3 we get our required inequalities (23). □

Corollary 2.5. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for Reimann-Liouville fractional integral holds:*

$$\begin{aligned} &\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right) \\ &\le \frac{\Gamma(\alpha+1)}{2(\wp_2-\wp_1)^\alpha}\left[I^\alpha_{\wp_1+}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2) + I^\alpha_{\wp_2-}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right] \\ &\le \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1) + \mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \end{aligned} \tag{24}$$

Proof. By using the idea $\varphi(\jmath) = \frac{\jmath^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3 we have our desired inequalities (24). □

Corollary 2.6. *Let $\mathcal{F}_1, \mathcal{F}_2 : [\wp_1, \wp_2] \to \mathbb{R}$ be a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then the following inequalities for k-Riemann-Liouville fractional* *integral holds:*

$$\begin{aligned}
&\mathcal{F}_1\left(\frac{\wp_1+\wp_2}{2}\right)\mathcal{F}_2\left(\frac{\wp_1+\wp_2}{2}\right)\\
&\le \frac{\alpha\Gamma_k(\alpha)}{2(\wp_2-\wp_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\wp_1+,k}\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)+I^{\alpha}_{\wp_2-,k}\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)\right]\\
&\le \frac{\mathcal{F}_1(\wp_1)\mathcal{F}_2(\wp_1)+\mathcal{F}_1(\wp_2)\mathcal{F}_2(\wp_2)}{2}. \qquad (25)
\end{aligned}$$

Proof. By using the idea $\varphi(\jmath)=\frac{\jmath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.3 we have our desired inequalities (25). □

Conclusion

In this paper, we derived inequalities of Hermite-Hadamard type for the product of two convex functions by using the generalized fractional integral. Interested reader can derive more inequalities of Hermite-Hadamard type with different approaches by using generalized fractional integral.

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