A new class of harmonic univalent functions associated with q-derivative defined by Hadamard product

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Abstract. In this paper, we introduce a class of harmonic univalent functions associated with q-derivative defined by Hadamard product. The object of the present paper is to determine coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class.
Keywords: harmonic univalent functions, q-derivative, extreme points.

1. Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [11]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.2}$$

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In [11] Clunie and Shell-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. Denote by V_H the subclass of S_H consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n, g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \qquad |b_1| < 1. \tag{1.3}$$

and $f(z)$ is the given by

$$f = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n}, \qquad |b_1| < 1. \tag{1.4}$$

For $0 < q < 1$, the q-derivative of a function $f \in$A is defined by (see [6], [7], [9], [16], [18], [25] and [26])

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & (z \neq 0) \\ f'(0), & (z = 0) \end{cases} \tag{1.5}$$

and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.5), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}, \tag{1.6}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad (0 < q < 1). \tag{1.7}$$

As $q \longrightarrow 1^-$, $[n]_q \longrightarrow n$. For a function $h(z) = z^n$, we obtain

$$D_q(h(z)) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q \, z^{n-1},$$

$$\lim_{q \longrightarrow 1^-} (D_q(h(z))) = \lim_{q \longrightarrow 1^-} \left([n]_q \, z^{n-1} \right) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative. For functions $f(z) \in V_H$ given by given by (1.4) and $F \in V_H$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} \overline{|B_n| z^n}, \qquad A_n, B_n \geq 0, \tag{1.8}$$

we recall the Hadamard product (or convolution) of f and F by

$$\begin{aligned} (f * F)(z) &= (h * H)(z) + \overline{(g * G)(z)} \\ &= z + \sum_{n=2}^{\infty} |a_n| \, |A_n| z^n + \sum_{n=1}^{\infty} \overline{|b_n| \, |B_n| z^n}. \end{aligned} \tag{1.9}$$

For $0 < q < 1, 1 < \gamma \leq 2, 0 \leq \lambda \leq 1, A_n, B_n \geq 0$ and for all $z \in U$, let $S_{H,q}(F,\lambda,\gamma)$ denote the family of harmonic functions $(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}$, where h and g are given by (1.2), H and G are given by (1.9) and satisfying the analytic criterion

$$\operatorname{Re}\left\{(1-\lambda)\frac{(f * F)(z)}{z} + \lambda D_q (f * F)(z)\right\} < \gamma. \tag{1.10}$$

Equivalently

$$\operatorname{Re}\left\{(1-\lambda)\frac{(h * H)(z) + \overline{(g * G)(z)}}{z} + \lambda\left[D_q(h * H)(z) + \overline{D_q (g * G)(z)}\right]\right\} < \gamma. \tag{1.11}$$

Let $\overline{S}_{H,q}(F,\lambda,\gamma)$ be the subclass of $S_{H,q}(F,\lambda,\gamma)$ consisting of functions $(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}$ given by (1.9).

We note that for suitable choices of F and λ we obtain the following subclasses:

(1) If we take $\lambda = 0$ and $F(z) = z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n + \sum_{n=1}^{\infty} \overline{\Psi_n(\alpha_1) z^n}$, where $\Psi_n(\alpha_1)$ is given by

$$\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \dots (\alpha r)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_s)_{n-1}\ (n-1)!} \tag{1.12}$$

($\alpha_i > 0, i = 1, \dots, r, \beta_j, j = 1, \dots, s; r \leq s+1; s, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}$, $z \in U$), the class $\lim_{q \longrightarrow 1^-} \overline{S}_H(z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n + \sum_{n=1}^{\infty} \overline{\Psi_n(\alpha_1) z^n}$, $0, \gamma)$, where $\Psi_n(\alpha_1)$ is given by (1.12), reduces to the class $\overline{S}_{H_{r,s}}([\alpha_1];\gamma)$ (see [8]).

(2) If we take $\lambda = 0$ and

$$F(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n}, (\eta \in \mathbb{N}_0, \mu \geq 0, l \geq 0),$$

the class $\lim_{q \longrightarrow 1^-} \overline{S_H}(z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left(\frac{1+l+\mu(n-1)}{l+1}\right)^{\eta} z^n}$, $\lambda, \gamma)$ the class reduces to reduces to the class $\overline{SI^{\eta}}(\mu, l;\gamma)$ (see [21]).

(3) If we take $\lambda = 0$ and

$$F(z) = z + \sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta} z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta} z^n}, (0 \leq \mu < 1, \eta \in \mathbb{N}_0),$$

the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}(z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta}z^n + (-1)^{\eta}\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^{\eta}z^n},\gamma)$$

reduces to the class $\overline{R}_H(\eta,\mu,\gamma)$ (see [24]).
(4) If we take $\lambda=1$ and $F(z)=\frac{z}{1-z}+\overline{\frac{z}{1-z}}$, the class $\lim_{q\longrightarrow 1^-}\overline{S}_H(\frac{z}{1-z}+\overline{\frac{z}{1-z}},1,\gamma)$ reduces to the class $\overline{S}_H(\gamma)$ (see [13]).
(5) If we take $F(z)=z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)z^n}$, where $\Upsilon_n^m(\alpha_1,l,\mu)$ is given by

$$\Upsilon_n^m(\alpha_1,l,\mu)=\left[\frac{1+l+\mu(n-1)}{l+1}\Psi_n(\alpha_1)\right]^m, m\in\mathbb{N}_0, \mu\geq 0, l\geq 0 \tag{1.13}$$

and $\Psi_n(\alpha_1)$ is given by (1.12), the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}\left(z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)z^n+\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)z^n},\lambda,\gamma\right)$$

reduces to the class $\overline{R}_{H_{r,s}}([\alpha_1],\eta,\mu,l,\gamma)=$

$$\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{D_{\mu,l}^{m,r,s}f(z)}{z}+\lambda\left(D_{\mu,l}^{m,r,s}f(z)\right)'\right)<\gamma,\right.$$
$$\left.\mu,l\geq 0; 0\leq\lambda\leq 1; r,s,m\in\mathbb{N}_0; 1<\gamma\leq 2; z\in U\right\},$$

where $D_{\mu,l}^{m,r,s}f(z)$ is the modified extended multiplier Dziok-Srivastava operator, defined as follows:

$$\begin{aligned} I(\eta,\mu,l)f(z) &= I^{\eta}(\mu,l)h(z)+I^{\eta}(\mu,l)\overline{g(z)} \\ &= z+\sum_{n=2}^{\infty}\Upsilon_n^m(\alpha_1,l,\mu)|a_n|z^n+\sum_{n=1}^{\infty}\overline{\Upsilon_n^m(\alpha_1,l,\mu)|b_n|z^n} \end{aligned}$$

and the operator $D_{\mu,l}^{m,r,s}$ was introduced and studied by El-Ashwah et al. (see [15]).
(6) If we take

$$F(z)=z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n},$$

$(0\leq\mu<1, m\in\mathbb{N}_0)$, the class $\lim_{q\longrightarrow 1^-}\overline{S_H}(z+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{\left[\frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}\right]^m z^n},\lambda,\gamma)$ reduces to the class $\overline{R}_H(m,\mu,\lambda,\gamma)=$

$$\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{\Omega^m f(z)}{z}+\lambda\left(\Omega^m f(z)\right)'\right)<\gamma,\right.$$
$$\left.0\leq\mu<1; 0\leq\lambda\leq 1; 1<\gamma\leq 2; m\in\mathbb{N}_0; z\in U\right\},$$

the operator Ω^m was introduced and studied by Dixit and Porwal (see [14]).
(7) If we take

$$F(z){=}z{+}\sum_{n=2}^{\infty}[1{+}\mu(n-1)]^{\eta}\,C\,(n,\delta)\,z^n{+}\,(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1{+}\mu(n-1)]^{\eta}\,C\,(n,\delta)\,z^n},$$

where $C\,(n,\delta)=\binom{n+\delta+1}{\delta}$, $\mu>0,\eta,\delta\in\mathbb{N}_0$, the class

$$\lim_{q\longrightarrow 1^-}\overline{S_H}\left(z+\sum_{n=2}^{\infty}[1+\mu(n-1)]^{\eta}\,C\,(n,\delta)\,z^n\right.$$
$$\left.+(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1+\mu(n-1)]^{\eta}\,C\,(n,\delta)\,z^n},\lambda,\gamma\right)$$

reduces to the class $\overline{S}_H(\mu,\eta,\delta,\lambda,\gamma)=$

$$\left\{f\in\overline{S_H}:\mathrm{Re}\left((1-\lambda)\,\frac{D\,(\eta,\delta,\mu)\,f(z)}{z}+\lambda\,(D\,(\eta,\delta,\mu)\,f(z))'\right)<\gamma\right.$$
$$\left.\mu>0;0\le\lambda\le 1;\eta,\delta\in\mathbb{N}_0;1<\gamma\le 2;z\in U\right\},$$

where the operator $D\,(\eta,\delta,\mu)\,f(z)$ is defined as follows (see [12]):

$$\begin{aligned}D\,(\eta,\delta,\mu)\,f(z)&=D^{\eta}_{\delta,\mu}h(z)+D^{\eta}_{\delta,\mu}\overline{g(z)}\\&=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]^{\eta}\,C\,(n,\delta)\,|a_n|\,z^n\\&\quad+(-1)^{\eta}\sum_{n=1}^{\infty}\overline{[1+\mu(n-1)]^{\eta}\,C\,(n,\delta)\,|b_n|\,z^n},\end{aligned}$$

the operator $D^{\eta}_{\delta,\mu}$ was introduced and studied by Al-Shaqsi and Darus (see [5]).

Also we note that:
(1) If we take $\lambda=1$, the class $\overline{S}_{H,q}(F,1,\gamma)$ reduces to the class $\overline{S}_{H,q}(F,\gamma)=\{f\in\overline{S_H}:\mathrm{Re}(D_q(f*F)(z))<\gamma,0<q<1,1<\gamma\le 2;z\in U\}$.
(2) If we take $F(z)=\frac{z}{1-z}+\overline{\frac{z}{1-z}}$, the class $\overline{S}_H(\frac{z}{1-z}+\overline{\frac{z}{1-z}},\lambda,\gamma)$ reduces to the class $\overline{S}_{H,q}(\lambda,\gamma)=\{f\in\overline{S_H}:\mathrm{Re}((1-\lambda)\dfrac{f(z)}{z}+\lambda D_qf(z))<\gamma,0<q<1,0\le\lambda\le 1;1<\gamma\le 2;z\in U\}$.
(3) If we take $F(z)=z+\sum_{n=2}^{\infty}\Gamma\,(\alpha_1,\beta_1,q,n)\,z^n+\sum_{n=1}^{\infty}\overline{\Gamma\,(\alpha_1,\beta_1,q,n)\,z^n}$, where $\Gamma\,(\alpha_1,\beta_1,q,n)$ is given by

$$\Gamma\,(\alpha_1,\beta_1,q,n)=\frac{(\alpha_1,q)_{n-1}\dots(\alpha_r,q)_{n-1}}{(q,q)_{n-1}\,(\beta_1,q)_{n-1}\dots(\beta_s,q)_{n-1}}\tag{1.14}$$

($\alpha_i>0,i=1,\dots,r,$ $\beta_j,j=1,\dots,s;$ $r=s+1;s,r\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,\dots\}$; $|q|<1,z\in U$), the class $\overline{S}_H(z+\sum_{n=2}^{\infty}\Gamma(\alpha_1,\beta_1,q,n)z^n+$

$\sum_{n=1}^{\infty}\overline{\Gamma(\alpha_1,\beta_1,q,n)z^n}, 0,\gamma)$, where $\Gamma(\alpha_1,\beta_1,q,n)$ is given by (1.14), reduces to the class

$$\begin{aligned}&\overline{R}_H(q,\alpha_1,\beta_1,\lambda,\gamma)\\&=\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{H_s^r[\alpha_1,\beta_1,q]f(z)}{z}+\lambda D_q\left(H_s^r[\alpha_1,\beta_1,q]f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1;\alpha_i,\beta_j>0;0\leq\lambda\leq1;1<\gamma\leq2;s,r\in\mathbb{N}_0;z\in U\right\},\end{aligned}$$

where $H_s^r[\alpha_1,\beta_1,q]f(z) = H_s^r(\alpha_1,\dots\alpha_r;\beta_1\dots\beta_s;q)f(z)$ and the operator $H_s^r[\alpha_1,\beta_1,q]$ was introduced and studied by Mohammed and Darus (see [20] and [1]), and $\lim_{q\longrightarrow1^-}H_s^r[\alpha_1,\beta_1,q]$ reduces to the operator $H_{q,s}(\alpha_1,\beta_1)$ is the modified Dziok-Srivastava operator (see [3] and [4]).

(4) If we take $F(z)=z+\sum_{n=2}^{\infty}\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n+\sum_{n=1}^{\infty}\overline{\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n}$ $(\eta>-1,)$, the class $\overline{S}_H(z+\sum_{n=2}^{\infty}\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n+\sum_{n=1}^{\infty}\overline{\frac{[n+\eta-1]_q!}{[\eta]_q![n-1]_q!}z^n},\lambda,\gamma)$ reduces to the class

$$\begin{aligned}\overline{S}_{H_q}(\eta,\lambda,\gamma)=&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{R_q^\eta f(z)}{z}+\lambda D_q\left(R_q^\eta f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1;\eta>-1;0\leq\lambda\leq1;1<\gamma\leq2;z\in U\right\},\end{aligned}$$

the operator R_q^η was introduced and studied by Aldweby and Darus (see [2]),

(5) If we take $F(z)=z+\sum_{n=2}^{\infty}[n]_q^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[n]_q^m z^n}$ $(0<q<1, m\in\mathbb{N}_0)$, the class $\overline{S_H}(z+\sum_{n=2}^{\infty}[n]_q^m z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[n]_q^m z^n},m,\lambda,\gamma)$ reduces to the class

$$\begin{aligned}\overline{R}_H(q,m,\lambda,\gamma)=&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{D_q^m f(z)}{z}+\lambda D_q\left(D_q^m f(z)\right)\right)<\gamma,\right.\\&\left.0<q<1,0\leq\lambda\leq1;1<\gamma\leq2;m\in\mathbb{N}_0;z\in U\right\},\end{aligned}$$

the operator $D_q^m f(z)$ was introduced and studied by Govindaraj and Sivasubramanian [17] and Murugusundaramoorthy and Vijaya [23] (see also Jahangiri [19]).

(6) If we take $F(z)=z+\sum_{n=2}^{\infty}[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n}$ $(0<q<1, l,\mu\geq0,\eta\in\mathbb{N}_0)$, the class $\overline{S_H}(z+\sum_{n=2}^{\infty}[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n+(-1)^m\sum_{n=1}^{\infty}\overline{[\frac{1+l+\mu([n]_q-1)}{l+1}]^\eta z^n},\mu,\eta,\lambda,\gamma)$ reduces to the class $\overline{R}_H(q,l,\mu,\eta,\lambda,\gamma)=$

$$\begin{aligned}&\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{I_q^\eta(\mu,l)f(z)}{z}+\lambda D_q\left(I_q^\eta(\mu,l)f(z)\right)\right)<\gamma\right.\\&\left.0<q<1;\eta\in\mathbb{N}_0;\mu,l\geq0;0\leq\lambda\leq1;1<\gamma\leq2;z\in U\right\},\end{aligned}$$

where the operator $I_q^\eta(\mu,l)$ was introduced and studied by Wongsaijai and Sukantamala (see [28]) and $\lim_{q\longrightarrow1^-}I_q^\eta(\mu,l)$ reduces to the operator $I(\eta,\mu,l)$, where $I(\eta,\mu,l)f(z)$ is the modified Catas operator (see [21]),

$$\begin{aligned} I(\eta,\mu,l)f(z) &= I^{\eta}(\mu,l)h(z)+\overline{I^{\eta}(\mu,l)g(z)} \\ &= z+\sum_{n=2}^{\infty}\left[\frac{1+l+\mu\left([n]_q-1\right)}{l+1}\right]^{\eta}|a_n|\,z^n \\ &\quad +(-1)^m\overline{\sum_{n=1}^{\infty}\left[\frac{1+l+\mu\left([n]_q-1\right)}{l+1}\right]^{\eta}|b_n|\,z^n}, \end{aligned}$$

where $I^{\eta}(\mu,l)$ is the extended multiplier transformation (see [10]).
(7) If we take $F(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n+(-1)^m\overline{\sum_{n=1}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n}$ $(0<q<1,\mu<2)$, the class

$$\overline{S_H}(z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n+(-1)^m\overline{\sum_{n=1}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\mu)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}z^n},\lambda,\gamma)$$

reduces to the class

$$\overline{R}_H(q,\mu,\lambda,\gamma)=\left\{f\in\overline{S_H}:\operatorname{Re}\left((1-\lambda)\frac{\Omega^{\mu}_{q,z}f(z)}{z}+\lambda D_q\left(\Omega^{\mu}_{q,z}f(z)\right)\right)<\gamma,\right.$$
$$\left.0<q<1;\mu<2;0\le\lambda\le 1;1<\gamma\le 2;z\in U\right\},$$

the operator $\Omega^m_{q,z}$ was introduced and studied by Murugusuurdaramoorthy et al. (see [22]) and $\lim_{q\longrightarrow 1^-}\Omega^m_{q,z}f(z)$ reduces to the operator $\Omega^m_z f(z)$, where the operator $\Omega^m_z f(z)$ was introduced and studied by Owa and Srivastava, (see [27]).

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $1<\gamma\le 2, 0\le\lambda\le 1, A_n, B_n\ge 0$ and $z\in U$.

Theorem 1. *Let $f=h+\overline{g}$ be such that $h(z)$ and $g(z)$ given by (1.2). Furthermore, let*

$$\sum_{n=2}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|A_n|\,|a_n|+\sum_{n=1}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|B_n|\,|b_n|\le\gamma-1. \tag{2.1}$$

where $a_1=1$ and $n(\gamma-1)\le[\lambda([n]_q-1)+1]|A_n|\le[\lambda([n]_q-1)+1]|B_n|$ for $n\ge 2$, *then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z)\in S_{H,q}(F,\lambda,\gamma)$.*

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{n=1}^{\infty} b_n\left(z_1^n-z_2^n\right)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n\left(z_1^n-z_2^n\right)}\right| \\ &> 1-\frac{\sum_{n=1}^{\infty} n\,|b_n|}{1-\sum_{n=2}^{\infty} n\,|a_n|} \\ &\geq 1-\frac{\sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1}\,|b_n|}{1-\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1}\,|a_n|} \geq 0,\end{aligned}$$

which proves univalence. Note that $f\ (z)$ is sense-preserving in U. This is because

$$\begin{aligned}\left|h'(z)\right| &\geq 1-\sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1} \\ &> 1-\sum_{n=2}^{\infty} n\,|a_n| \geq 1-\sum_{n=2}^{\infty} \frac{[\lambda(n-1)+1]|A_n|}{\gamma-1}\,|a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{[\lambda(n-1)+1]|B_n|}{\gamma-1}\,|b_n| \geq \sum_{n=1}^{\infty} n\,|b_n| \\ &> \sum_{n=1}^{\infty} n\,|b_n|\left|z^{n-1}\right| \geq \left|g'(z)\right|.\end{aligned}$$

Now, we will show that $f\ (z) \in S_H(F,\lambda,\gamma)$. We only need to show that if (2.1) holds then the condition (1.7) is satisfied. Using the fact that $Re\{w\} < \gamma$ if and only if $|w-1| < |w-(2\gamma-1)|$, it suffices to show that

$$\left|\frac{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q(h*H)(z)+\overline{D_q(g*G)(z)}\right]-1}{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q(h*H)(z)+\overline{D_q(g*G)(z)}\right]-(2\gamma-1)}\right| < 1.$$

We have

$$\begin{aligned}&\left|\frac{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda[D_q(h*H)(z)+\overline{D_q(g*G)(z)}]-1}{(1-\lambda)\frac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda[D_q(h*H)(z)+\overline{D_q(g*G)(z)}]-(2\gamma-1)}\right| \\ &= \left|\frac{\sum_{n=2}^{\infty}[\lambda(n-1)+1]A_n a_n z^{n-1}+\sum_{n=1}^{\infty}[\lambda(n-1)+1]B_n\overline{b_n z^{n-1}}}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda(n-1)+1]A_n a_n z^{n-1}+\sum_{n=1}^{\infty}[\lambda(n-1)+1]B_n\overline{b_n z^{n-1}}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z|^{n-1}+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n||z|^{n-1}}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z|^{n-1}-\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n||z|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|}{2(\gamma-1)-\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|-\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|},\end{aligned}$$

which is bounded above by 1 by using (2.1). This completes the proof of Theorem 1. The harmonic univalent functions of the form

$$f(z) = z+\sum_{n=2}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}x_n z^n+\sum_{n=1}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}\overline{y_n z^n}, \tag{2.2}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class $S_{H,q}(F,\lambda,\gamma)$ for all $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| \le 1$ because coefficient inequality (2.1) holds.

Theorem 2. *A function* $f(z)$ *of the form* (1.3) *is in the class* $\overline{S}_{H,q}(F,\lambda,\gamma)$ *if and only if*

$$\text{(2.3)} \quad \sum_{n=2}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|A_n|\,|a_n|+\sum_{n=1}^{\infty}\left[\lambda\left([n]_q-1\right)+1\right]|B_n|\,|b_n|\le\gamma-1.$$

where $a_1 = 1$ and $n(\gamma-1) \le [\lambda([n]_q-1)+1]|A_n| \le [\lambda([n]_q-1)+1]|B_n|$ for $n \ge 2$.

Proof. Since $\overline{S}_{H,q}(F,\lambda,\gamma) \subset S_{H,q}(F,\lambda,\gamma)$, we only need to prove the "only if" part of this theorem. To this end , for functions $f(z)$ of the form (1.3), we notice that the condition

$$\operatorname{Re}\left\{(1-\lambda)\,\tfrac{(h*H)(z)+\overline{(g*G)(z)}}{z}+\lambda\left[D_q\left(h*H\right)(z)+D_q\overline{(g*G)\,(z)}\right]\right\}<\gamma,$$

is equivalent to

$$\operatorname{Re}\left\{1+\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n|z^{n-1}+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n|\overline{z^{n-1}}\right\}$$
$$\le 1+\sum_{n=2}^{\infty}[\lambda([n]_q-1)+1]|A_n||a_n||z^{n-1}|+\sum_{n=1}^{\infty}[\lambda([n]_q-1)+1]|B_n||b_n||\overline{z^{n-1}}|<\gamma.$$

Letting $z \to 1^-$ along the real axis, we obtain the inequality (2.3). This completes the proof of Theorem 2.

3. Distortion theorem

Theorem 3. *Let the function* $f(z)$ *defined by* (1.3) *belongs to the class* $\overline{S}_{H,q}(F,\lambda,\gamma)$ and $(\lambda q+1)|A_2| \le [\lambda([n]_q-1)+1]|A_n| \le [\lambda([n]_q-1)+1]|B_n|$ for $n \ge 2$. *Then for* $|z| = r < 1$, *we have*

$$\text{(3.1)} \quad \begin{aligned}&(1-|b_1|)\,r-\left(\tfrac{\gamma-1}{(\lambda q+1)|A_2|}-\tfrac{|B_1|}{(\lambda q+1)|A_2|}\,|b_1|\right)r^2\\ &\le|f(z)|\le(1+|b_1|)\,r+\left(\tfrac{\gamma-1}{(\lambda q+1)|A_2|}-\tfrac{|B_1|}{(\lambda q+1)|A_2|}\,|b_1|\right)r^2\end{aligned}$$

for $|b_1| \le \frac{\gamma-1}{|B_1|}$.*The results are sharp with equality for the functions* $f(z)$ *defined by*

$$\text{(3.2)} \quad f(z)=z+|b_1|\,\overline{z}+\left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|}-\frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right)\overline{z}^2$$

and

$$f(z)=z-|b_1|\,\overline{z}-\left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|}-\frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right)z^2. \tag{3.3}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Since

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_nz^n},$$

then

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)r^n\le(1+|b_1|)r+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)r^2\\
&=(1+|b_1|)r+\frac{\gamma-1}{(\lambda q+1)\,|A_2|}\sum_{n=2}^{\infty}\left(\frac{(\lambda q+1)\,|A_2|}{\gamma-1}\,|a_n|+\frac{(\lambda q+1)\,|A_2|}{\gamma-1}\,|b_n|\right)r^2\\
&\le(1+|b_1|)r+\tfrac{\gamma-1}{(\lambda q+1)|A_2|}\sum_{n=2}^{\infty}\left(\tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1}\,|a_n|+\tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1}\,|b_n|\right)r^2\\
&\le(1+|b_1|)r+\frac{\gamma-1}{(\lambda q+1)\,|A_2|}\left(1-\frac{|B_1|}{\gamma-1}\,|b_1|\right)r^2\\
&=(1+|b_1|)r+\left(\frac{\gamma-1}{(\lambda q+1)\,|A_2|}-\frac{|B_1|}{(\lambda q+1)\,|A_2|}\,|b_1|\right)r^2.
\end{aligned}$$

The functions $f(z)$ given by (3.2) and (3.3), respectively, for $|b_1|\le\frac{\gamma-1}{|B_1|}$ show that the bounds given in Theorem 3 are sharp.

4. Extreme points

Theorem 4. *Let $f(z)$ be given by* (1.3)*. Then $f(z)\in\overline{S}_{H,q}(F,\lambda,\gamma)$ if and only if*

$$f(z)=\sum_{n=1}^{\infty}\left(\mu_nh_n(z)+\eta_ng_n(z)\right), \tag{4.1}$$

where $h_1(z)=z$,

$$h_n(z)=z+\frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}z^n,\qquad(n=2,3,\ldots) \tag{4.2}$$

and

$$g_n(z)=z+\frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}\overline{z^n},\quad(n=1,2,\ldots), \tag{4.3}$$

$\mu_n \geq 0, \eta_n \geq 0, \sum_{n=1}^{\infty} (\mu_n + \eta_n) = 1$. *In particular, the extreme points of the class* $\overline{S}_H(F, \lambda, \gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$, *respectively.*

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (\mu_n h_n(z) + \eta_n g_n(z)) \\ &= z + \sum_{n=2}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}\mu_n z^n + \sum_{n=1}^{\infty} \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}\eta_n \overline{z^n}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1}\left(\frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}\mu_n\right) \\ &+\sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1}\left(\frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}\eta_n\right) = \sum_{n=2}^{\infty}\mu_n + \sum_{n=1}^{\infty}\eta_n = 1-\mu_1 \leq 1 \end{aligned}$$

and so $f(z) \in \overline{S}_{H,q}(F, \lambda\gamma)$.

Conversely, if $f(z) \in \overline{S}_{H,q}(F, \lambda, \gamma)$, then

$$|a_n| \leq \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}, \ (n \geq 2)$$

and

$$|b_n| \leq \frac{\gamma-1}{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}, \ (n \geq 1).$$

Setting

$$\mu_n = \frac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1}|a_n|, \quad (n = 2, 3, \ldots)$$

and

$$\eta_n = \frac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1}|b_n|, \quad (n = 1, 2, \ldots).$$

Since $0 \leq \mu_n \leq 1$ $(n = 2, 3, \ldots)$ and $0 \leq \eta_n \leq 1$ $(n = 1, 2, \ldots)$, $\mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n + \sum_{n=1}^{\infty}\eta_n \geq 0$, then, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of the Theorem 4.

Now, we show that the class $\overline{S}_{H,q}(F, \lambda, \gamma)$ is closed under convex combinations of its members.

Theorem 5. *The class* $\overline{S}_{H,q}(F, \lambda, \gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{S}_{H,q}(F, \lambda, \gamma)$, where f_i is given by

$$f_i = z + \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z^n}.$$

Then, by using Theorem 2, we have

$$\sum_{n=2}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right] |A_n|}{\gamma - 1} |a_{n_i}| z^n + \sum_{n=1}^{\infty} \frac{\left[\lambda\left([n]_q - 1\right) + 1\right] |B_n|}{\gamma - 1} |b_{n_i}| \overline{z}^n \leq 1. \tag{4.4}$$

For $\sum_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \overline{z^n}. \tag{4.5}$$

Then by (4.4), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) + \sum_{n=1}^{\infty} \tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|A_n|}{\gamma-1} |a_{n_i}| + \sum_{n=1}^{\infty} \tfrac{\left[\lambda\left([n]_q-1\right)+1\right]|B_n|}{\gamma-1} |b_{n_i}|\right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.2) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{S}_{H,q}(F, \lambda, \gamma)$. This completes the proof of the Theorem 5.

Remark 2. Specializing F, in the above results, we obtain the corresponding results for the corresponding classes $\overline{S}_{H,q}(F,\gamma), \overline{S}_{H,q}(\lambda,\gamma), \overline{R}_{H_{r,s}}(q,\alpha_1,\beta_1,\lambda,\gamma)$, $\overline{S}_{Hr}(\eta,\lambda,\gamma)$ $\overline{R}_H(q,m,\lambda,\gamma), \overline{R}_H(q,l,\mu,\eta,\lambda,\gamma)$ and $\overline{R_H}(q,\mu,\lambda,\gamma)$ defined in the introduction.

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