1-factorization of small regular graphs

Xiaoxia Zhang*
Department of Mathematics
Xinyang Normal University
Xinyang, 464000
China
xiaoxiazhang_1983@163.com

Jingen Yang
Department of Mathematics
Xinyang Normal University
Xinyang 464000
China

Mingfang Huang
Department of Mathematics
School of Science
Wuhan University of Technology
Wuhan, 430000
China

Abstract. It is a well-known conjecture that if a regular simple graph G of order $2n$ has degree $\Delta(G)$ satisfying $\Delta(G) \geq n$, then G is 1-factorizable. By the colour exchange theory, Cariolaro [J. London Math. Soc., 77 (2007), 387-404] proved the validity of this conjecture for regular graphs of even order at most 10. In this paper, we shall provide a slightly simple proof of this result.

Keywords: 1-factorization, regular graph.

1. Introduction

Graphs considered in this paper are finite simple graphs. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and the maximum degree of G, respectively. A graph G is *regular* if $\delta(G) = \Delta(G)$. A cycle is a connected 2-regular graph. A n-cycle is a cycle on n vertices, denoted by C_n. $K^+_{m,n-m}$ denotes the simple graph obtained from the complete bipartite graph $K_{m,n-m}$ by adding an edge between two vertices of degree $n-m$. If $E_1 \subset E(G)$, by $G - E_1$ we shall denote the graph obtained from G by deleting all the edges in E_1. The operation $+$ used in this paper to express the disjoint union of two graphs and $E_G[H_1 + H_2]$ is used to denote the edge cut of $V(H_1)$ and $V(H_2)$ in G, where $H_1 + H_2$ is a spanning subgraph of G. More terminologies and notations not defined here are all from [1].

∗. Corresponding author

A *k-factor* of G is a k-regular spanning subgraph of G, and a *k-factorization* of G is a set F of edge-disjoint k-factors $F_1, F_2, \cdots, F_n$ such that $G = F_1 \cup F_2 \cup \cdots \cup F_n$ and in which each k-factor F_i is called a *decomposition factor* of G, where $1 \leq i \leq n$. G is said to be *k-factorizable* if it has a k-factorization. In fact, a *1-factor* of G is just a perfect matching of G. Many colouring problems ask for a decomposition of a regular graph of even order into disjoint 1-factors. We call such a decomposition a 1-factorization. A well-known, longstanding conjecture (which first appeared in a paper by Chetwynd and Hilton [2], but which may go back to G.A. Dirac in the early 1950s) is as follows.

Conjecture 1.1. (1-*Factorization Conjecture*) *A regular graph G of order $2n$ satisfying $\Delta(G) \geq n$ is 1-factorizable.*

This conjecture considered very hard, once solved would have an important impact on graph theory as well as other branches of mathematics. Many experts and scholars have devoted to this study, yet the vast majority of works existing in the literature have concentrated on proving the truth of Conjecture 1.1 for nearly complete graphs [2, 5, 11, 12] and for graphs of relatively large degree [2, 3, 6, 9]. It is worth mentioning that there was a breakthrough in 1-factorization by Perkovic and Reed [10], who proved (by probabilistic methods) that Conjecture 1.1 is asymptotically true and an approximate version of which was recently put forward by Csaba et.al [4]. These results provide some evidence in support of Conjecture 1.1. However, it occurs to us that all the known methods for proving the existence of a 1- factorization are very hard to follow.

Deciding whether a given graph is 1-factorizable is a intractable problem in general, which is known to be NP-complete [8]. The difficulty is how to determine the decomposition form of 1- factorization. An obvious, but useful, fact is that if we have a collection of edge-disjoint 1-factors of G, say $\{F_1 \cup \cdots \cup F_k\}$, then G is 1-factorizable if $G-(F_1 \cup \cdots \cup F_k)$ is 1-factorizable, where $1 \leq k < \Delta(G)$. In many cases, however, even if $G-(F_1 \cup \cdots \cup F_k)$ is not 1-factorizable, G is also 1-factorizable. This suggests that the pre-existing collection $\{F_1 \cup \cdots \cup F_k\}$ must contain some 1-factor F_j which is not a decomposition factor of G, where $1 \leq j \leq k$. We take L_3 (see Figure 1) for example. Three vertical bars constitute a 1-factor (denoted by F_1), but not a decomposition factor of L_3, that's because $L_3 - F_1 = C_3 + C_3$ is not 1-factorizable. Motivated by this, we investigated 1-factorization of small regular graphs and proved Conjecture 1.1 holds for regular graphs of even order at most 10, as follows.

Theorem 1.2. *Let G be a regular graph of order $2n$ satisfying $\Delta(G) \geq n$, where $n \leq 5$. Then G is 1-factorizable.*

This result has been proved by Cariolaro [7] who present a new theory on colour exchange and apply it to the class of regular graphs of even order at most 10. In this paper, we are aim to provide a slightly simple proof.

2. Preliminaries

This section we shall establish three useful theorems which are all from the book by Bondy and Murty [1]. The first one is a well-known theorem of Dirac.

Theorem 2.1. *Let G be a graph with $|V(G)| \geq 3$ and $\delta(G) \geq |V(G)|/2$. Then G is a hamiltonian graph.*

It is known that every k-regular bipartite graph with $k > 0$ has a perfect matching. In fact, we can further get the following stronger conclusion which is an exercise of chapter 5 in the book by Bondy and Murty [1].

Theorem 2.2. *For $k > 0$, each of the following holds.*
(1) *every k-regular bipartite graph is 1-factorizable.*
(2) *every $2k$-regular graph is 2-factorizable.*

Proof. (1) Let G be a k-regular bipartite graph. Then G has a perfect matching, say M_1. We consider the graph $G - M_1$ which is a $(k-1)$-regular bipartite graph. Let M_2 be a perfect matching of $G - M_1$. Next we consider the graph $G - M_1 - M_2$. And so it goes on, we get finally k edge-disjoint 1-factors $M_1, M_2, \cdots, M_k$ and $G = M_1 \cup M_2 \cup \cdots \cup M_k$. So G is 1-factorizable.

(2) Let G be a $2k$-regular graph and $V(G) = \{v_1, v_2, \cdots, v_v\}$. Then G admits an Euler tour C. Next we construct a simple bipartite graph G^* from C with bipartition (X, Y). Let $X = \{x_1, x_2, \cdots, x_v\}$ and $Y = \{y_1, y_2, \cdots, y_v\}$. For $i, j \in \{1, 2, \cdots, v\}$, $x_i y_j \in E(G^*)$ if and only if $v_i v_j \in E(C)$ and $v_i v_j$ is oriented from v_i to v_j. For any $u \in V(C)$, let $E^+(u)$ and $E^-(u)$ denote the edges with tail and head at u, respectively. Then $|E^+(u)| = |E^-(u)|$ since C is an Euler tour. Thus G^* is a k-regular bipartite graph. By (1), G^* is 1-factorizable. Note that each 1-factor of G^* is corresponding to a 2-factor of G. Thus G is 2-factorizable. □

Theorem 2.2 shows that every 2-regular graph without odd cycles is 1-factorizable and every 4-regular graph is 2-factorizable. For a 3-regular graph, there is an important result as follows.

Theorem 2.3. *Every 3-regular graph without cut edges has a perfect matching.*

3. Proof of Theorem 1.2

Let G be a regular graph of order $2n$ satisfying $\Delta(G) \geq n$, where $n \leq 5$. First of all, we make a significant claim.

Claim 1 G is a $\Delta(G)$-edge connected regular graph.

Proof of Claim 1. Let X be a nontrivial edge cut of G. We shall prove that $|X| \geq \Delta(G)$. Let G_1, G_2 be two nontrivial components of $G - X$ and $|V(G_1)| \leq |V(G_2)|$. Then $|V(G_1)| \leq n$ and $|X| = \Delta(G)|V(G_1)| - 2|E(G_1)|$. It follows that $|X| \geq \Delta(G)|V(G_1)| - |V(G_1)|(|V(G_1)| - 1)$. If $|X| < \Delta(G)$, then

$\Delta(G)|V(G_1)| - |V(G_1)|(|V(G_1)| - 1) < \Delta(G)$. It follows that $\Delta(G)(|V(G_1)| - 1) < |V(G_1)|(|V(G_1)| - 1)$. Note that $|V(G_1)| > 1$. Thus we get that $|V(G_1)| > \Delta(G) \geq n$, a contradiction. So $|X| \geq \Delta(G)$. This shows that G is a $\Delta(G)$-edge connected regular graph.

Next we shall handle each case of $n \leq 5$, respectively. It is trivil for $n = 1$. When $n \geq 2$, by Theorem 2.1, G is a hamiltonian graph. By the definition, G contains a hamiltonian cycle of order $2n$ as a spanning subgraph. So G must have two edge-disjoint perfect matchings. Let M be a perfect matching of G. If $\Delta(G - M) \geq n$ and $G - M$ is 1-factorizable, then G is also 1-factorizable. Hence it is sufficient to prove that G is 1-factorizable if G is a regular graph of order $2n$ satisfying $\Delta(G) = n$.

For $n = 2$, $|V(G)| = 4$ and $\Delta(G) = 2$. Then G is C_4 which is 1-factorizable.

For $n = 3$, $|V(G)| = 6$ and $\Delta(G) = 3$. If G is a 3-regular bipartite graph, by Theorem 2.2(1), G is 1-factorizable. Next we suppose that G contains some odd cycles. By Claim 1, G has no cut edges. Hence it follows that G has a perfect matching by Theorem 2.3. Let M be a perfect matching of G. Then $G - M = C_3 + C_3$. Thus G is isomorphic to the graph L_3 as shown in Figure 1. For the sake of narrative convenience, we use bold lines to mark a perfect matching of L_3 (the same below). It is easy to find that deleting the perfect matching we can get an even cycle C_6 which is 1-factorizable. Hence G is 1-factorizable.

L_3 $\qquad$ H_1

Figure 1. Two graphs with odd cycles

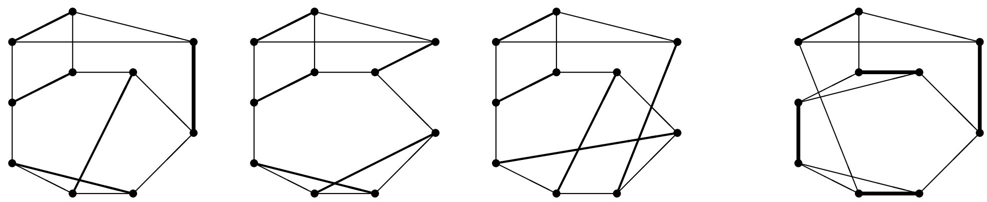
For $n = 4$, $|V(G)| = 8$ and $\Delta(G) = 4$. Let M_1 and M_2 be two perfect matchings of G and let $H = G - M_1 - M_2$. Then H is a 2-regular graph. If H is 1-factorizable, then G is also 1-factorizable, we are done. Next we suppose that H is not 1-factorizable. Then H is $C_3 + C_5$. This indicates that one of two perfect matchings, say M_1, is not a decomposition factor of G. Let $H_1 = H \cup M_1 = G - M_2$. Then H_1 is a 3-regular graph as shown in Figure 1. As above, it is easy to prove that H_1 is 1-factorizable. It follows that G is also 1-factorizable.

For $n = 5$, $|V(G)| = 10$ and $\Delta(G) = 5$. Let $M_1^{'}$ and $M_2^{'}$ be two perfect matchings of G and let $H^{'} = G - M_1^{'} - M_2^{'}$. Then $H^{'}$ is a 3-regular graph. If $H^{'}$ is 1-factorizable, then G is also 1-factorizable, we are done. Next we suppose that $H^{'}$ is not 1-factorizable.

Claim 2. $H^{'}$ has a perfect matching.

Proof of Claim 2. If $H^{'}$ has no cut edges, then by Theorem 2.3, $H^{'}$ has a perfect matching. Next we suppose that $H^{'}$ contains a cut edge e. Let R_1 and R_2 be two components of $H^{'} - e$. Note that $H^{'}$ is a 3-regular graph, there must be $|V(R_1)| = |V(R_2)| = 5$ and $R_1 \cong R_2 \cong K_{2,3}^{+}$. Let $e = uv$ and $u \in V(R_1)$, $v \in V(R_2)$. Note that $R_1 - u$ has a perfect matching $M_1^{''}$ and $R_2 - v$ has a perfect matching $M_2^{''}$. So $M_1^{''} \cup \{uv\} \cup M_2^{''}$ is a perfect matching of $H^{'}$.

Let $M_3^{'}$ be a perfect matching of $H^{'}$ and let $H^{''} = H^{'} - M_3^{'}$. Then $H^{''}$ is a 2-regular graph and contains just two odd cycles. For otherwise, $H^{'}$ is 1-factorizable, a contradiction. Thereby $H^{''}$ maybe $C_3 + C_7$, $C_3 + C_3 + C_4$ or $C_5 + C_5$.



(a) (b) (c) (d)

Figure 2. Four types of $H^{'}$ with $H^{''} = C_3 + C_7$

Claim 3 $H^{''} = C_5 + C_5$, and $H^{'}$ has one cut edge or $H^{'}$ is a Peterson graph (denoted by P_{10}).

Proof of Claim 3. If $H^{''} = C_3 + C_7$, then $H^{'}$ is isomorphic to one of the four graphs depicted in Figure 2. If $H^{''} = C_3 + C_3 + C_4$, then $H^{'}$ is isomorphic to one of the three graphs depicted in Figure 3. In both cases, we can prove that $H^{'}$ is 1-factorizable the same as above mentioned, a contradiction. So $H^{''} = C_5 + C_5$. If $H^{'}$ has no cut edge, then $|E_{H^{'}}[C_5 + C_5]| = 3$ or 5. We list all possible cases of $H^{'}$ in Figure 4. Similarly, we can check that each graph in Figure 4 except for the Peterson graph is 1-factorizable. Hence we assume that $H^{'}$ has one cut edge or $H^{'}$ is a Peterson graph.

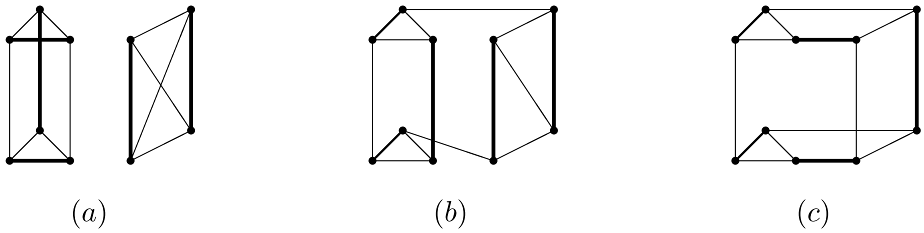


Figure 3. Three types of $H^{'}$ with $H^{''} = C_3 + C_3 + C_4$

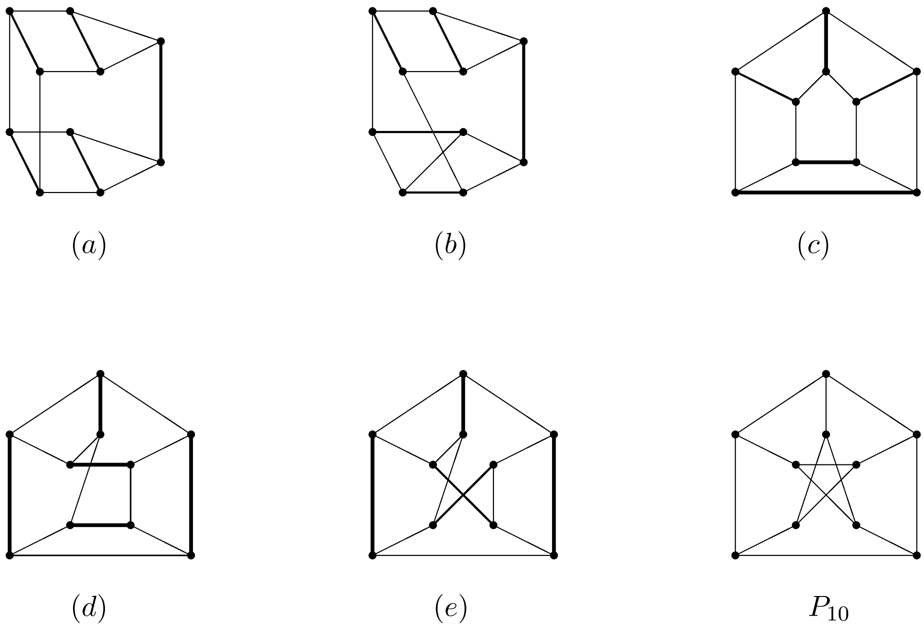


Figure 4. Six types of $H^{'}$ with $H^{''} = C_5 + C_5$

Finally, we consider the graph $H_i^{'} = H^{''} \cup M_i^{'}$ which is also a 3-regular graph, where $i \in \{1, 2\}$. Note that $G = H_1^{'} \cup M_2^{'} \cup M_3^{'}$ or $G = H_2^{'} \cup M_1^{'} \cup M_3^{'}$. If $H_1^{'}$ or $H_2^{'}$ is 1-factorizable, we are done. Therefore we suppose that both $H_1^{'}$ and $H_2^{'}$ are not 1-factorizable. As discussed in Claim 3, $H_i^{'}$ has one cut edge or $H_i^{'}$ is a Peterson graph. However, it is impossible that each of the three graphs $H^{'}$, $H_1^{'}$ and $H_2^{'}$ has one cut edge, that's because $|E_G[C_5 + C_5]| = 3$, contrary to Claim 1. We assume, without loss of generality, that $H^{'}$ is a Peterson graph and each of

$H_1^{'}$ and $H_2^{'}$ has one cut edge. Let $G^{'} = H^{'} \cup M_1^{'}$. Then $G^{'}$ is a 4-regular graph which is 2-factorizable follows from Theorem 2.2(2). Figure 5 illustrates $G^{'}$ (the bold lines form into $M_1^{'}$) and its two edge-disjoint 2-factors F_1 and F_2. It is clear that both F_1 and F_2 are 1-factorizable. It follows that $G^{'}$ is 1-factorizable. Note that $G = G^{'} \cup M_2^{'}$. It is obvious that G is also 1-factorizable. This completes our proof.

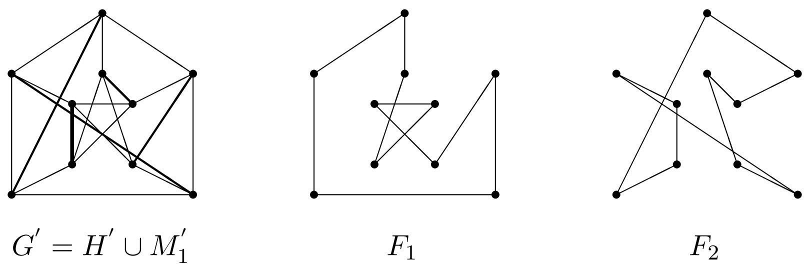


Figure 5. Graph $G^{'}$ and its two edge-disjoint 2-factors

Acknowledgement

The research of the first author is supported by the Natural Science Foundation of China (Grant No.11701496) and Nanhu Scholars Program for Young Scholars of XYNU. The Third author is supported by the Fundamental Research Funds for the Central Universities (WUT: 2020IB010) and in part by the NSFC (11861069).

References

[1] J.A. Bondy, U.S. R. Murty, *Graph theory with application,* North-Holland, New York, 1976.

[2] A.G. Chetwynd, A.J.W. Hilton, *Regular graphs of high degree are 1-factorizable*, Proc. London Math. Soc., 50 (1985), 193-206.

[3] A.G. Chetwynd, A.J.W. Hilton, *1-factorizing regular graphs of high degree-an improved bound*, Discrete Math., 75 (1989), 103-112.

[4] B. Csaba, D. Kühn, A. Lo, D. Osthus, A. Treglown, *Proof of the 1-factorization and Hamilton decomposition conjectures*, Scuola Normale Superiore, 16 (2014), 477-482.

[5] A.G. Chetwynd, A.J.W. Hilton, *The edge-chromatic class of regular graphs of degree 4 and their complements*, Discrete Appl. Math., 16 (1987), 125-134.

[6] D. Cariolaro, A.J.W. Hilton, *An application of tuttes theorem to 1-factorization of regular graphs of high degree*, Discrete Mathematics, 309 (2009), 4736-4745.

[7] D. Cariolaro, *1-factorization of regular graphs by colour exchange*, J. London Math. Soc., 77 (2007), 387-404.

[8] I. Holyer, *The NP-completeness of edge-colouring*, SIAM J. Comput., 10 (1981), 718-720.

[9] T. Niessen, L. Volkmann, *Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree*, J. Graph Theory, 14 (1990), 225-246.

[10] L. Perković, B. Reed, *Edge coloring regular graphs of high degree*, Discrete Math., 165/166 (1997), 567-578.

[11] Z. Song, *Chromatic index critical graphs of odd order with five major vertices*, J. Comb. Math. Comb. Comput., 41 (2002), 161-186.

[12] Z. Song, H.P. Yap, *Chromatic index critical graphs of even order with five major vertices*, Graphs and Combin., 21 (2005), 239-246.

Accepted: 21.06.2019