On the condition number of integral equations in the elastic two-dimensional case using the cross multipole coefficients

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Abstract. The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified Green's function technique. In this work, we generalize a new criterion of optimality of the perturbed fundamental solution based on the minimization of the condition number of the modified integral equations using the cross multipole coefficients.
Keywords: cross multipole coefficients, modified Green's function, integral equations, linear elasticity, condition number.

1. Introduction

The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified Green's function technique, where the simple and cross multipole coefficients must satisfy some suitable and mild conditions [3]. Some criteria to determine an optimal choice for these multipole coefficients are developed recently. The first criterion is based on the minimization of the norm of the modified integral operator [2], [5], [11], [13], and [5], and motivated by enlarging the radius of convergence of the used numerical method (successive approximations). The second criterion is based on the minimization of the norm of the modified Green's function [9], and motivated by the minimization of the norm of the difference between the modified and exact Green's function. In [1], Argyropoulos et al. have presented another criterion based on the minimization of

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the condition number of the boundary integral equations describing the problem. In [8], we have developed a new criterion based on the minimization of the norm of the surface stress operator, or the norm of the modified traction operator using simple multipole coefficients, motivated by the minimization of the norm of the difference between the modified and exact kernel of the integral operator. In this work we generalize the same criterion developed by Argyropoulos et al. In [1], for the case of cross multipole coefficients. This paper is organized as follows, in the next section, we present the mathematical formulation of the exterior Neumann boundary value problem using the modified Green's function technique, in section 3 we present our main results which are the determination of the optimal choice of the simple and cross multipole coefficients minimizing the condition number of the modified boundary integral equations describing the problem.

2. Formulation of the problem

An exterior Neumann boundary value problem in two-dimensional elastic case can be described through a boundary integral equation of the form [3]:

$$\left(\frac{1}{2}I+\bar{K}_0^*\right)(\varphi)(p)=f(p)\,,\ p\in\partial D, \tag{2.1}$$

where f is a Hölder continuous density, and the integral operator K_0 is defined as :

$$(K_0\varphi)(p)=\frac{1}{2\pi}\int\limits_{\partial D}T_pG_0(p,\ q)\ \varphi(q)\ ds_q,\ p\in\partial D, \tag{2.2}$$

G_0 is the Green's function (fundamental solution), and T is the surface stress operator.

Using the modified Green's function technique, by introduce a regular solution [3], our boundary value problem can be described through a modified boundary integral equation of the form::

$$\left(\frac{1}{2}I+\bar{K}_1^*\right)(\varphi)(p)=f(p)\,,\ p\in\partial D, \tag{2.3}$$

where the modified integral operator K_1 is defined as:

$$(K_1\varphi)(p)=\frac{1}{2\pi}\int_{\partial D}T_pG_1(p,\ q)\ \varphi(q)\ ds_q,\ p\in\partial D, \tag{2.4}$$

and the modified Green's function is written as:

$$G_1(p,\ Q)=\frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2}\left[F_m^{\sigma l}(P)\otimes\hat{F}_m^{\sigma l}(Q)+a_m^{\sigma l}\ F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q)\right.$$

$$\left.+(-1)^{\sigma+l}b_m\ F_m^{\sigma l}(P)\otimes F_m^{(3-\sigma)(3-l)}(Q)\right], \tag{2.5}$$

where

$$F_m^{\sigma 1}(P) = \text{grad}\left(H_m^1(kr_p)\ E_m^\sigma(\theta_p)\right), \quad (2.6)$$
$$F_m^{\sigma 2}(P) = rot\left(H_m^1(Kr_p)\ E_m^\sigma(\theta_p)\ \hat{e}_3\right).$$

$\hat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [3], and

$$E_m^\sigma(\theta_p) = \sqrt{\varepsilon_m}\begin{cases}\cos(m\theta_p), & \sigma = 1\\ \sin(m\theta_p), & \sigma = 2\end{cases}, \text{ with } \varepsilon_m = \begin{cases}1, & m = 0,\\ 2, & m > 0,\end{cases}$$

$a_m^{\sigma l}$ and b_m are the simple and cross multipole coefficients, which must satisfy the following conditions:

$$\bar{b}_m\left(a_m^{\sigma 1} + \frac{1}{2}\right) + b_m\left(\bar{a}_m^{\sigma 2} + \frac{1}{2}\right) = 0, \quad (2.7)$$

and

$$\left|a_m^{\sigma l} + \frac{1}{2}\right|^2 + |b_m|^2 - \frac{1}{4} < 0,\ \forall m = 0:\infty \text{ and } \forall\sigma,\ l = 1,\ 2.$$

3. Main results

3.1 Case of the circle

As shown in [4], it is externally to get explicit results for the multipole coefficients which minimize the operator norms for arbitrary boundaries ∂D. Similar discussions are given in [1] for the condition number of integral equations in the elastic threedimensional case. Nevertheless, the special result for minimizing the condition number when ∂D is a circle serves as a guide to an explicit coefficient choice which leads to well conditioned integral equations for perturbations of circular domains. So we examine first the circular case, by considering the question of how to choose the simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m in the modified Green's function given by (2.5), minimizing the condition number of the modified boundary integral equations describing the problem (2.3). We introduce the operator:

$$M : L_2(\partial D) \longrightarrow L_2(\partial D) \quad / \ M = \frac{1}{2}I + \overline{K}_1^*, \quad (3.1)$$

and its L_2 adjoint M^*. It is well known that, the condition number which is given by the relation [4]:

$$Cond(M) = \|M\| \,.\, \left\|M^{-1}\right\|$$

with respect to the L_2 norm can be expressed as:

$$Cond(M) = \sqrt{\frac{\lambda_{\max}^M}{\lambda_{\min}^M}}$$

where $\lambda_{\max}^M$ and $\lambda_{\min}^M$ denote respectively, the largest and the smallest spectral value of the self-adjoint operator M^*M.

Theorem 1. *There are uniquely defined simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m which minimize the condition number of the modified boundary integral equation* (2.3)*. These simple and cross multipole coefficients are given by the relations* (3.2) *and* (3.3)*:*

$$a_m^{\sigma l} \cdot \frac{i}{4\mu K^2} = \frac{\overline{\beta}_m^{\sigma l} \cdot g_m^{\sigma l} - \alpha_m^{(3-\sigma)(3-l)} \cdot f_m^{\sigma l}}{\Delta_m^{\sigma l/}}, \tag{3.2}$$

$$(-1)^{\sigma+l} b_m \cdot \frac{i}{4\mu K^2} = \frac{\beta_m^{\sigma l} \cdot f_m^{\sigma l} - \alpha_m^{\sigma l} \cdot g_m^{\sigma l}}{\Delta_m^{\sigma l/}}, \tag{3.3}$$

where

$$\Delta_m^{\sigma l/} = \alpha_m^{\sigma l} \cdot \alpha_m^{(3-\sigma)(3-l)} - \left|\beta_m^{\sigma l}\right|^2, \tag{3.4}$$

$$\alpha_m^{\sigma l} = \left\|TF_m^{\sigma l}\right\|^2, \tag{3.5}$$

$$\beta_m^{\sigma l} = \left\langle TF_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)}\right\rangle, \tag{3.6}$$

$$f_m^{\sigma l} = \left\langle \overline{K}_0^* T\overline{F}_m^{\sigma l}, F_m^{\sigma l\perp}\right\rangle, \tag{3.7}$$

$$g_m^{\sigma l} = \left\langle \overline{K}_0^* T\overline{F}_m^{(3-\sigma)(3-l)}, F_m^{\sigma l\perp}\right\rangle. \tag{3.8}$$

Proof. It is easily proved in [10] that the following relations hold for a circle with radius a:

$$F_m^{\sigma 1}(p) = kH_m'(ka)\, P_m^\sigma(\theta_p) + \frac{m}{a} H_m(ka) Q_m^\sigma(\theta_p) \tag{3.9}$$

$$(-1)^\sigma F_m^{\sigma 2}(p) = \frac{m}{a} H_m(ka) P_m^{(3-\sigma)}(\theta_p) + kH_m'(ka)\, Q_m^{(3-\sigma)}(\theta_p), \tag{3.10}$$

where

$$P_m^\sigma(\theta_p) = E_m^\sigma(\theta_p)\, \overrightarrow{r} \quad \text{and} \quad Q_m^\sigma(\theta_p) = (-1)^\sigma E_m^{(3-\sigma)}(\theta_p)\, \overrightarrow{\theta}, \tag{3.11}$$

$$T_p G_1(p,q) = [T_p G_1(p,q)]^t. \tag{3.12}$$

In view of (3.12) we conclude that $K_1 = \overline{K}_1^*$.

Consider now the modified double layer potentials V_n^v and U_n^v with densities P_n^v and Q_n^v :

$$V_n^v(p) = \frac{1}{2\pi} \int_{R_q=a} T_p G_1(p,q) . P_n^v(\theta_p) . ds_q, \tag{3.13}$$

$$U_n^v(p) = \frac{1}{2\pi} \int_{R_q=a} T_p G_1(p,q) . Q_n^v(\theta_q) . ds_q. \tag{3.14}$$

Exploiting the orthogonality relations for the vectors P_n^v and Q_n^v, we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 1}(q).P_n^v(\theta_q).a.d\theta_q = kaH'_m(ka)\delta_{mn}\delta_{\sigma\nu}, \tag{3.15}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 2}(q).P_n^v(\theta_q).a.d\theta_q = (-1)^{\sigma} mH_m(Ka)\delta_{mn}\delta_{(3-\sigma)\nu}, \tag{3.16}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 1}(q).Q_n^v(\theta_q).a.d\theta_q = mH_m(ka)\delta_{mn}\delta_{\sigma\nu}, \tag{3.17}$$

$$\frac{1}{2\pi}\int_0^{2\pi} TF_m^{\sigma 2}(q).Q_n^v(\theta_q).a.d\theta_q = (-1)^{(3-\nu)}Ka)H'_m(Ka)\delta_{mn}\delta_{\sigma(3-\nu)}. \tag{3.18}$$

Substituting the expressions (3.15) to (3.18) in (3.13) and (3.14), we obtain:

$$\begin{aligned} P_n^v(p) = \tfrac{i}{4\mu K^2} kaH'_n(Ka) &\left[\begin{array}{c} T\widehat{F}_n^{\nu 1}(p) + a_n^{\nu 1} TF_n^{\nu 1}(p) \\ +(-1)^{\nu+1} b_n TF_n^{(3-\nu)2}(p) \end{array}\right] \\ +\tfrac{i}{4\mu K^2}(-1)^{3-\nu} nH_n(Ka) &\left[\begin{array}{c} T\widehat{F}_n^{(3-\nu)2}(p) + a_n^{(3-\nu)2} TF_n^{(3-\nu)2}(p) \\ +(-1)^{3-\nu} b_n TF_n^{\nu 1}(p) \end{array}\right], \end{aligned} \tag{3.19}$$

$$\begin{aligned} U_n^v(p) = \tfrac{i}{4\mu K^2} nH_n(ka) &\left[\begin{array}{c} TF_n^{\nu 1}(p) + a_n^{\nu 1} TF_n^{\nu 1}(p) \\ +(-1)^{\nu+1} b_n TF_n^{(3-\nu)2}(p) \end{array}\right] \\ +\tfrac{i}{4\mu K^2}(-1)^{3-\nu} KaH'_n(Ka) &\left[\begin{array}{c} T\widehat{F}_n^{(3-\nu)2}(p) + a_n^{(3-\nu)2} TF_n^{(3-\nu)2}(p) \\ +(-1)^{3-\nu} b_n TF_n^{\nu 1}(p) \end{array}\right]. \end{aligned} \tag{3.20}$$

In order to find the eigenvalues of M, the following relation has to be satisfied:

$$MU(p) = \lambda U(p), \quad R_p = a. \tag{3.21}$$

Taking into account that $\{P_n^v, Q_n^v\}$ is a basis in $(L_2(\partial D))^2$, we can express $U(p)$ as a linear combination of these vectors, so:

$$U(p) = \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2} (\alpha_m^{\alpha} P_m^{\alpha}(p) + \beta_m^{\alpha} Q_m^{\alpha}(p)). \tag{3.22}$$

To calculate $MU(p)$, we must calculate:

$$MP_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) P_n^v(p), \text{ and } MQ_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) Q_n^v(p).$$

If we apply the Neumann boundary condition to V_n^v, we obtain [11]:

$$TV_n^v(p) = \left(\frac{1}{2}I + K_1\right) P_n^v(p) \qquad R_p = a, \tag{3.23}$$

using $K_1 = \overline{K}_1^*$, (3.23) becomes:

$$MP_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) P_n^v(p) = TV_n^\nu(p) \quad R_p = a. \tag{3.24}$$

In the same way, we can obtain:

$$MQ_n^v(p) = \left(\frac{1}{2}I + \overline{K}_1^*\right) Q_n^v(p) = TU_n^\nu(p) \quad R_p = a. \tag{3.25}$$

Exploiting the following relations [12]:

$$\begin{aligned} TF_n^{v1}(p) = k^2 \left(2\mu H_n''(ka) - \lambda H_n(ka)\right) P_n^v(\theta_p) \\ +\tfrac{2\mu n}{a} \left(kH_n'(ka) - \tfrac{H_n(ka)}{a}\right) Q_n^v(\theta_p), \end{aligned} \tag{3.26}$$

$$\begin{aligned} TF_n^{v2}(p) = \mu K^2 \left(2H_n''(Ka) + H_n(Ka)\right)(-1)^\nu P_n^{(3-v)}(\theta_p) \\ +\tfrac{2\mu n}{a} \left(KH_n'(Ka) - \tfrac{H_n(Ka)}{a}\right)(-1)^\nu Q_n^{(3-v)}(\theta_p), \end{aligned} \tag{3.27}$$

$$\begin{aligned} T\widehat{F}_n^{v1}(p) = k^2 \left(2\mu J_n''(ka) - \lambda J_n(ka)\right) P_n^v(\theta_p) \\ +\tfrac{2\mu n}{a} \left(kJ_n'(ka) - \tfrac{J_n(ka)}{a}\right) Q_n^v(\theta_p), \end{aligned} \tag{3.28}$$

$$\begin{aligned} TF_n^{v2}(p) = \mu K^2 \left(2J_n''(Ka) + J_n(Ka)\right)(-1)^\nu P_n^{(3-v)}(\theta_p) \\ +\tfrac{2\mu n}{a} \left(KJ_n'(Ka) - \tfrac{J_n(Ka)}{a}\right)(-1)^\nu Q_n^{(3-v)}(\theta_p), \end{aligned} \tag{3.29}$$

and using the notations:

$$\begin{aligned} b_n^{v1} = k^2 \left(2J_n''(ka) + J_n(ka)\right) + a_n^{v1} k^2 \left(2H_n''(ka) + H_n(ka)\right) \\ +b_n \mu K^2 \left(2H_n''(Ka) + H_n(Ka)\right), \end{aligned} \tag{3.30}$$

$$\begin{aligned} b_n^{v2} = \mu K^2 \left(2J_n''(Ka) + J_n(Ka)\right) + a_n^{(3-v)2} \mu K^2 \left(2H_n''(Ka) + H_n(Ka)\right) \\ +b_n k^2 \left(2\mu H_n''(ka) - \lambda H_n(ka)\right), \end{aligned} \tag{3.31}$$

$$\begin{aligned} b_n^{v3} = \tfrac{2\mu n}{a} \left(kJ_n'(ka) - \tfrac{J_n(ka)}{a}\right) + a_n^{v1} \tfrac{2\mu n}{a} \left(kH_n'(ka) - \tfrac{H_n(ka)}{a}\right) \\ +b_n \tfrac{2\mu n}{a} \left(KH_n'(Ka) - \tfrac{H_n(Ka)}{a}\right), \end{aligned} \tag{3.32}$$

$$\begin{aligned} b_n^{v4} = \tfrac{2\mu n}{a} \left(KJ_n'(Ka) - \tfrac{J_n(Ka)}{a}\right) + a_n^{(3-v)2} \tfrac{2\mu n}{a} \left(KH_n'(ka) - \tfrac{H_n(Ka)}{a}\right) \\ +b_n \tfrac{2\mu n}{a} \left(kH_n'(ka) - \tfrac{H_n(ka)}{a}\right), \end{aligned} \tag{3.33}$$

then, we obtain:

$$TV_n^v(p) = \tfrac{i}{4\mu K^2} kaH_n'(Ka)\left[b_n^{\nu 1} P_n^v(\theta_p) + b_n^{\nu 3} Q_n^v(\theta_p)\right] - \tfrac{i}{4\mu K^2} nH_n(Ka)\left[b_n^{\nu 2} P_n^v(\theta_p) + b_n^{\nu 4} Q_n^v(\theta_p)\right], \tag{3.34}$$

$$TU_n^v(p) = \tfrac{i}{4\mu K^2} nH_n(ka)\left[b_n^{\nu 1} P_n^v(\theta_p) + b_n^{\nu 3} Q_n^v(\theta_p)\right] - \tfrac{i}{4\mu K^2} KaH_n'(Ka)\left[b_n^{\nu 2} P_n^v(\theta_p) + b_n^{\nu 4} Q_n^v(\theta_p)\right]. \tag{3.35}$$

In view of (3.24), (3.25), (3.34), (3.35), and (3.21) leads to the equations:

$$\left(\lambda - \tfrac{i}{4\mu K^2}\left[kaH_n'(ka) b_n^{\nu 1} + nH_n(Ka) b_n^{\nu 2}\right]\alpha_n^v\right) - \tfrac{i}{4\mu K^2}\left[nH_n(ka) b_n^{\nu 1} + KaH_n'(Ka) b_n^{\nu 2}\right]\beta_n^v = 0, \tag{3.36}$$

$$-\tfrac{i}{4\mu K^2}\left[kaH_n'(ka) b_n^{\nu 3} + nH_n(Ka) b_n^{\nu 4}\right]\alpha_n^v + \left(\lambda - \tfrac{i}{4\mu K^2}\left[nH_n(ka) b_n^{\nu 3} + KaH_n'(Ka) b_n^{\nu 4}\right]\beta_n^v\right) = 0. \tag{3.37}$$

If we use the following notations:

$$\begin{aligned} A_n^{\nu 1} &= \left[kaH_n'(ka) b_n^{\nu 1} + nH_n(Ka) b_n^{\nu 2}\right], \\ A_n^{\nu 2} &= \left[kaH_n'(ka) b_n^{\nu 3} + nH_n(Ka) b_n^{\nu 4}\right], \\ A_n^{\nu 3} &= \left[nH_n(ka) b_n^{\nu 1} + KaH_n'(Ka) b+_n^{\nu 2}\right], \\ A_n^{\nu 4} &= \left[nH_n(ka) b_n^{\nu 3} + KaH_n'(Ka) b_n^{\nu 4}\right]. \end{aligned}$$

Then (3.36) and (3.37) can be rewritten as follows:

$$\left(\lambda - \frac{i}{4\mu K^2} A_n^{\nu 1}\alpha_n^\nu\right) - \frac{i}{4\mu K^2} A_n^{\nu 3}\beta_n^\nu = 0, \tag{3.38}$$

$$-\frac{i}{4\mu K^2} A_n^{\nu 2}\alpha_n^\nu + \left(\lambda - \frac{i}{4\mu K^2} A_n^{\nu 4}\beta_n^\nu\right) = 0. \tag{3.39}$$

In order to get a non-trivial solution of the above system, its determinant must vanish. So the eigenvalues must satisfy:

$$\lambda^2 - \frac{i}{4\mu K^2}\left(A_n^{\nu 1} + A_n^{\nu 4}\right)\lambda + \left(\frac{i}{4\mu K^2}\right)^2\left(A_n^{\nu 1} A_n^{\nu 4} - A_n^{\nu 2} A_n^{\nu 3}\right) = 0. \tag{3.40}$$

Obviously, in order to minimize the condition number we have to choose the multipole coefficients, in such a way that all eigenvalues become equal to 1. Then the condition number is 1. From (3.40), using the same technique developed in [1], in order that all the eigenvalues be equal to 1, we obtain the relations:

$$\left(\frac{i}{4\mu K^2}\right)^2\left(A_n^{\nu 1} A_n^{\nu 4} - A_n^{\nu 2} A_n^{\nu 3}\right) = e^{2i\theta_n^\nu}, \tag{3.41}$$

$$-\frac{i}{4\mu K^2}\left(A_n^{\nu 1}+A_n^{\nu 4}\right)=\rho_n^\nu e^{i\theta_n^\nu}, \tag{3.42}$$

where ρ_n^ν and θ_n^ν are arbitrary real numbers satisfying the inequalities:

$$0\leq \rho_n^\nu \leq 2 \text{ and } 0\leq \theta_n^\nu \leq 2\pi.$$

Indeed, using (3.41) and (3.42), (3.40) becomes:

$$\lambda^2+\rho_n^\nu e^{i\theta_n^\nu}\lambda+e^{2i\theta_n^\nu}=0. \tag{3.43}$$

Which admits the solutions:

$$\lambda_{1,2}=\frac{-\left(\rho_n^\nu e^{i\theta_n^\nu}\right)\pm ie^{i\theta_n^\nu}\sqrt{4-\left(\rho_n^\nu\right)^2}}{2}.$$

Note here that we have : $|\lambda_1|=|\lambda_2|=1$.

Obviously there are infinitely many choices of multipole coefficients $a_m^{\sigma l}$ and b_m which satisfy the imposed conditions. If we choose $a_m^{\sigma l}$ and b_m as the coefficients which minimize the norm of the modified integral operator [10], after some calculations we obtain:

$$A_n^{\nu 1}=A_n^{\nu 4}=-4\mu K^2 \text{and } A_n^{\nu 2}=A_n^{\nu 3}=0. \tag{3.44}$$

For these values (3.44) and for $\rho_n^\nu=2$, $\theta_n^\nu=\pi$, we find that (3.40) has a double root $\lambda=1$. □

The above choice of the multipole coefficients does not satisfy the inequalities (2.7) imposed on the coefficients by the uniqueness theorem [3]. But as in [10], it has been proved that with this choice, the norm of the modified integral operator is equal to zero. So the boundary integral equation is uniquely solvable.

3.2 Case of the perturbation of the circle

As in [8] and [10], we can consider a family of non-circular boundaries defined parametrically by the relation:

$$R_\varepsilon=a+\varepsilon\varphi\left(\theta_p\right), \quad 0\leq\theta_p\leq\pi, \tag{3.45}$$

where φ and $\frac{\partial\varphi}{\partial\theta}$ are all bounded. We will use the estimates for the multipole vectors [10]:

$$F_m^{\sigma l}(p_\varepsilon)=F_m^{\sigma l}(p_a)+O(\varepsilon). \tag{3.46}$$

$$TF_m^{\sigma l}(p_\varepsilon)=TF_m^{\sigma l}(p_a)+O(\varepsilon), \tag{3.47}$$

where p_ε is a point in the perturbed circle, while p_a describe points on the circle of radius a. In [10] it has been proved that the boundary integral operator K_1^ε is a perturbation of the boundary integral operator K_1^a defined on the circle:

$$K_1^\varepsilon = K_1^a + O(\varepsilon).$$

In view of this estimates it is straightforward that the eigenvalues of the perturbed operator M_ε are perturbations of the eigenvalues of the original operator M_a. So:

$$Cond(M_\varepsilon) = Cond(M_a) + O(\varepsilon) = 1 + O(\varepsilon).$$

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