A categorical approach to vitally dense monomorphisms of S-acts

M. Hezarjaribi Dastaki
Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran
Iran
masoome.hezarjaribi@srbiau.ac.ir

H. Rasouli*
Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran
Iran
hrasouli@srbiau.ac.ir

Abstract. In this paper we consider a new type of closure operators on acts over a commutative monoid, namely vital closure operator, to get the class of vitally dense monomorphisms derived from this closure operator and investigate injectivity and essentiality with respect to this class of monomorphisms. We study some categorical properties of vitally dense monomorphisms such as limits and colimits. It is proved that the three notions of essentiality for vitally dense monomorphisms are the same, and considering the relations between injectivity, retractivity, essentiality and injective envelopes, we show that injectivity well behaves in regard to such kind of monomorphisms of acts.

Keywords: S-act, vitally dense monomorphism, vitally dense injective, vitally dense essential.

1. Introduction and preliminaries

Closure operators are mostly applicable in topology and algebra (see [3]). Various kinds of closure operators can be found in many papers. For instance, Ebrahimi [5] considered ideal closure operators of M-sets. Also sequential closure operators of acts over semigroups were investigated in [9, 10, 11, 12]. Essentiality with respect to monomorphisms or an arbitrary class of morphisms of a category is a fundamental concept in algebra and category theory which is closely related to injectivity. In fact, injective objects are characterized and injective envelops are defined by means of essentiality (see, for example, [1, 2, 4, 13]).

∗. Corresponding author

Here we introduce vital closure operators of acts over a commutative monoid to get the class of vitally dense monomorphisms arising from them. We study some elementary properties of vital closure operators and consider some categorical properties of vitally dense monomorphisms including limits and colimits. There are three different notions of essentiality with respect to a subclass $\mathcal{M}$ of monomorphisms in the literature. Moreover, Banaschewski [1] proposed three propositions mainly about the relations between injectivity, retractivity, essentiality and injective envelopes, all with respect to a class $\mathcal{M}$ of morphisms, and if they hold true in the given category, one says that $\mathcal{M}$-injectivity "well behaves". Here, considering $\mathcal{M}$ to be the class $\mathcal{M}_v$ of vitally dense monomorphisms of acts, we show that these three notions of essentiality are in fact equivalent and injectivity well behaves in regard to this class of monomorphisms.

In the following, we briefly give some definitions and preliminaries needed in the sequel.

Let S be a monoid. A (*right*) *S-act* is a non-empty set A together with a map $A \times S \to A, (a,s) \mapsto as$, such that for all $a \in A, s,t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. In this case, we say that A is an extension of B. An element θ in an S-act A is said to be a *zero* or *fixed element* if $\theta s = \theta$ for all $s \in S$. Let A and B be two S-acts. A mapping $f : A \to B$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. We denote the category of all S-acts and homomorphisms between them by **Act-**S. In this category, monomorphisms are exactly one-to-one homomorphisms and *epimorphisms* are exactly onto homomorphisms. A *congruence* ρ on an S-act A is an equivalence relation on A such that $a\rho a'$ implies that $as\rho a's$ for $a, a' \in A$ and $s \in S$. If ρ is a congruence on A, then the factor set $A/\rho = \{[a]_\rho : a \in A\}$ is an S-act, called the *factor act* of A by ρ, with the action given by $[a]_\rho s = [as]_\rho$ for $a \in A$ and $s \in S$.

Let A be an S-act. For any $H \subseteq A \times A$, the *congruence generated by* H is denoted as $\rho(H)$ which is the smallest congruence on A containing H. One can see that $x\rho(H)y$ if and only if either $x = y$ or there exist $s_1, s_2, \ldots, s_n \in S$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ such that $(a_i, b_i) \in H$ or $(b_i, a_i) \in H$ and

$$\begin{array}{cccccccc} x = a_1 s_1 & & b_2 s_2 = a_3 s_3 & & \cdots & & b_n s_n = y \\ & b_1 s_1 = a_2 s_2 & & b_3 s_3 = a_4 s_4 & & \cdots & \end{array}$$

Let $\{A_i \mid i \in I\}$ be a family of S-acts. The *product* of this family, denoted by $\prod_{i\in I} A_i$, is their cartesian product with the componentwise action. Also the *coproduct* $\coprod_{i\in I} A_i$ of this family is their disjoint union with natural action. For a family $\{A_i \mid i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number. For undefined terms and notations about S-acts, we refer to [6, 8].

2. Vital closure operator

In this section we introduce and study a closure operator on **Act-**S where S is a commutative monoid. This leads to the notion of vitally dense monomorphism for S-acts which is the subject of this study.

Let $\mathcal{C}$ be a category. Recall that a family $C = (C_B)_{B\in\mathcal{C}}$, with $C_B : \mathrm{Sub}B \to \mathrm{Sub}B$, taking any subobject $A \leq B$ to a subobject $C_B(A)$, is called a *closure operator* on $\mathcal{C}$ if it satisfies the following:

1. (*Extension*) $A \leq C_B(A)$,
2. (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,
3. (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Moreover, a closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Hereditary* if for every S-act B and $A_1 \leq A_2 \leq B$, $C_{A_2}(A_1) = C_B(A_1) \cap A_2$.

(c) *Additive* if for every S-act B, $C_B(A_1 \cup A_2) = C_B(A_1) \cup C_B(A_2)$.

(d) *Productive* if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B(A) = \prod_i C_{B_i}(A_i)$.

(e) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for every S-act B and $A \leq B$.

(f) *Discrete* if $C_B(A) = A$ for every S-act B and $A \leq B$.

Definition 2.1. For a commutative monoid S, a family $C^v = (C^v_B)_{B\in\mathbf{Act}-S}$ with $C^v_B : \mathrm{Sub}B \to \mathrm{Sub}B$ is defined as

$$C^v_B(A) = \{b \in B : bs \in A \ \text{ for some } \ s \in S\}$$

for any subact A of B.

It is easy to show that C^v is a closure operator on **Act-**S which is called a *vital closure operator*. Indeed, for any $b \in C^v_B(A)$ and $t \in S$, there exists $s \in S$ with $bs \in A$ and so $(bt)s = b(ts) = b(st) = (bs)t \in A$ by commutativity of S. Then $bt \in C^v_B(A)$, which means that $C^v_B(A)$ is a subact of B. The extension and monotonicity properties are clear. For continuity, take a homomorphism $f : B \to D$ and $b \in C^v_B(A)$. Then $bs \in A$ for some $s \in S$ and then $f(b)s = f(bs) \in f(A)$. Hence, $f(b) \in C^v_D(f(A))$, i.e. $f(C^v_B(A)) \leq C^v_D(f(A))$.

Clearly, if $A \leq B \leq D$, then $C^v_B(A) \leq C^v_D(A)$.

From now on, S stands for a commutative monoid.

A C^v*-dense* subact A of an S-act B is called *vitally dense* or v*-dense* for short, i.e. $C^v_B(A) = B$. A homomorphism $f : A \to B$ is said to be *vitally dense* or v*-dense* if $f(A)$ is a v-dense subact of B. The class of all v-dense monomorphisms of acts is denoted as $\mathcal{M}_v$.

We now prove some properties of this closure operator.

Theorem 2.2. *The closure operator C^v is weakly hereditary, hereditary, additive, idempotent and also discrete if S is a group. Moreover, C^v is productive whenever S is a finite commutative monoid.*

Proof. We just prove some parts of this result; the remainder are also straightforward. For idempotency, take any $b \in C^v_B(C^v_B(A))$ where $A \leq B$. Then $bs \in C^v_B(A)$ for some $s \in S$. This gives that there exists $t \in S$ such that $(bs)t \in A$. Thus $b(st) \in A$, which means that $b \in C^v_B(A)$. For hereditariness, let $A_1 \leq A_2 \leq B$ and $a \in C^v_{A_2}(A_1)$. Then $as \in A_1, a \in A_2$ for some $s \in S$. Thus $as \in A_1, a \in B$. Hence, $a \in C^v_B(A_1) \cap A_2$. Conversely, let $a \in C^v_B(A_1) \cap A_2$. Then $a \in A_2, as \in A_1$ for some $s \in S$. Thus $a \in C^v_{A_2}(A_1)$. For the last part, consider a family of subacts A_i of B_i and take $A = \prod_i A_i$ and $B = \prod_i B_i$. We must show that $C^v_B(A) = \prod_i C^v_{B_i}(A_i)$. Let $b = \{b_i\} \in \prod_i C^v_{B_i}(A_i)$. Then for any i, there exists $s_i \in S$ for which $b_i s_i \in A_i$. Since S is finite, so is the number of s_i's. Using commutativity of S, take the product of all s_i's, namely s. Then $b_i s \in A_i$ for each i and so $bs = \{b_i\}s = \{b_i s\} \in A$. This means that $\prod_i C^v_{B_i}(A_i) \leq C^v_B(A)$. The reverse inequality is clear. □

3. Categorical properties of vitally dense monomorphisms

In this section some categorical properties including limits and colimits with respect to the class $\mathcal{M}_v$ of all v-dense monomorphisms of acts derived from the vital closure operator are investigated.

An important property for a class $\mathcal{M}$ of morphisms of a category is being the composition closed, that is, the composition of any two morphisms of $\mathcal{M}$ belongs to $\mathcal{M}$. Also $\mathcal{M}$ is said to be *right* (*left*) *cancellable* if for monomorphisms f and g with $gf \in \mathcal{M}$ one has $g \in \mathcal{M}$ ($f \in \mathcal{M}$) (see, for example, [2]). In the following, we study these properties for the class $\mathcal{M}_v$.

Lemma 3.1. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two homomorphisms where g is a monomorphism. Then gf is a v-dense homomorphism if and only if so are f and g. In particular, $\mathcal{M}_v$ is closed under composition as well as right and left cancellable.*

Proof. Assume that gf is a v-dense homomorphism. We show that $f(A)$ and $g(B)$ are v-dense subacts of B and C, respectively. For any $b \in B$, $g(b) \in C$. Since $gf(A)$ is a v-dense subact of C, there exist $s \in S$ and $a \in A$ for which $g(b)s = gf(a)$. So $g(bs) = gf(a)$ and then $bs = f(a) \in f(A)$. Now let $c \in C$. Then there exists $s \in S$ with $cs \in gf(A)$ and so $cs \in g(B)$. Conversely, let f and g be v-dense homomorphisms. It must be shown that $gf(A)$ is a v-dense subact of C. Let $c \in C$. Since $g(B)$ is v-dense in C, there exist $s \in S$ and $b \in B$ such that $cs = g(b)$. Also, since $f(A)$ is v-dense in B, there exist $t \in S$ and $a \in A$ for which $bt = f(a)$. Now we have $c(st) = (cs)t = g(b)t = g(bt) = gf(a) \in gf(A)$, as desired. □

It is clear that any epimorphism is a v-dense homomorphism. So, using Lemma 3.1, the following is obtained.

Corollary 3.2. *The composition of a v-dense homomorphism with an epimorphism is a v-dense homomorphism.*

Proposition 3.3. *Let S be a finite monoid. Then the class $\mathcal{M}_v$ is closed under products and direct sums.*

Proof. Let $(f_i : A_i \to B_i)_{i\in I}$ be a family of v-dense monomorphisms. Consider the commutative diagram

$$\begin{array}{ccc} \prod_{i\in I} A_i & \xrightarrow{f} & \prod_{i\in I} B_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

We show that $f = (f_i)_{i\in I} : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is a v-dense monomorphism. Let $b = (b_i)_{i\in I} \in \prod_{i\in I} B_i$. Since each f_i is v-dense, there exists $s_i \in S$ for $i \in \{1, 2, \cdots, n\}$ such that $b_i s_i \in \mathrm{Im} f_i$. Now let $s := s_1 s_2 \cdots s_n$. We have $bs = (b_i s)_{i\in I} \in \mathrm{Im} f$. Hence, f is v-dense. It is obvious that f is a monomorphism. So $f \in \mathcal{M}_v$. For the second assertion, let $(f_i : A_i \to B_i)_{i\in I}$ be a family of v-dense monomorphisms where A_i's and B_i's have a unique fixed element 0. Then, using the first part, $f = \bigoplus_{i\in I} f_i = (\prod_{i\in I} f_i)|_{\bigoplus_{i\in I} A_i} : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$ is a v-dense monomorphism. More precisely, if $(b_i)_{i\in I} \in \bigoplus_{i\in I} B_i$, then there exists $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ and $s \in S$ with $f((a_i)_{i\in I}) = (b_i)_{i\in I} s$. But, $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$, because for all i with $b_i = 0$ we have $0 = b_i s = f_i(a_i)$, and so $a_i = 0$ since f_i is a monomorphism. □

Proposition 3.4. *The class $\mathcal{M}_v$ is closed under coproducts.*

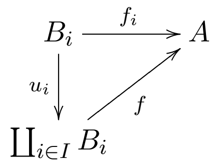
Proof. Consider the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u'_i} \\ \coprod_{i\in I} A_i & \xrightarrow{f} & \coprod_{i\in I} B_i \end{array}$$

in which $(f_i : A_i \to B_i)_{i\in I}$ is a family of v-dense monomorphisms. Let $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the homomorphism satisfying $f(u_i(a_i)) = u'_i f_i(a_i)$, for $a_i \in A_i$, which exists by the universal property of coproducts. In fact, $f(a_i, i) = (f_i(a_i), i)$. We claim that f is a v-dense monomorphism. Let $b \in \coprod_{i\in I} B_i$. Then $b \in B_i$ for some $i \in I$ and $b = u'_i(b_i)$. Since f_i is v-dense, there exists $a_i \in A_i$ and $s \in S$ for which $f_i(a_i) = bs$, and hence $bs = u'_i f_i(a_i) = f u_i(a_i) \in \mathrm{Im} f$, which means that f is v-dense. Also, f is a monomorphism since u'_i and f_i, $i \in I$ are monomorphisms. □

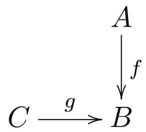
Proposition 3.5. *Let $(f_i : B_i \to A)_{i\in I}$ be a family of v-dense homomorphisms. Then $f : \coprod_{i\in I} B_i \to A$ is a v-dense homomorphism.*

Proof. Consider the diagram



where $f : \coprod_{i\in I} B_i \to A$ is the homomorphism obtained by the universal property of coproducts. For any $a \in A$, since f_i is v-dense, there exist $b_i \in B$ and $s \in S$ such that $f_i(b_i) = as$. So $as = f_i(b_i) = fu_i(b_i)$ and hence f is a v-dense homomorphism. $\square$

Recall that the pullback of a given diagram

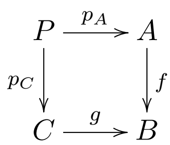


in **Act-**S is the subact $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ (if non-empty) of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps.

A class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the next result, we study this property for v-dense monomorphisms of acts.

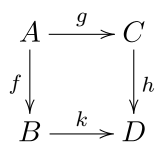
Proposition 3.6. *The class $\mathcal{M}_v$ is pullback stable.*

Proof. Consider the pullback diagram



where $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ and $p_C : P \to C$ and $p_A : P \to A$ are restrictions of the projection maps. Assume that $g \in \mathcal{M}_v$. We show that $p_A \in \mathcal{M}_v$. Let $a \in A$. Then $f(a) \in B$ and so it follows from the v-density of g that there exist $s \in S$ and $c \in C$ such that $g(c) = f(a)s = f(as)$. Hence, $(c, as) \in P$ and we have $as = p_A(c, as)$, which means that p_A is a v-dense homomorphism. $\square$

For a class $\mathcal{M}$ of morphisms of a category, it is said that *pushouts transfer $\mathcal{M}$-morphisms* if for a pushout diagram



if $g \in \mathcal{M}$, then so is k.

The pushout of a given diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

in **Act-**S is the factor act $Q = (B \sqcup C)/\theta$, where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also the pushout maps are given as $h = \gamma u_C : C \to Q, k = \gamma u_B : B \to Q$, where $\gamma : B \sqcup C \to Q$ is the canonical epimorphism. Multiple pushouts in **Act-**S are constructed in a similar way.

Proposition 3.7. *In* **Act-**S*, pushouts transfer* v*-dense monomorphisms.*

Proof. Consider the pushout diagram (as described above)

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & Q \end{array}$$

where g is a v-dense monomorphism. We show that k is also a v-dense monomorphism. By [7], k is a monomorphism. So it remains to prove that k is v-dense. Let $[x]_\theta \in Q$. If $x = u_B(b)$ for some $b \in B$, then $[x]_\theta = k(b) \in \mathrm{Im}(k)$. If $x = u_C(c)$ for some $c \in C$, using that g is v-dense, $cs = g(a)$ for some $s \in S$ and $a \in A$ which implies that $[x]_\theta s \in \mathrm{Im}(k)$. □

For a class $\mathcal{M}$ of morphisms of a category, we say that multiple pushouts transfer $\mathcal{M}$-morphisms if in the multiple pushout $(Q, (B_i \stackrel{d'_i}{\to} Q)_{i\in I})$ of a family $\{d_i : A \to B_i \mid i \in I\}$ of $\mathcal{M}$-morphisms, $d'_i \in \mathcal{M}$, for every $i \in I$.

Similarly to the pushouts, we have the following result.

Proposition 3.8. *Multiple pushouts transfer* v*-dense monomorphisms.*

Proof. Let $\{d_i : A \to B_i \mid i \in I\}$ be a family of v-dense monomorphisms. Recall that the multiple pushout of this family is $(\coprod_{i\in I} B_i)/\theta$, where θ is the congruence on $\coprod_{i\in I} B_i$ generated by all pairs $H = \{(u_i d_i(a), u_j d_j(a)) : i, j \in I, a \in A\}$, where for each $i \in I$, $u_i : B_i \to \coprod_{i\in I} B_i$ is the i-th coproduct injection map. Also the multiple pushout maps are $d'_i = \gamma u_i : B_i \to (\coprod_{i\in I} B_i)/\theta$ where $\gamma : \coprod_{i\in I} B_i \to (\coprod_{i\in I} B_i)/\theta$ is the natural epimorphism. The same argument as in the proof of [10, Proposition 3.15] gives that d'_i is a monomorphism for each $i \in I$. To show that each $d'_i d_i$ (and hence each d'_i by Lemma 3.1) is v-dense, let

$b \in (\coprod_{i\in I} B_i)/\theta$. Then there exist $j \in I$ and $b_j \in B_j$ such that $b = [u_j(b_j)]_\theta$. Since d_j is v-dense, there exist $a \in A$ and $s \in S$ such that $d_j(a) = b_j s$. Now we get

$$bs = [u_j(b_j)]_\theta s = [u_j(b_j s)]_\theta = d'_j(b_j s) = d'_j d_j(a) = d'_i d_i(a) \in \mathrm{Im}(d'_i d_i),$$

as required. □

Definition 3.9. For a class $\mathcal{M}$ of morphisms of a category $\mathcal{A}$, we say that $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for each set indexed family $\{m_i : A \to A_i \mid i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all m_i's, that is, there are $d_i : A_i \to B$ with $d_i m_i = m$. In addition, if d_i's belong to $\mathcal{M}$, it is said that $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property.

Corollary 3.10. *The category* **Act-**S *has* $\mathcal{M}_v$*-bounds and* $\mathcal{M}_v$*-amalgamation property.*

Proof. Let $\{h_i : A \to B_i \mid i \in I\}$ be a set indexed family in $\mathcal{M}_v$ and $h : A \to B = (\coprod_{i\in I} B_i)/\theta$ be the multiple pushout of h_i's. Then h factors over all h_i's and is a v-dense monomorphism by Proposition 3.8. The second assertion also follows from Proposition 3.8. □

This following is devoted to the study of the behaviour of v-dense monomorphisms with respect to colimits and directed colimits. To this end, let us recall some preliminaries.

Let $\mathcal{A} : \mathbf{I} \to$ **Act-**S be a diagram in **Act-**S ($\mathbf{I}$ is a small category and $\mathcal{A}$ is a functor) determining the acts $\mathcal{A}(\alpha) = A_\alpha \in$ **Act-**S for $\alpha \in I = \mathrm{Obj}(\mathbf{I})$ and homomorphisms $g_{\alpha\beta} : A_\alpha \to A_\beta$ for $\alpha \to \beta$ in $\mathrm{Mor}(\mathbf{I})$. The *colimit* of this diagram is obtained as $\varinjlim_\alpha A_\alpha =: (\coprod_{\alpha\in I} A_\alpha)/\theta$, where θ is the congruence generated by

$$H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) : a_\alpha \in A_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}.$$

The colimit homomorphisms are $g_\alpha := \gamma_\theta u_\alpha : A_\alpha \to \varinjlim_\alpha A_\alpha$ where u_α's are the coproduct injection maps and γ_θ is the canonical epimorphism.

A *directed system* of S-acts and homomorphisms is a family $(B_\alpha)_{\alpha\in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha\le\beta\in I}$ of homomorphisms such that given $\alpha \le \beta \le \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, and also $g_{\alpha\alpha} = id$. Note that the *directed colimit* of a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\le\beta\in I})$ in **Act-**S is given as $\varinjlim_\alpha B_\alpha = (\coprod_{\alpha\in I} B_\alpha)/\rho$, where the congruence ρ is defined by $(b_\alpha, b_\beta) \in \rho$ if there exists $\delta \ge \alpha, \beta$ such that $u_\delta g_{\alpha\delta}(b_\alpha) = u_\delta g_{\beta\delta}(b_\beta)$, where u_δ's are injection maps of the coproduct.

Notice that the family $g_\alpha = \gamma_\rho u_\alpha : B_\alpha \to \varinjlim_{\alpha\in I} B_\alpha$ of homomorphisms satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \le \beta$, where $\gamma_\rho : \coprod_{\alpha\in I} B_\alpha \to \varinjlim_{\alpha\in I} B_\alpha$ is the canonical epimorphism.

Theorem 3.11. *The class of v-dense monomorphisms is closed under colimits.*

Proof. Assume that $\mathcal{A}, \mathcal{B} : \mathbf{I} \to \mathbf{Act}\text{-}S$ be diagrams determining the S-acts A_α, B_α for $\alpha \in I = \mathrm{Obj}(\mathbf{I})$ and homomorphisms $g_{\alpha\beta} : A_\alpha \to A_\beta, \acute{g}_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta \in \mathrm{Mor}(\mathbf{I})$. Consider the colimits of these diagrams with the colimit homomorphisms $g_\alpha = \gamma_\theta u_\alpha : A_\alpha \to \varinjlim_\alpha A_\alpha = (\coprod_{\alpha \in I} A_\alpha)/\theta, \acute{g}_\alpha = \gamma_{\acute{\theta}} \acute{u}_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha = (\coprod_{\alpha \in I} B_\alpha)/\acute{\theta}$ where θ is the congruence generated by $H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) \mid a_\alpha \in A_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}$ and $\acute{\theta}$ is the congruence generated by $\acute{H} = \{(\acute{u}_\alpha(b_\alpha), \acute{u}_\beta \acute{g}_{\alpha\beta}(b_\alpha)) \mid b_\alpha \in B_\alpha, \alpha \to \beta \in \mathrm{Mor}(\mathbf{I})\}$. Suppose that $\{f_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ is a family of v-dense monomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f_\alpha} & B_\alpha \\ {\scriptstyle g_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \acute{g}_{\alpha\beta}} \\ A_\beta & \xrightarrow{f_\beta} & B_\beta \end{array}$$

We show that $f = \varinjlim_\alpha f_\alpha$ with $f[u_\alpha(a_\alpha)]_\theta = [\acute{u}_\alpha f_\alpha(a_\alpha)]_{\acute{\theta}}$ is a v-dense monomorphism. We have the following commutative diagram:

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\quad f_\alpha \quad} & B_\alpha \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow{\scriptstyle \acute{g}_\alpha} \\ (\coprod_{\alpha \in I} A_\alpha)/\theta & \xrightarrow{f} & (\coprod_{\alpha \in I} B_\alpha)/\acute{\theta} \end{array}$$

Let $x = [\acute{u}_\alpha(b_\alpha)]_{\acute{\theta}} \in \varinjlim_\alpha B_\alpha$ for some $\alpha \in I$ and $b_\alpha \in B_\alpha$. Since f_α is v-dense, there exist $s \in S, a \in A$ such that $f_\alpha(a_\alpha) = b_\alpha s$. Hence, $xs = [\acute{u}_\alpha(b_\alpha s)]_{\acute{\theta}} = \acute{g}_\alpha(b_\alpha s) = \acute{g}_\alpha f_\alpha(a_\alpha) = f g_\alpha(a_\alpha) = f[u_\alpha(a_\alpha)]_\theta$. Also f is a monomorphism since so is f_α. $\square$

Proposition 3.12. *The category* $\mathbf{Act}\text{-}S$ *has* $\mathcal{M}_v$*-directed colimits.*

Proof. Consider a directed system $((B_\alpha)_{\alpha \in I}, (g_{\alpha\beta})_{\alpha \leq \beta \in I})$ of S-acts and homomorphisms, and the colimit homomorphisms $g_\alpha : B_\alpha \longrightarrow \varinjlim_\alpha B_\alpha$. Take v-dense monomorphisms $h_\alpha : A \longrightarrow B_\alpha, \alpha \in I$ with $g_{\alpha\beta} h_\alpha = h_\beta$ for $\alpha \leq \beta \in I$. Let $h : A \longrightarrow \varinjlim_\alpha B_\alpha$ be the directed colimit of h_α for $\alpha \in I$. That is, $h = \varinjlim_\alpha h_\alpha = g_\alpha h_\alpha = g_\beta h_\beta = g_\gamma h_\gamma = \cdots$. Let $\alpha \in I$. Since h_α is a monomorphism, so is h. Now since g_α is an epimorphism, it is v-dense and then h is v-dense. $\square$

Definition 3.13. For a class $\mathcal{M}$ of morphisms of a category $\mathcal{A}$, we say that $\mathcal{A}$ satisfies $\mathcal{M}$-*chain* condition if for any directed system $((A_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta \in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha \in I}$ with $g_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 3.14. *The category* **Act-**S *fulfills the* $\mathcal{M}_v$*-chain condition.*

Proof. Take $A = \varinjlim_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be the colimit maps. Then, applying Proposition 3.12, the assertion holds. □

4. Vital dense essentiality and vital dense injectivity

In this section we proceed with the study of essentiality with respect to the class $\mathcal{M}_v$. We show that the three notions of essentiality introduced in [2] coincide for the class $\mathcal{M}_v$. We investigate the relations between v-dense essentiality, v-dense injectivity and v-dense absolute retractivity. It is proved that maximal v-dense essential extensions, minimal v-dense injective v-dense extensions and v-dense injective hulls of acts are equivalent and exist which will be called the v-dense injective envelopes. In other words, injectivity well behaves with respect to v-dense monomorphisms of acts in the sense of [1].

Recall that for a subclass $\mathcal{M}$ of the class $\mathcal{M}ono$ of monomorphisms of a category $\mathcal{A}$ and $A \overset{m}{\to} B \in \mathcal{M}$ one usually uses one of the following definitions to say that m is essential:

(1) $A \overset{m}{\to} B \overset{f}{\to} C \in \mathcal{M} \Rightarrow f \in \mathcal{M}$

(2) $A \overset{m}{\to} B \overset{f}{\to} C \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$ (essential monomorphism)

(3) $A \overset{m}{\to} B \overset{f}{\to} C \in \mathcal{M} \Rightarrow f \in \mathcal{M}ono$

Here, considering $\mathcal{A}$ to be the category **Act-**S and $\mathcal{M}$ to be the class $\mathcal{M}_v$ of all v-dense monomorphisms of acts, we show that the above types of essentialities are the same.

Theorem 4.1. *Let* $f : A \to B \in \mathcal{M}_v$. *Then the following are equivalent:*
(i) *Any homomorphism* $g : B \to C$ *for which* $gf \in \mathcal{M}_v$ *belongs to* $\mathcal{M}_v$.
(ii) *Any homomorphism* $g : B \to C$ *for which* $gf \in \mathcal{M}_v$ *is a monomorphism.*
(iii) *Any homomorphism* $g : B \to C$ *for which* gf *is a monomorphism is itself a monomorphism.*
(iv) *For every congruence* ρ *on* B, $\rho \cap (f(A) \times f(A)) = \Delta_{f(A)}$ *implies* $\rho = \Delta_B$.

Proof. The implication (i) ⇒ (ii) is clear.
(ii) ⇒ (iii) Let $g : B \to D$ be a homomorphism such that gf is a monomorphism. Since $gf : A \to g(B)$ is a v-dense monomorphism, by the assumption, $g : B \to g(B)$ and so g is a monomorphism.
(iii) ⇔ (iv) See Lemma 3.1.15 in [8].
(iv) ⇔ (i) Let gf be a v-dense monomorphism. By (iii) ⇔ (iv), g is a monomorphism. Now, using Lemma 3.1, g is a v-dense monomorphism. □

Definition 4.2. We say that a v-dense monomorphism $f : A \to B$ is a *v-dense essential monomorphism* or *$\mathcal{M}_v$-essential* if it satisfies any one of the equivalent conditions of Theorem 4.1.

It follows by Theorem 4.1 that:

Corollary 4.3. *A monomorphism f is $\mathcal{M}_v$-essential if and only if it is a v-dense as well as essential monomorphism.*

Lemma 4.4. *The composition of $\mathcal{M}_v$-essentials is an $\mathcal{M}_v$-essential.*

Proof. Let $f : A \to B$ and $g : B \to C$ be $\mathcal{M}_v$-essentials. We show that so is gf. Using Lemma 3.1, gf is a v-dense monomorphism. Suppose that $h : C \to D$ is a homomorphism for which $h(gf) = (hg)f$ is a monomorphism. It must be shown that h is a monomorphism. Since f is $\mathcal{M}_v$-essential, hg and hence h is a monomorphism because g is $\mathcal{M}_v$-essential. □

Lemma 4.5. *If gf is $\mathcal{M}_v$-essential and g is a monomorphism, then g is an $\mathcal{M}_v$-essential.*

Proof. Using Lemma 3.1, g is v-dense. In light of Corollary 4.3, it suffices to prove that g is essential. Let h be a homomorphism such that hg is a monomorphism. Then, since f is also a monomorphism, $h(gf) = (hg)f$ is a monomorphism. But gf is $\mathcal{M}_v$-essential and so h is a monomorphism. Thus g is $\mathcal{M}_v$-essential. □

Theorem 4.6. *The category* **Act-**S *fulfills Banaschewski's $\mathcal{M}_v$-condition, in the sense that for any $\mathcal{M}_v$-monomorphism $f : A \to B$ there exists a homomorphism $g : B \to C$ such that gf is $\mathcal{M}_v$-essential.*

Proof. Apply Zorn's lemma to the poset

$$\Sigma = \{\theta \in \mathrm{Con}(B) \mid A \xrightarrow{f} B \xrightarrow{\pi_\theta} B/\theta \in \mathcal{M}_v\}$$

with inclusion as the order. Then for a maximal element $\overline{\theta}$ of Σ, $\pi_{\overline{\theta}} f$ is a v-dense monomorphism. In view of Corollary 4.3, it remains to prove that $\pi_{\overline{\theta}} f$ is essential. Let $h : B/\theta \to D$ be a homomorphism such that $h\pi_{\overline{\theta}} f$ is a monomorphism. We show that h is a monomorphism. Let $h([b]_{\overline{\theta}}) = h([\acute{b}]_{\overline{\theta}})$, $[b]_{\overline{\theta}} \neq [\acute{b}]_{\overline{\theta}}$ for $[b]_{\overline{\theta}}, [\acute{b}]_{\overline{\theta}} \in B/\theta$. Define the following relation on B:

$$b\rho\acute{b} \quad \Leftrightarrow \quad [b]_{\overline{\theta}} \ (\ker h) \ [\acute{b}]_{\overline{\theta}}$$

It is clear that ρ is a congruence on B and $\overline{\theta} \subset \rho$. We show that $\pi_\rho f$ is a v-dense monomorphism. Since π_ρ is an epimorphism and f is v-dense, $\pi_\rho f$ is a v-dense homomorphism by Corollary 3.2. Suppose that $\pi_\rho f(a) = \pi_\rho f(\acute{a})$ for $a, \acute{a} \in A$. Then we have $(f(a), f(\acute{a})) \in \rho$ and so $h([f(a)]_{\overline{\theta}}) = h([f(\acute{a})]_{\overline{\theta}})$. This implies that $h\pi_{\overline{\theta}} f(a) = h\pi_{\overline{\theta}} f(\acute{a})$ and so $a = \acute{a}$ because $h\pi_{\overline{\theta}} f$ is a monomorphism. Hence, $\rho \in \Sigma$ which contradicts the maximality of $\overline{\theta}$. □

By a *v-dense retract* of an S-act B, we mean a v-dense subact A of B together with a homomorphism from B to A which maps A identically.

Remark 4.7. If A is a proper v-dense retract of B, then B is not a v-dense essential extension of A. Indeed, consider a retraction $g : B \to A$ and take $b \in B \setminus A$. Now $g(b) \in A$. Since $g(g(b)) = g(b)$, it follows that g is not a monomorphism whereas $g|_A$ is a monomorphism.

Let A be an S-act. Then A is said to be *v-dense injective* if it is injective with respect to all v-dense monomorphisms. Also A is called *v-dense absolute retract* if A is a v-dense retract of each of its v-dense extensions. Clearly, a v-dense retract of any v-dense injective S-act is v-dense injective.

By virtue of Proposition 3.7, Theorem 4.6 and [2, Theorem 3.6], the following result is obtained.

Theorem 4.8. *Let S be a monoid. Then the following are equivalent for each S-act A:*
(i) *A is v-dense injective.*
(ii) *A is v-dense absolute retract.*
(iii) *A has no proper v-dense essential extension.*

Definition 4.9. Let A be an S-act and E be a v-dense extension of A. Then E is called

(i) a *maximal v-dense essential extension* of A if E is a v-dense essential extension of A and not contained properly in any other v-dense essential extension of A.

(ii) a *minimal v-dense injective v-dense extension* of A if E is v-dense injective and contains no proper subact B which is a v-dense injective extension of A.

(iii) an *$\mathcal{M}_v$-injective hull* of A if E is v-dense injective as well as a v-dense essential extension of A.

Lemma 4.10. *If B is a v-dense essential extension of an S-act A and A is embedded into some (v-dense) injective S-act Q, then B can also be embedded into Q.*

Proof. Straightforward. □

As we shall see, the next result ensures the existence of v-dense injective envelopes of S-acts.

Theorem 4.11. *Every S-act has a maximal v-dense essential extension.*

Proof. Let A be an S-act and Q be an injective extension of A. By Lemma 4.10, we may assume that both A and all its v-dense essential extensions are subacts of Q. Let P be the set of all v-dense essential extensions of A. Consider P as a partially ordered set under inclusion. For any chain $(A_i)_{i\in I}$ in P, $\bigcup_{i\in I} A_i \in P$ is an upper bound. Using Zorn's lemma, P has a maximal element M which is in fact a maximal v-dense essential extension of A. □

Lemma 4.12. *If B is a maximal v-dense essential extension of an S-act A, then B is v-dense injective.*

Proof. In light of Theorem 4.8, it suffices to show that B has no proper v-dense essential extension. If there exists an S-act C such that $B \subset C$ and C is a v-dense essential extension of B, then C is a v-dense essential extension of A by Lemma 4.4 which contradicts the assumption. □

Theorem 4.13. *Let A be an S-act and E be a v-dense extension of A. The following are equivalent:*

(i) *E is a maximal v-dense essential extension of A.*

(ii) *minimal v-dense injective v-dense extension of A.*

(iii) *E is an $\mathcal{M}_v$-injective hull of A.*

Proof. (i) ⇒ (iii) Follows from Lemma 3.1, Proposition 3.7, Theorem 4.6 and [2, Theorem 3.8(iv)].

(iii) ⇒ (ii) Follows from Lemma 3.1 and [2, Theorem 3.8(v)].

(ii) ⇒ (i) Let E be a minimal v-dense injective v-dense extension of A and B be a maximal v-dense essential extension of A which exists by Theorem 4.11. Consider the diagram

$$\begin{array}{ccc} A & \overset{j}{\hookrightarrow} & B \\ {\scriptstyle i}\downarrow & & \\ E & & \end{array}$$

where i and j are the v-dense embeddings. Since E is v-dense injective, there exists a homomorphism $k : B \to E$ such that $kj = i$. Thus kj is a monomorphism. By v-dense essentiality of B, k is a monomorphism and so B is a v-dense subact of E by Lemma 3.1. Since B is a maximal v-dense essential extension of A, it is v-dense injective by Lemma 4.12. Therefore, $B = E$ by the minimality of E. □

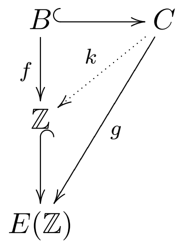
An S-act E is called a *v-dense injective envelope* of an S-act A if it satisfies one of the equivalent conditions in Theorem 4.13.

It is well-known that each S-act is embedded into an injective envelope. Here, using Theorems 4.11 and 4.13, we present a counterpart of this result for the vitally dense case.

Corollary 4.14. *For every S-act there exists a v-dense injective envelope unique up to isomorphism.*

Finally, the following example demonstrates that the notion of v-dense injective envelope is different from the usual notion of injective envelope for acts.

Example 4.15. Consider $\mathbb{N}$ and $\mathbb{Z}$ as $(\mathbb{N}_0, +)$-acts with usual addition as the actions. It is clear that $\mathbb{Z}$ is a v-dense extension of $\mathbb{N}$. Since $\mathbb{Z}$ has no fixed element, it is not injective and hence not an injective envelope of $\mathbb{N}$. However, we claim that $\mathbb{Z}$ is a v-dense injective envelope of $\mathbb{N}$. First we show that $\mathbb{Z}$ is an essential extension of $\mathbb{N}$. Let $f : \mathbb{Z} \to A$ be a homomorphism such that $f|_{\mathbb{N}}$ is a monomorphism. Suppose that $f(z) = f(z')$ for $z, z' \in \mathbb{Z}$. There exist $n, n' \in \mathbb{N}_0$ such that $z + n, z' + n' \in \mathbb{N}$. Thus $z + (n + n'), z' + (n + n') \in \mathbb{N}$ and $f(z + (n + n')) = f(z' + (n + n'))$ so that $z + (n + n') = z' + (n + n')$ and then $z = z'$ which shows that f is a monomorphism. Now we prove that $\mathbb{Z}$ is v-dense injective. Suppose that B is a v-dense subact of C, $f : B \to \mathbb{Z}$ is a homomorphism and $E(\mathbb{Z})$ is an injective envelope of $\mathbb{Z}$. It follows from the injectivity of $E(\mathbb{Z})$ that there exists a homomorphism $g : C \to E(\mathbb{Z})$ such that the following diagram commutes:



We claim that Im$g \subseteq \mathbb{Z}$. Indeed, for any $c \in C$, there exists $n \in \mathbb{N}_0$ such that $cn \in B$ since B is v-dense in C. Then $g(cn) = f(cn) \in \mathbb{Z}$ and there exists $x \in \mathbb{Z}$ such that $g(c)n = g(cn) = x + n$. This implies that $g(c) = x$, otherwise consider the homomorphism $h : E(\mathbb{Z}) \to E(\mathbb{Z})$ defined by $h(y) = yn$ for any $y \in E(\mathbb{Z})$. We have $h(g(c)) = g(c)n = x + n = h(x)$ and $g(c) \neq x$, which means that h is not a monomorphism. Since $E(\mathbb{Z})$ is an essential extension of $\mathbb{Z}$, $h|_{\mathbb{Z}}$ is not a monomorphism, which is a contradiction. Now taking $k := g : C \to \mathbb{Z}$, we have $k|_B = f$, which shows that $\mathbb{Z}$ is v-dense injective.

Acknowledgments

The authors would like to express their appreciation to the referee for carefully reading the paper and making helpful comments.

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Accepted: 16.06.2019