Semirings of $0, 1$-preserving endomorphisms of semilattices

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Abstract. In the paper, various endomorphism semirings of semilattices are investigated. In particular, various conditions are found under which these endomorphism semirings are simple.

Keywords: semiring, semimodule, semilattice, endomorphism.

1. Introduction

A semiring is an algebraic structure possessing two associative binary operations, usually denoted as addition and multiplication, where the addition is commutative and the multiplication distributes over the addition from both sides.

Let S be a semiring. An element $a \in S$ is *additively neutral (absorbing)* if $a + b = b$ $(a + b = a)$ for every $b \in S$. We write $0_S \in S$ $(o_S \in S)$ if S contains the uniquely determined additively neutral (absorbing) element 0_S (o_S). On the other hand, $0_S \notin S$ $(o_S \notin S)$ means that S contains no element of that kind.

An element $a \in S$ is called *right multiplicatively absorbing* if $ba = a$ for every $b \in S$. The set of right multiplicatively absorbing elements will be denoted by

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$\underline{R}(S)$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is an ideal and, in fact, $\underline{R}(S)$ is the smallest right ideal of the semiring S.

The semiring S is called (*congruence-*)*simple* if S has just two congruences.

A commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ ($M \times S \to M$) is called *left* (*right*) *S-semimodule*. In the sequel, if not specified, semimodules are left S-semimodules.

A semimodule M is called *faithful* if for all $a, b \in S,\ a \neq b$, there is $x \in M$ with $ax \neq bx$, and M is called *simple* if there are just two congruence relations on M.

We are going to study various endomorphism semirings of some semilattices. Semirings of that type are additively idempotent, and therefore, throughout the paper all semirings and semimodules are assumed to be *additively idempotent*. It means that the respective additive semigroups are semilattices. Now, given a semilattice M $(= M(+))$, the basic order $\leq$ is defined on M by $x \leq y$ iff $x + y = y$. If $0_M \in M$ ($o_M \in M$) then 0_M (o_M) is the smallest (greatest) element of M. A non-empty subset A of M is an *ideal* if $A + M = A$. The ideal A is said to be *prime* if $M \setminus A$ is a subsemilattice of M.

If $o_M \in M$ ($0_M \in M$) then we will say that M satisfies (1β) ((2β)) if every infinite strictly increasing (decreasing) sequence $x_1 < x_2 < x_3 < \dots$ ($x_1 > x_2 > x_3 > \dots$) of elements from M is upwards (downwards) cofinal in $M \setminus \{o_M\}$ ($M \setminus \{0_M\}$).

2. Characteristic semimodules

Let S be a semiring, $|S| \geq 3$, and M be a (left S-)semimodule. We shall say that M is a *characteristic semimodule* if the following three conditions are satisfies:

(a) M is faithful and $|M| \geq 3$;

(b) $0_M, o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$;

(c) there is a mapping $\underline{\varepsilon} : N = M \setminus \{o_M\} \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x, y, z \in M$, $x \neq o_M$, $y \leq x$, $z \nleq x$.

Throughout this section, let M be a characteristic semimodule. By [1, 2.2, 2.3], the semimodule M is simple, $\underline{\varepsilon}$ is an injective mapping of N into $R = \underline{R}(S)$, $|R| \geq 2$, $o_S \in S$ and $\underline{\varepsilon}(0_M) = o_S \in R$. By [1, 2.4], we have $RM = \{0_M, o_M\}$.

The semimodule M will be called *nearly critical* if for all $a, b \in R$, $a < b$, and $x \in K = M \setminus \{0_M, o_M\}$ there is at least one $c \in S$ such that $acx \neq bcx$ (then $acx = 0_M$ and $bcx = o_M$).

Lemma 2.1. *Assume that* $\underline{\varepsilon}(N) = R$. *Then* M *is nearly critical iff the following condition is satisfied:*

(3β) *For all* $u \in N$ *and* $v, w \in K$, $u < v$, *there is at least one* $c \in S$ *such that* $cw \leq v$ *and* $cw \nleq u$.

Proof. It is easy. □

Lemma 2.2. *Assume that $R \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$ and that the set N has no minimal element. Then M is nearly critical iff (3β) is true.*

Proof. It is easy (take into account that for every $u \in N$ there is $v \in K$ with $u < v$). □

The semimodule M will be called *critical* if $Sx = M$ for every $x \in K$. Now, if M is critical then M is nearly critical and $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M are just all subsemimodules of M.

Lemma 2.3. *M is critical iff the following condition is satisfied:*

(4β) For all $x, y \in K$ there is $c \in S$ with $cx = y$.

Proof. It is obvious. □

Remark 2.4. (i) [2, 2.8] If the semiring S is simple then M is nearly critical.
(ii) [2, 2.9] If the semiring S is simple, M is not critical and (1β) is true then both S and M are infinite, $o_N \in N$, $\underline{\varepsilon}(o_N) = 0_S \in S$, $P = M \setminus \{o_N\}$ is a subsemimodule of M and P is a critical semimodule. By 2.3, for all $x, y \in M \setminus \{0_M, o_N, o_M\}$ there is $c \in S$ with $cx = y$.

Lemma 2.5. *The right semimodule R_S is faithful.*

Proof. Let $a, b \in S$ be such that $ca = cb$ for every $c \in R$. If $x \in M$ is such that $ax \in N$ then $0_M = \underline{\varepsilon}(ax)ax = \underline{\varepsilon}(ax)bx$, and hence $bx \in N$ and $bx \leq ax$. Using symmetry, we conclude that $ax = bx$ for every $x \in M$. Since ${}_SM$ is faithful, we get $a = b$. □

Lemma 2.6. *If the right S-semimodule R_S is simple then the following condition is true:*

(5β) For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $c \in S$ such that $cw \leq v$, $cw \nleq u$ and $cz \nleq v$ for every $z \nleq w$.

Proof. Using the right hand forms of [3, 4.10, 4.12(i)], we find $c \in S$ such that $\underline{\varepsilon}(v)c = \underline{\varepsilon}(w) < \underline{\varepsilon}(u)c$. □

Lemma 2.7. *Assume that the right S-semimodule R_S is simple. Then the condition (3β) is true and either $|R| = 2$ or the set $S \setminus \{o_S\}$ is not a subsemiring of S.*

Proof. (3β) follows immediately from (5β). If $S \setminus \{o_S\}$ is a subsemiring of S then the relation $((R \setminus \{o_S\}) \times (R \setminus \{o_S\}) \cup \mathrm{id}_R$ is a congruence of R_S. □

Lemma 2.8. *Assume that the right S-semimodule R_S is simple and that $0_S \in S$ (or $0_R \in R$). Then $0_S = 0_R$ and the following condition is satisfied:*

(6β) *For all* $u \in N \setminus \{o_N\}$ *and* $w \in K$ *there is at least one* $c \in S$ *such that* $cw \in N$, $cw \not\leq u$ *and* $cz = o_M$ *for every* $z \not\leq w$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $0_S c = \underline{\varepsilon}(w) < \underline{\varepsilon}(u)c$. □

Lemma 2.9. *If* $o_N \in N$ *then* (5β) *implies* (6β).

Proof. It is obvious. □

Lemma 2.10. *Assume that the right* S*-semimodule* R_S *is simple and that* $0_S \in S$. *Then the following condition is satisfied:*

(7β) *For all* $u \in N$ *and* $v \in K$, $u < v$, *there is at least one* $c \in S$ *such that* $cK \leq v$ *and* $cz \not\leq u$ *for at least one* $z \in K$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $\underline{\varepsilon}(v)c = 0_S < \underline{\varepsilon}(u)c$. □

Lemma 2.11. *If* $o_N \in N$ *then* (5β) *implies* (7β).

Proof. It is obvious. □

Lemma 2.12. *Assume that the right* S*-semimodule* R_S *is simple and that* $0_S \in S$. *Then the following condition is satisfied:*

(8β) *For every* $u \in N \setminus \{o_N\}$ *there is at least one* $c \in S$ *such that* $cK \subseteq N$ *and* $cz \not\leq u$ *for at least one* $z \in K$.

Proof. Proceeding similarly as in the proof of 2.6, we find $c \in S$ such that $0_S c = 0_S < \underline{\varepsilon}(u)c$. □

Lemma 2.13. *If either* $o_N \in N$ *or the set* N *has no maximal element then* (7β) *implies* (8β).

Proof. It is obvious. □

Lemma 2.14. *Assume that* (2β) *is true and* $\underline{\varepsilon}(N) = R$. *Then the right* S*-semimodule* R_S *is simple iff* (5β) *is true and either* $|R| = 2$ *or* $S \setminus \{o_S\}$ *is not a subsemiring of* S.

Proof. If R_S is simple then we use 2.6 and 2.7. If $|R| = 2$ then R_S is simple anyway. Now, assume that (5β) is true and $S \setminus \{o_S\}$ is not a subsemiring of S. Taking into account the right hand form of [3,4.13], we have to show that for all $a, b, d \in R$, $a < b$, $d \neq o_S$, there is $c \in S$ with $ac = d < bc$. But $a, b, d \in \underline{\varepsilon}(N)$ and it remains to use (5β). □

Lemma 2.15. *Assume that* (2β) *is true,* $0_S \in S$ *and* $R \setminus \{0_S\} \subseteq \underline{\varepsilon}(N)$. *Then the right* S*-semimodule* R_S *is simple iff the conditions* (5β), (6β), (7β) *and* (8β) *are satisfied and either* $|R| = 2$ *or* $S \setminus \{o_S\}$ *is not a subsemiring of* S.

Proof. Using 2.6, 2.7, 2.8, 2.10 and 2.12, we proceed similarly as in the proof of 2.14. □

Remark 2.16. (cf. 2.15) If (1β) is true, $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ then $o_N \notin N$, $N+N = N$ and the set N has no maximal element. Now, (7β) implies (8β).

Lemma 2.17. *Assume that* (1β) *is true and the set* N *has no maximal element (equivalently,* $o_N \notin N$ *and* $N+N=N$*). If* $a \in S$ *then* $\underline{\varepsilon}(v) \leq a$ *for at least one* $v \in N$ *iff* $o_M \in aK$.

Proof. If $\underline{\varepsilon}(v) \leq a$ then $v < u$ for some $u \in K$ and $au = o_M$. Conversely, if $A \cap K \neq \emptyset$, where $A = \{\, x \in M \mid ax \neq o_M \,\}$, then $A + A = A$ and if $v = o_A \in A$ then $\underline{\varepsilon}(v) \leq a$. On the other hand, if $o_A \notin A$ then $A = N$ and $o_M \notin aK$, a contradiction. □

Lemma 2.18. *Assume that* (1β) *is true and* $N+N \neq N$*. If* $a \in S$ *then* $\underline{\varepsilon}(v) \leq a$ *for at least one* $v \in N$ *iff there is a maximal element* $w \in N$ *such that* $ax = o_M$ *for* $x \nleq w$ *(then* $\underline{\varepsilon}(w) \leq a$*).*

Proof. Observing that there is no infinite strictly increasing sequence of elements in N, our result easily follows. □

3. Endomorphism semirings (a)

Throughout this section, let M $(= M(+))$ be a semilattice containing at least three elements and such that $0_M \in M$ and $o_M \in M$. The set $\underline{E}$ $(= \underline{E}_{0,1}(M))$ of endomorphisms of M that preserve the elements 0_M and o_M is a unitary semiring; we have $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in \underline{E}$ and $x \in M$, and $1_{\underline{E}} = \mathrm{id}_M$ is the multiplicatively neutral element of $\underline{E}$.

The set $(\underline{D}_{0,1} =)$ $\underline{D} = \{\, f \in \underline{E} \mid f(M) = \{0_M, o_M\} \,\}$ is an ideal of the semiring $\underline{E}$ and the following result is obvious.

Proposition 3.1. (i) *If* $f \in \underline{D}$ *then the set* $\{\, x \in M \mid f(x) = o_M \,\}$ *is a prime ideal of the semilattice* M.
(ii) *If* A *is a prime ideal of* M*,* $f(A) = \{o_M\}$ *and* $f(M \setminus A) = \{0_M\}$ *then* $f \in \underline{D}$. □

For every $x \in N = M \setminus \{o_M\}$, the set $A_x = \{\, y \in M \mid y \nleq x \,\}$ is a prime ideal of M and the corresponding endomorphism will be denoted by q_x. A_x is the principal prime ideal determined by x, $q_x(u) = 0_M$ for $u \leq x$ and $q)x(v) = o_M$ for $v \nleq x$. We put $(\underline{B}_{0,1} =)$ $\underline{B} = \{\, q_x \mid x \in N \,\}$ and $(\underline{C}_{0,1} =)$ $\underline{C} = \{\, q_{x_1} + \cdots + q_{x_n} \mid n \geq 1, x_i \in N \,\}$. Clearly, $\underline{C}$ is just the subsemiring of $\underline{D}$ generated by the set $\underline{B}$ and we have the following results:

Proposition 3.2. $\underline{C} = \underline{B}$ *iff the ordered set* $M(\leq)$ *is a lattice.* □

Proposition 3.3. *Assume that the condition* (1β) *is true. Then* $\underline{D} = \underline{B}$ *if and only if either* $o_N \in N$ *or* $N + N \neq N$. □

Proposition 3.4. *If* $N+N = N$ *then* $\xi \in \underline{D}$, *where* $\xi(N) = \{0_M\}$ *and* $\xi(o_M) = o_M$. *If* $o_N \in N$ *then* $\xi = q_{o_N} \in \underline{B}$. □

Proposition 3.5. *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$. *Then:*
(i) $\underline{C} \subseteq E$.
(ii) $\underline{R}(E) = E \cap \underline{D}$.
(iii) *The right* E*-semimodule* $\underline{R}(E)_E$ *is faithful.*
(iv) $q_{0_M} = o_E \in E$.
(v) *If* $0_E \in E$ *then* $0_E \in \underline{R}(E)$ *and* $N + N = N$.
(vi) *If* $v \in N$ *then* $q_v = 0_E$ *if and only if* $v = o_N \in N$ *(then* $0_E = q_{o_N} = \xi$*).*
(vii) *If* $N + N = N$ *and* $\xi \in E$ *then* $\xi = 0_E$.
(viii) *If* $N + N = N$ *and* $\xi \notin E$ *then* $0_E \notin E$.
(ix) *If* $e \in E$ *is right multiplicatively neutral then* $e = 1_E = \mathrm{id}_M$. □

Proposition 3.6. *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$ *and* $|E| \geq 3$. *Then the (left* E*-)semimodule* ${}_EM$ *is characteristic (via* $\underline{\varepsilon}(x) = q_x$ *for every* $x \in N$*).* □

Proposition 3.7. [2, 1.1] *Let* E *be a subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$. *Then* E *is simple if and only if the right semimodule* $\underline{R}(E)_E$ *is simple and* $E \subseteq \underline{R}(E) + \underline{E}$. □

Proposition 3.8. *Let* E *be a simple subsemiring of* $\underline{E}$ *such that* $\underline{B} \subseteq E$ *and* $|E| \geq 3$. *Then:*
(1) *For all* $u \in N = M \setminus \{o_M\}$ *and* $v, w \in K = M \setminus \{0_M, o_M\}$, $u < v$, *there is at least one* $f \in E$ *such that* $f(w) \leq v$, $f(w) \nleq u$ *and* $f(z) \nleq v$ *for every* $z \nleq w$;
(2) *If* $0_E \in E$ (see 3.5) *then for all* $u \in N \setminus \{o_N\}$ *and* $w \in K$ *there is at least one* $f \in E$ *such that* $f(w) \in K$, $f(w) \nleq u$ *and* $f(z) = o_M$ *for every* $z \nleq w$;
(3) *If* $0_E \in E$ *then for all* $u \in N$ *and* $v \in K$, $u < v$, *there is at least one* $f \in E$ *such that* $f(N) \leq v$ *and* $f(z) \nleq u$ *for at least one* $z \in K$;
(4) *If* $0_E \in E$ *then for every* $u \in N \setminus \{o_N\}$ *there is at least one* $f \in E$ *such that* $f(N) \subseteq N$ *and* $f(z) \nleq u$ *for at least one* $z \in K$.

Proof. See 2.6, 2.8, 2.10 and 2.12. □

4. Endomorphism semirings (b)

Let M be a semilattice such that $|M| \geq 4$, $0_M \in M$, $o_M \in M$ and the condition (1β) is true (i.e., every infinite strictly increasing sequence of elements from M is upwards cofinal in $N = M \setminus \{o_M\}$). Let E be a subsemiring of $\underline{E}$ such that $\underline{B} \subseteq E$ (see the preceding section).

Proposition 4.1. (i) *The ordered set* $M(\leq)$ *is a lattice.*
(ii) $\underline{B} = \underline{C}$.

Proof. It is easy. □

Proposition 4.2. (i) *If $N + N \neq N$ then $o_N \notin N$, there is no infinite strictly increasing sequence of elements in M, $0_E \notin E$ and $\underline{R}(E) = \underline{B} = \underline{D}$.*
(ii) *If $o_N \in N$ then $N = N + N$, there is no infinite strictly increasing sequence of elements in M, $0_E = \xi = q_{o_N} \in E$ and $\underline{R}(E) = \underline{B} = \underline{D}$.*
(iii) *If $o_N \notin N = N + N$ and $\xi \notin E$ then the set N has no maximal element, $0_E \notin E$, $\xi \notin \underline{B}$, $\underline{R}(E) = \underline{B}$ and $\underline{D} = \underline{B} \cup \{\xi\}$.*
(iv) *If $o_N \notin N = N + N$ and $\xi \in E$ then the set N has no maximal element, $\xi = 0_E \in E \setminus \underline{B}$ and $\underline{R}(E) = \underline{D} = \underline{B} \cup \{\xi\}$.*

Proof. It is easy (use (1β), 3.3 and 3.5(ii),(v),(vi),(vii),(viii)). □

Proposition 4.3. *$0_E \in E$ if and only if either $o_N \in N$ (then $0_E = \xi = q_{o_N}$) or $o_N \notin N = N + N$ and $\xi \in E$ (then $0_E = \xi$).* □

Corollary 4.4. *$\underline{R}(E) = \underline{B} = \underline{\varepsilon}(N)$ except for the case when $o_N \notin N = N + N$ and $\xi \in E$. In that case, we have $\underline{R})E) = \underline{B} \cup \{\xi\} = \underline{D}$, $\xi \notin \underline{B}$ and $\underline{\varepsilon}(N) = \underline{B} = \underline{R}(E) \setminus \{0_E\}$.* □

Corollary 4.5. *$\underline{R}(E) = \underline{D}$ except for the case when $o_N \notin N = N + N$ and $\xi \notin E$. In that case, we have $\underline{R}(E) = \underline{B}$ and $\underline{D} = \underline{B} \cup \{\xi\}$.* □

Proposition 4.6. *The semimodule ${}_EM$ is nearly critical if and only if the following condition is satisfied:*

1. *For all $u \in N$ and $v, w \in K$, $u < v$, there is at least one $f \in E$ such that $f(w) \leq v$ and $f(w) \nleq U$.*

Proof. Combine 2.1, 2.2 and 4.4. □

Proposition 4.7. *The semimodule ${}_EM$ is critical if and only if E operates transitively on the set K.*

Proof. This is obvious (see 2.3). □

Proposition 4.8. *Assume that the semiring E is simple, but E does not operate transitively on K. Then:*
(i) *E and M are infinite.*
(ii) *$o_N \in N$ and $M \setminus \{o_N\}$ is a subsemimodule of M.*
(iii) *E operates transitively on the set $K \setminus \{o_N\}$.*
(iv) *$E(x) = M \setminus \{o_N\}$ for every $x \in K \setminus \{o_N\}$.*
(v) *$M \setminus \{o_M\} \subseteq E(o_N)$.*

Proof. (i), (ii) and (iv) follow from 2.4(ii) and 4.7, and (iv) follows from (iii). As concerns (v), choose $v \in K \setminus \{o_N\}$. By 3.8(1), there is $f \in E$ with $0_M \neq f(o_N) \leq v$. Then $Ef(o_N) = M \setminus \{o_N\}$ by (iii). □

Proposition 4.9. (i) *If* $N+N \neq N$ *then* $\underline{B}+\underline{E} = \{\, f \in \underline{E} \mid$ *there is a maximal element* $w \in N$ *such that* $f(x) = o_M$ *for* $x \nleq w \,\}$.
(ii) *If* $o_N \notin N = N+N$ *then* $\underline{B}+\underline{E} = \{\, f \in \underline{E} \mid o_M \in f(K) \,\}$.
(iii) *If* $o_N \in N$ *then* $\underline{B}+\underline{E} = \underline{E}$.
(iv) *If* $\mathrm{id}_M \in \underline{B}+\underline{E}$ *then* $o_N \in N$.

Proof. Use 2.17 and 2.18. □

5. Endomorphism semirings (c)

This section is an immediate continuation of the foregoing section. Here, moreover, we will assume that the condition (2β) is true (i.e., every infinite strictly decreasing sequence of elements from M is downwards cofinal in $L = M \setminus \{0_M\}$. Note that if E is simple then the conditions 3.8(1),(2),(3) and (4) are satisfied and, besides, E operates transitively on K or 4.8 takes place.

Theorem 5.1. *Assume that* $0_E \notin E$ (see 4.2). *If the set* $L = M \setminus \{0_M\}$ *has at least one minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{B}+\underline{E}$ (see 3.7, 4.8(i),(ii)) *and the condition* 4.6(1) *is satisfied. If the set* L *has no minimal element then the semiring* E *is simple if and only if* $E \subseteq \underline{B}+\underline{E}$, 4.6(1) *is true and, moreover, the following condition is satisfied:*

(1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq o_M$ *and* $f(x) \nleq v$ *for every* $x \in K$.

Proof. First, let E be simple. By 3.7, the right semimodule $\underline{R}(E)_E$ is simple and we use 2.7 and 4.6 to show that 4.6(1) is true. Furthermore, $\underline{R}(E) = \underline{B}$ by 4.4 and $E \subseteq \underline{B}+\underline{E}$ follows from 3.7. Finally, by [2, 3.2.3(ii)], $o_E = q_v f$ for some $v \in K$ and $f \in E \setminus \{o_E\}$. That is, the condition (1) is true.

Now, we proceed conversely to show that E is simple. We have $E \subseteq \underline{R}(E)+\underline{E}$ and the semimodule ${}_EM$ is nearly critical by 4.6. It remains to use [2, 3.2.1, 3.2.2]. □

Theorem 5.2. *Assume that* $o_N \in N$ *(then* $0_E = q_{o_N} \in E$*). If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 4.6(1) *is satisfied. If the set* L *has no minimal element then the semiring* E *is simple if and only if the conditions* 4.6(1) *and* 5.1(1) *are satisfied.*

Proof. Using [2, 3.2.5, 3.2.6], we proceed in the very same way as in the proof of 5.1. □

Theorem 5.3. *Assume that* $0_E \in E$ *and* $o_N \notin N$ (see 4.3). *If the set* L *has at least one minimal element then the semiring* E *is simple if and only if the condition* 4.6(1) *is satisfied and, moreover, the following condition is satisfied as well:*

(1) *There are* $f \in E$ *and* $v, w \in K$ *such that* $f(w) \neq 0_M$ *and* $f(x) \leq v$ *for every* $x \in K$.

If the set L has no minimal element the the semiring E is simple if and only if the conditions (1), 4.6(1) *and* 5.1(1) *are satisfied.*

Proof. Using [2, 3.2.7 3.2.8], we proceed similarly as in the proof of 5.1. □

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