Generalized Runge-Kutta integrators for solving fifth-order ordinary differential equations

Mohammed S. Mechee*
Department of Mathematics
Faculty of Computer Science and Mathematics
University of Kufa, Najaf
Iraq
mohammeds.abed@uokufa.edu.iq

F.A. Fawzi
Department of Mathematics
Faculty of Computer Science and Mathematics
Tikrit University
Salahaddin
Iraq
firasadil01@tu.edu.iq

Abstract. In this paper, RKM integrators for solving special fifth-order ordinary differential equations (ODEs) have been generalized to solve two classes of general quasi-linear fifth-order ODEs which denoted by GRKM. The novel contribution of this work is the generalizing of RKM integrators by derivation two numerical methods for solving these classes of ODEs. The algebraic equations of order conditions for the proposed GRKM methods have derived up to the order-seventh using Taylor expansion approach. Based on these order conditions, two GRKM methods of fifth- and sixth-order with three- stages have derived. The numerical methods have been tested using two classes of problems in order to compare them with existing methods which show that the proposed integrators are less than existing RK methods in term of time complexity of function evaluations.
Keywords: Runge-Kutta method (RK), Runge-Kutta Nystrom method (RKN), direct Runge-Kutta method (RKD), Runge-Kutta-Mohammed method (RKM), RKT-RKTG- and RKTGG- and RKFD-method, fifth-order, ordinary differential equations, ODEs, Taylor expansion.

1. Introduction

The differential equations (DEs) are the most important tools in mathematical modelling. For examples, the phenomena of physics, fluid and heat flow, motion of objects, vibrations, chemical reactions and nuclear reactions have been modeled by systems of DEs. Many applications of DEs of different types, particularly ODEs of different orders, can be found in the mathematical modeling of real-life problems. Fourth-order DEs often arise in many fields of applied

*. Corresponding author

science such as mechanics, quantum chemistry, electronic and control engineering. Also, beam theory [12], fluid dynamics [2] and [13], ship dynamics [23] and neural networks [14]. Many numerical and analytical approximation for solving such DEs of various orders have studied in the literature. Most of the solutions of mathematical models of these applications should be approximated.

However, in the applications of DEs, the researchers used to developed the numerical and approximated methods for the solutions of type of DEs.

The methods of solutions of DEs are always not able to solve many types of DEs or they can solve some types of DEs indirectly. This reason make us to study the derivation of more direct numerical methods for this propose. In order to apply indirect numerical method to solve a ODE of higher-order, the ODE should be transformed into a system of first-order ODEs. Many researchers developed the family of RK methods for solving first- , second-, third-, fourth- , and fifth-order ODEs. The literature review of the methods of solutions of mathematical models which contains fifth-order BVPs [21], and [3]. For review of the methods of RK type, the first one-step method for solving first-order ODEs, was introduced by Runge in 1895. Also, Heun constructed one-step method in 1900, and Kutta formulated the general scheme of RK methods in 1901 [4]. [20] and [22] have derived Runge-Kutta-Nystrm (RKN) methods for solving second-order ODEs while the direct numerical methods with variable step-size have been derived by [6]. [19] has developed a singly diagonally implicit RKN method with periodical solutions while a symbolic derivation of order conditions of RKN method has introduced by [8].

For solving third-order ODEs, [15] and [24] have derived direct numerical methods of RK type for solving special third-order ODEs using one-step method with constant step-size, one-step direct integrators of RK type with variable step-size of orders 6(5), 5(4) and 4(3) have derived by [19] while [9] has constructed five-stage fourth-order Runge-Kutta type method for directly solving general third-order ODEs of the form $y''' = f(x, y, y', y'')$.

However, for solving special fourth- and fifth-order ODEs using one-step method with constant step-size, [17] and [11] have derived one-step direct numerical integrators of RK type with constant step-size while [15] derived one-step direct numerical integrators of RK type with constant step-size for solving special fifth-order ODEs. Finally, [10] have developed direct processes of explicit Runge-Kutta type (RKT) as solutions for any fourth-order ODEs of the structure $w''''(t) = f(x, w(t), w'(t), w''(t))$.

2. Preliminary

Here, we give the definition for a general quasi linear fifth-order ODE as the follows:

$$v^{(5)}(x) = f(x, v(x), v'(x), v''(x), v'''(x), v''''(x)), x \geq x_0, \tag{1}$$

Equation (1) has some special forms. In this paper, we have derived numerical methods of RK type for two of them.

2.1 Class one of quasi-linear fifth-order ODEs

The first class of general quasi-linear fifth-order ODE with no explicit dependence on the second-, third-, and fourth-derivatives $v^{(i)}(x)$, for $i = 2, 3, 4$ has the following form:

$$v^{(5)}(x) = f(x, v(x), v'(x)), \quad x \geq x_0, \tag{2}$$

with the initial conditions (ICs) $v^{(i)}(x_0) = \alpha^i$ where $\alpha^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_N^i]$ for $i = 1, 2, \ldots, 5$ where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, v(x) = [v_1(x), v_2(x), ..., v_N(x)]$ and

$$f(x, v(x), v'(x)) = [f_1(x, v_1(x), v_1'(x)), f_2(x, v_2(x), v_2'(x)), \ldots, f_N(x, v_N(x), v_N'(x))].$$

2.1.1 Proposed GRKM methods

The proposed GRKM integrator with s-stages for solving class one of general quasi-linear fifth-order ODEs can be written as follows:

$$z_{n+1} = z_n + hz_n' + \frac{h^2}{2!}z_n'' + \frac{h^3}{3!}z_n''' + \frac{h^4}{4!}z_n^{(4)} + h^5 \sum_{i=1}^{s} b_i k_i, \tag{3}$$

$$z_{n+1}' = z_n' + hz_n'' + \frac{h^2}{2!}z_n''' + \frac{h^3}{3!}z_n^{(4)} + h^4 \sum_{i=1}^{s} b_i' k_i, \tag{4}$$

$$z_{n+1}'' = z_n'' + hz_n''' + \frac{h^2}{2!}z_n^{(4)} + h^3 \sum_{i=1}^{s} b_i'' k_i, \tag{5}$$

$$z_{n+1}''' = z_n''' + hz_n^{(4)} + h^2 \sum_{i=1}^{s} b_i''' k_i, \tag{6}$$

$$z_{n+1}^{(4)} = z_n^{(4)} + h \sum_{i=1}^{s} b_i'''' k_i, \tag{7}$$

where

$$k_1 = f(x_n, z_n) \tag{8}$$

and

$$\begin{aligned} k_i = f(x_n + c_i h, z_n + c_i h z_n' + c_i^2 \frac{h^2}{2!} z_n'' + c_i^3 \frac{h^3}{3!} z_n''' + c_i^4 \frac{h^4}{4!} z_n^{(4)} + c_i^5 \frac{h^5}{5!} z_n^{(5)} \\ +h^6 \sum_{m=1}^{i-1} a1_{im} k_m, z_n' + c_n h z_n'' + c_n^2 \frac{h^2}{2!} z_n''' + c_i^3 \frac{h^3}{3!} z_n'''' + h^4 \sum_{l=1}^{i-1} a2_{im} k_m), \end{aligned} \tag{9}$$

for $i = 2, 3, \cdots, s$. The coefficients of the GRKM integrators can be formulated as : $b_i, b_i', b_i'', b_i''', b_i''''$ and $c_i, a1_{i,j}, a2_{i,j}$ for $i, j = 1, 2, \cdots, s$ are real. GRKM is an explicit method if $a1_{i,j} = a2_{i,j} = 0$ for $i \leq j$ and otherwise GRKM is an implicit method. Using Butcher notation, GRKM coefficients can be expressed in the Table 1:

Table 1: The Butcher Tableau GRKM Method

c	A1
	A2
	b^T
	b'^T
	b''^T
	b'''^T
	b''''^T

2.2 Class two of general quasi-linear fifth-order ODEs

The second class of general quasi-linear fifth-order ODE with no explicit dependence on the third-, fourth-derivatives $v^{(i)}(x)$, for $i = 3, 4$ has following form:

$$v^{(5)}(x) = f(x, v(x), v'(x), v''(x)), \quad x \geq x_0, \tag{10}$$

with the initial conditions (ICs) $v^{(i)}(x_0) = \alpha^i$ where $\alpha^i = [\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_N]$ for $i = 0, 1, \ldots, 4$, $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, v(x) = [v_1(x), v_2(x), ..., v_N(x)]$ and $f(x, v(x), v'(x), v''(x)) = [f_1(x, v_1(x), v'_1(x), v''_1(x)), f_2(x, v_2(x), v'_2(x), v''_2(x)), \ldots, f_N(x, v_N(x), v'_N(x), v''_N(x))]$.

2.2.1 Proposed GRKM methods

The proposed GRKM integrator with s-stages for solving the class two of general quasi-linear fifth-order ODEs as equations (3)-(8) and the following equation:

$$\begin{aligned} k_i = f(&x_n + c_i h, z_n + c_i h z'_n + c_i^2 \frac{h^2}{2!} z''_n + c_i^3 \frac{h^3}{3!} z'''_n + c_i^4 \frac{h^4}{4!} z_n^{(4)} \\ &+ c_i^5 \frac{h^5}{5!} z_n^{(5)} + h^6 \sum_{m=1}^{i-1} a1_{im} k_m, \\ &z'_n + c_n h z''_n + c_n^2 \frac{h^2}{2!} z'''_n + c_i^3 \frac{h^3}{3!} z''''_n + h^4 \sum_{l=1}^{i-1} a2_{im} k_m, \\ &z''_n + c_n h z'''_n + c_n^2 \frac{h^2}{2!} z''''_n + h^3 \sum_{l=1}^{i-1} a3_{im} k_m), \end{aligned} \tag{11}$$

for $i = 2, 3, \cdots, s$.

Here, the coefficients $a3_{i,j}$ for $i, j = 1, 2, \cdots, s$ are real. The additional condition of GRKM to be an explicit method if $a3_{i,j} = 0$ for $i \leq j$ and otherwise

GRKM is an implicit method. Using Butcher notation, GRKM coefficients can be expressed in Table 2:

Table 2: The Butcher Tableau GRKM Method

c	A1
	A2
	A3
	b^T
	b'^T
	b''^T
	b'''^T
	b''''^T

3. Derivation of order conditions

To determine the order conditions and then the coefficients of the first proposed numerical integrator indicated by equations (3)-(9), GRKM formula is expanded using the approach of Taylor's series expansion. This expansion, after some algebraic adaptations, is equated to the the solution which is proposed by Taylor expansion. From the direct expansion of the local truncation error we have found the general order conditions for the GRKM method based on the derivation of order conditions for the RK method presented in [7]. In the same manner, using equations (3)-(8) and Equation (11), we derived the following order conditions (12)-(18) of the proposed direct integrators of GRKM type using Maple software.

Order conditions for y

$$\sum b_i = \frac{1}{120}, \sum b_i c_i = \frac{1}{720}, \sum b_i c_i^2 = \frac{1}{2520}, \sum b_i c_i^3 = \frac{1}{840}. \tag{12}$$

Order conditions for y'

$$\begin{aligned} &\sum b'_i = \frac{1}{6}, \sum b'_i c_i = \frac{1}{24}, \sum b'_i c_i^2 = \frac{1}{120}, \\ &\sum b'_i c_i^3 = \frac{1}{630}, \sum \sum_{i,j=1}^{s} b'_i a_{ij} = \frac{1}{5040}, \sum b'_i c_i^4 = \frac{1}{210}. \end{aligned} \tag{13}$$

Order conditions for y''

$$\begin{aligned} &\sum b''_i = \frac{1}{6}, \sum b''_i c_i = \frac{1}{24}, \sum b''_i c_i^2 = \frac{1}{60}, \sum b''_i c_i^3 = \frac{1}{120}, \\ &\sum_{i,j=1}^{s} b''_i a_{ij} = \frac{1}{720}, \sum b''_i c_i^4 = \frac{1}{210}, \sum_{i,j=1}^{s} b''_i a_{ij} c_i = \frac{1}{5040}. \end{aligned} \tag{14}$$

Order conditions for y'''

$$\sum b_i''' = \frac{1}{2}, \sum b_i''' c_i = \frac{1}{6} \sum b_i''' c_i^2 = \frac{1}{12} \sum b_i''' c_i^3 = \frac{1}{20} \sum b_i''' c_i^4 = \frac{1}{30},$$

$$\sum_{i,j=1}^{s} b_i''' a_{ij} = \frac{1}{5040}, \sum b_i''' c_i^5 = \frac{1}{42}. \tag{15}$$

Order conditions for y''''

$$\sum b_i'''' = 1, \sum b_i'''' c_i = \frac{1}{2}, \sum b_i'''' c_i^2 = \frac{1}{3}, \sum b_i'''' c_i^3 = \frac{1}{4}, \sum b_i'''' c_i^4 = \frac{1}{5},$$

$$\sum b_i'''' c_i^5 = \frac{1}{6}, \tag{16}$$

$$a2_{21} = \frac{1}{125}(16\sqrt{6}(a2_{3,1} + a2_{32}) - 131(a2_{31} - a2_{32}) + \frac{16 - 3\sqrt{6}}{20}), \tag{17}$$

$$a3_{21} = \frac{85}{2000}\sqrt{6}, a3_{31} = \frac{1}{500}(\frac{51}{10}\sqrt{6} + \frac{33}{2500}), a3_{32} = \frac{51}{1250} + \frac{11}{1250}\sqrt{6}. \tag{18}$$

3.1 Derivation of the proposed methods

Using the algebraic order conditions (12)-(17), we get first proposed GRKM integrator for solving ODE in Equation (2) (see Table 3). However, adding the order condition (18) to obtain the second proposed GRKM integrator for solving ODE in Equation (10) (see Table 4)

Table 3: The Butcher Tableau GRKM Method

0	0		
$\frac{3}{5} - \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	0	
$\frac{3}{5} + \frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	$-\frac{1}{2}$	0
	0		
	$\frac{1}{18}$	0	
	$\frac{1}{18}$	$\frac{-532}{5625} - \frac{133}{22500}\sqrt{6}$	0
	$-\frac{37}{360}$	$\frac{1}{18}$	$\frac{1}{18}$
	$\frac{17}{360} - \frac{7}{360}\sqrt{6}$	$\frac{1}{18}$	$-\frac{11}{180} + \frac{7}{360}\sqrt{6}$
	$\frac{1}{18}$	$\frac{1}{18} + \frac{1}{48}\sqrt{6}$	$\frac{1}{18} - \frac{1}{48}\sqrt{6}$
	$\frac{1}{9}$	$\frac{7}{36} + \frac{1}{18}\sqrt{6}$	$\frac{7}{36} - \frac{1}{18}\sqrt{6}$
	$\frac{1}{9}$	$\frac{4}{9} + \frac{1}{36}\sqrt{6}$	$\frac{4}{9} - \frac{1}{36}\sqrt{6}$

4. Implementations

In this section, two classes problems are tested. The numerical results are compared with classical methods when the same set of problems is reduced to a

Table 4: The Butcher Tableau GRKM Method

0	0		
$\frac{3}{5}-\frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	0	
$\frac{3}{5}+\frac{1}{10}\sqrt{6}$	$\frac{1}{18}$	$-\frac{1}{2}$	0
	0		
	$\frac{12}{625}-\frac{3}{2500}\sqrt{6}$	0	
	$\frac{1}{2}$	$\frac{-1}{2}$	0
	0		
	$\frac{27}{500}-\frac{19}{100}\sqrt{6}$	0	
	$\frac{33}{2500}+\frac{51}{5000}\sqrt{6}$	$\frac{51}{1250}+\frac{11}{1250}\sqrt{6}$	0
	$-\frac{37}{360}$	$\frac{1}{18}$	$\frac{1}{18}$
	$\frac{17}{360}-\frac{7}{360}\sqrt{6}$	$\frac{1}{18}$	$-\frac{11}{180}+\frac{7}{360}\sqrt{6}$
	$\frac{1}{18}$	$\frac{1}{18}+\frac{1}{48}\sqrt{6}$	$\frac{1}{18}-\frac{1}{48}\sqrt{6}$
	$\frac{1}{9}$	$\frac{7}{36}+\frac{1}{18}\sqrt{6}$	$\frac{7}{36}-\frac{1}{18}\sqrt{6}$
	$\frac{1}{9}$	$\frac{4}{9}+\frac{1}{36}\sqrt{6}$	$\frac{4}{9}-\frac{1}{36}\sqrt{6}$

system of first-order equations and solved using existing RK of the same order [5].

4.1 First proposed method

In this subsection we have tested three problems involving $y''''' = f(x, y, y')$.

Example 4.1 (Homogenous ODE)**.**

$$y^{(5)}(t) = \cos(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0$. Exact solution: $y(t) = \sin(t), b = \pi$.

Example 4.2 (Linear ODE)**.**

$$y^{(5)}(t) = y'(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0$. Exact solution: $y(t) = \sin(t), b = \pi$.

Example 4.3 (Non-linear ODE)**.**

$$y^{(5)}(t) = -120y'^{3}(t), \quad 0 < t \leq b.$$

Initial conditions, $y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = -6, y^{(4)}(0) = 24$. Exact solution: $y(t) = \frac{1}{1+t}, b = 0.1$.

4.2 Second proposed method

In this subsection we have tested three problems involving $y''''' = f(x, y, y', y'')$.

Example 4.4 (Homogenous ODE). $y^{(5)}(t) = \cos(t), \ \ 0 < t \leq b.$
Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0.$
Exact solution: $y(t) = \sin(t), b = 1.$

Example 4.5 (Homogenous ODE). $y^{(5)}(t) = y(t) + y'(t) + y''(t);, \ \ 0 < t \leq b.$
Initial conditions, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0.$
Exact solution: $y(t) = \sin(t), b = 0.1.$

Example 4.6 (Nonlinear ODE). $y^{(5)}(t) = y^6(t) + y'^3(t) - 30y''^2(t), \ \ 0 < t \leq b.$
Initial conditions, $y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = -6, y^{(4)}(0) = 24.$
Exact solution: $y(t) = \frac{1}{1+t}, b = 10.$

5. Discussion and conclusion

In this study, RKM integrators have been generalized for solving two classes of general quasi-linear fifth-order ODEs. The contribution of this work is the generalizing of the integrators for derivation of two numerical methods for solving these classes which are denoted as GRKM. The objective of this work is to establish direct explicit integrators of RK type for solving two classes of fifth-order ODEs. For this purpose, we generalized the integrators RK, RKN, RKD, RKT, RKM and RKFD which are used for solving special first-, second-, third-, fourth- and fifth-order ODEs. We have generalized three-stages, fifth-order and four-stages sixth-order RKM methods. Numerical results using the constructed methods have been compared with existing methods, see Figure 1. They have clearly shown the advantage and the efficiency of the new methods. They are accurate as well-known existing methods. However, they are more efficient as they require less function evaluations. As such, these methods are more cost effective, in terms of computation time, than existing methods. The new methods agree very well with well-known existing methods in the literature and required less function evaluations. As such, these methods are more cost effective in terms of computation time than other existing methods. Hence, we can conclude that the new GRKM pairs are computationally very efficient in solving fifth-order ODEs.

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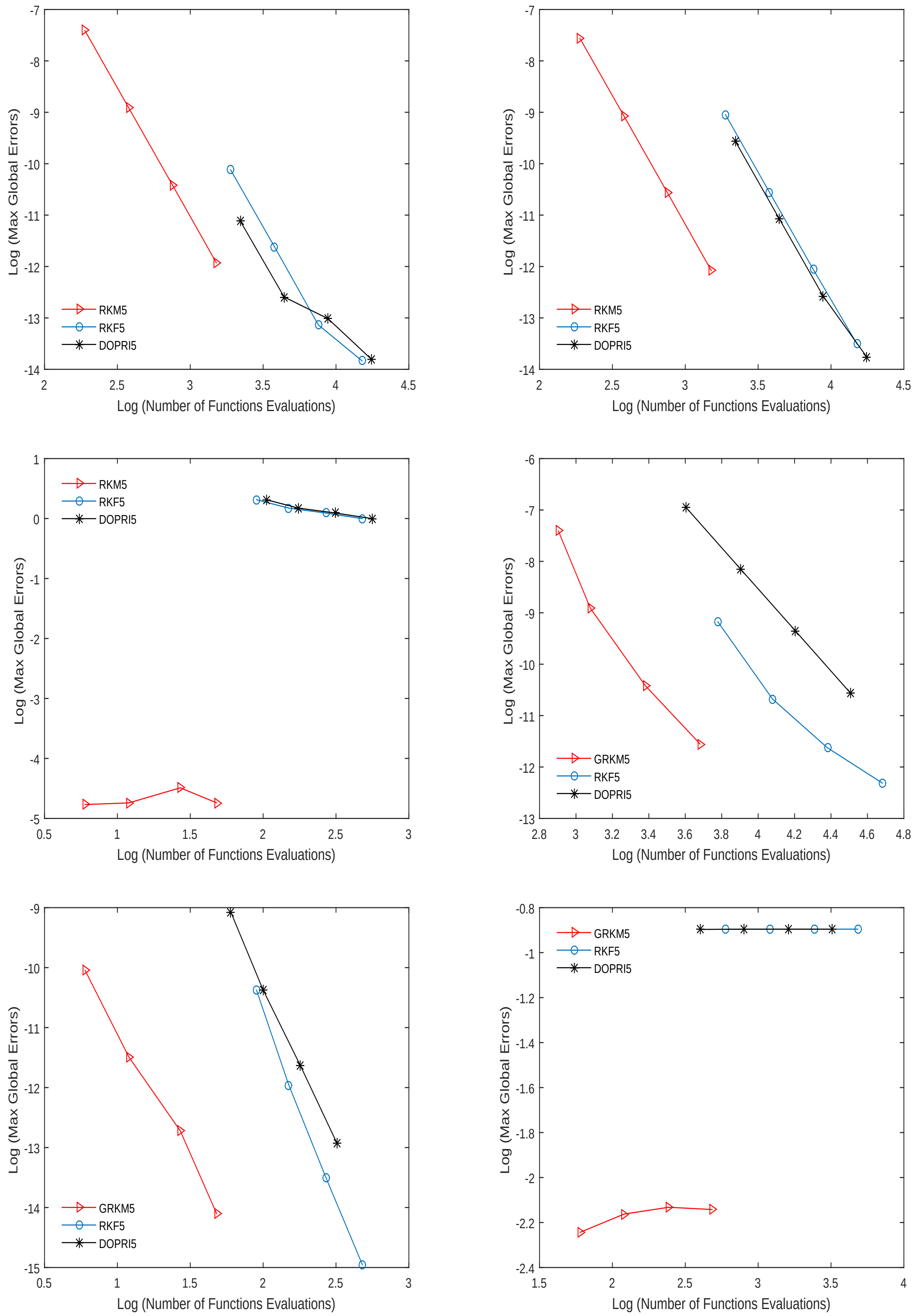

Figure 1: A comparison of numerical solutions for proposed with existing methods for examples (a) 4.1, (b) 4.2, (c) 4.3 and (d) 4.4, (e) 4.5 and (f) 4.6

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