A new study on rough soft lattices based on ideals

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Abstract. By using a novel relation θ_I based on an ideal I of a lattice L, which is a congruence relation, we investigate the roughness of soft lattices under this special ideal of L, such as rough soft sublattices, rough soft ideals and rough soft filters.
Keywords: lattice, ideal, filter, rough soft set, rough soft lattice (ideal, filter).

1. Introduction

In 1982, Pawlak [23] introduced the concept of rough set theory, a new mathematical approach to deal with uncertain knowledge, has recently received widespread attention on the research clues in both of theory and applications. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applications. Thus, some more general models have been proposed, such as [31, 32, 33]. Nowadays, rough set theory has been applied to many areas, such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [12, 11, 28]. In particular, many researchers applied this theory to algebraic structures. For example, Biswas and Nanda [4] proposed the notion of rough subgroups. Jun [16] studied roughness of ideals in BCK-algebras. Davvaz [9] studied roughness of rings. Xiao and Zhang [26] introduced the concept of rough prime ideals and rough fuzzy prime ideals in a semigroups. Liu [18] investigated the algebraic structures of generalized rough set theory. Davvaz and Mahdavipour [10] investigated rough modules. Rasouli and Davvaz [24] investigated roughness in MV-algebra. Ali et al. [3] investigated roughness in hemirings. Further, some authors applied rough set theory in a fuzzy algebraic system. For example, Kuroki [17] studied lower and upper

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approximations in a fuzzy group. In particular, Davvaz [9] constructed a t-level relation based on a fuzzy ideal and showed that $U(\mu, t)$ is a congruence relation on rings. In 2016, Zhan et al. [29] investigated roughness in n-ary semigroups based on fuzzy ideals.

In 1999, Molodtsov [21] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, the research on soft set theory is progressing rapidly. In 2003, Maji et al. [19] proposed some basic operations. Further, Ali et al. [1] revised some operations. In 2011, Ali [3] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [5, 6, 7, 14, 15, 20, 25]. Zhan and Zhu [30] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [14, 15]. In particular, in 2017, Zhu and Hu [34] investigated the relationship among soft sets, rough sets, fuzzy sets and lattices. The notion of soft rough fuzzy lattices (ideals, filters) over lattices was introduced. In 2018, Zhu and Hu [35] introduced the notion of Z-soft rough fuzzy BCI-algebras (ideals). Moreover, they studied roughness in BCI-algebras with respects to a Z-soft approximation space. Some new Z-soft rough fuzzy operations over BCI-algebras were explored. In particular, Z-lower and Z-upper soft rough fuzzy BCI-algebras (ideals) were investigated.

In mathematics, a lattice is a poset in which any two elements have a unique supremum and an infimum. We can also characterize lattices as algebraic structures satisfying certain axiomatic identities. Based on [9], in this paper, we consider the relationships between rough sets, soft sets and lattices. By using a special relation θ_I based on an ideal I of a lattice L, we present a definition of the lower and upper approximations of a subset of a lattice with respect to an ideal. Further, we characterize rough soft lattices (ideals, filters).

This paper is organized as follows. In Section 2, we recall some concepts and results on lattices, soft sets and rough sets. In Section 3, we study some operations of lower and upper approximations of lattices. Rough soft lattices (ideals, filters) based on ideals are investigated in Section 4.

2. Preliminaries

In this section, we recall some basic notions and results about lattices, soft sets and rough sets.

2.1 Lattices

A poset $(L, \leq)$ is called a lattice if it satisfies the condition that for any α, β in L, both $\alpha \vee \beta \in L$ and $\alpha \wedge \beta \in L$. L is said to be distributive if it satisfies the distributive law: $\forall x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 2.1 ([8])**.** Let L be a lattice and $\emptyset \subsetneq X \subseteq L$. Then X is a sublattice of L if $x, y \in L$, $x \vee y \in X$ and $x \wedge y \in X$.

Definition 2.2 ([8])**.** Let L be a lattice and $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if:

(1) $a, b \in I$ implies $x \vee y \in I$,

(2) $a \in L, b \in I$ and $a \leq b$ imply $a \in I$.

Definition 2.3 ([8])**.** Let L be a lattice and $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if:

(1) $a, b \in F$ implies $x \wedge y \in I$,

(2) $a \in L, b \in F$ and $a \geq b$ imply $a \in F$.

2.2 Soft sets

Definition 2.4 ([21])**.** A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \longrightarrow P(U)$ is a set-valued mapping.

For a soft set (F, A), the set $\text{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A). Hence, a null soft set is indeed a soft set with an empty support, and we say that a soft set (F, A) id non-null if $\text{Supp}(F, A) \neq \emptyset$.

Definition 2.5 ([13])**.** Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U. The inclusion symbol "$\subseteq$" of $\mathfrak{S}$ and $\mathfrak{I}$ by $\mathfrak{S} \subseteq \mathfrak{I}$, is defined as:

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.6 ([1])**.** Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U. The inclusion symbol "$\subseteq$" of $\mathfrak{S}$ and $\mathfrak{I}$ by $\mathfrak{S} \subseteq \mathfrak{I}$, is defined as:

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.7 ([1])**.** Let U be an initial universe set, E be the universe set of parameters and $A \subseteq E$.

(1) (N, A) is called a relative null soft set if $N(x) = \emptyset$.

(2) (W, A) is called a relative whole soft set if $W(x) = U$ for all $x \in A$.

(3) The relative complement of soft set $\mathfrak{S} = (F, A)$, denoted by $(F, A)^r$, is defined as the soft set (F^r, A), where $F^r(x) = U - F(x)$ for all $x \in A$.

Definition 2.8 ([1])**.** Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be two soft sets over a common universe U.

(1) The restricted intersection of (F, A) and (G, B), denoted by $(F, A) \Cap (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$, and $H(x) = F(x) \cap G(x)$ for all $x \in C$.

(2) The extended intersection of (F, A) and (G, B), denoted by $(F, A)\widetilde{\sqcap}(G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

(3) The restricted union of (F, A) and (G, B), denoted by $(F, A) \Cup (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $H(x) = F(x) \cup G(x)$ for all $c \in C$.

(4) The extended union of (F, A) and (G, B), denoted by $(F, A)\widetilde{\sqcup}(G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

2.3 Rough sets

Definition 2.9 ([23])**.** Let ρ be an equivalence relation on the universe U and (U, ρ) be a Pawlak approximation space. A subset $A \subseteq U$ is called definable if $\underline{\rho}(A) = \overline{\rho}(A)$; in the opposite case, i.e., if $\underline{\rho}(A) \neq \overline{\rho}(A)$, A is said to be a rough set, where two operators are defined as:

$$\underline{\rho}(A) = \{x \in U | [x]_\rho \subseteq A\},$$

$$\overline{\rho}(A) = \{x \in U | [x]_\rho \cap A \neq \emptyset\}.$$

Definition 2.10 ([13])**.** Let (U, ρ) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (U, ρ) are denoted by $\underline{\rho}(\mathfrak{S}) = (\underline{F}, A)$ and $\overline{\rho}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with

$$\underline{F}(x) = \underline{\rho}(F(x)) = \{y \in U | [y]_\rho \subseteq F(x)\},$$

$$\overline{F}(x) = \overline{\rho}(F(x)) = \{y \in U | [y]_\rho \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

If $\underline{\rho}(\mathfrak{S}) = \overline{\rho}(\mathfrak{S})$, $\mathfrak{S}$ is called definable; otherwise, $\mathfrak{S}$ is called a rough soft set.

Definition 2.11 ([22])**.** Let (F, A) be a soft set over a lattice L. Then (F, A) is said to be a soft lattice (ideal, filter) over L if $F(x)$ is a sublattice (ideal, filter) of L, for all $x \in A$.

Theorem 2.12 ([27])**.** Let I be an ideal of a lattice L and define a relation θ_I on L by

for all $a, b \in L$, $a \equiv b(\text{mod } \theta_I) \Leftrightarrow \exists d \in I \ s.t. \ a \vee d = b \vee d.$

Then the following statement hold:

(1) θ_I is a join-congruence on L;

(2) θ_I is a congruence on L if and only L is distributive.

3. Rough soft lattices based on ideals

In this section, we introduce the concept of rough soft lattices based on ideals, and we investigate the operations of lower and upper rough approximations of soft sets over lattices.

An equivalence relation θ on L is called a joint-congruence and meet-congruence if, for all $a, b, c, d \in L$, $a \equiv b(\text{mod } \theta)$ and $c \equiv d(\text{mod } \theta)$ imply $a \vee c \equiv b \vee d(\text{mod } \theta)$ and $a \wedge c \equiv b \wedge d(\text{mod } \theta)$, respectively. θ is called congruence relation if it is both a joint-congruence and a meet-congruence. It is obvious that $\{x \vee y | x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \vee b]_\theta$, $\{x \wedge y | x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \wedge b]_\theta$. Let A and B be subsets of L. We define the join and meet of two non-empty subsets in a lattice as follows:

$$A \vee B = \{a \vee b | a \in A, b \in B\}, A \wedge B = \{a \wedge b | a \in A, b \in B\}.$$

In the following, L is always a distributive lattice.

Let I be an ideal of L. It follows from Theorem 2.12 that θ_I is a congruence on L. Therefore, when $U = L$ and ρ is the above equivalence relation, then we use (L, θ_I) instead of approximation space (U, ρ).

Definition 3.1. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over L. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (L, θ_I) are denoted by: $\underline{\theta_I}(\mathfrak{S}) = (\underline{F}_I, A)$ and $\overline{\theta_I}(\mathfrak{S}) = (\overline{F}_I, A)$, respectively, which are soft sets over L with

$$\underline{F}_I(x) = \underline{\theta_I}(F(x)) = \{y \in U | [y]_{\theta_I} \subseteq F(x)\},$$

$$\overline{F}_I(x) = \overline{\theta_I}(F(x)) = \{y \in U | [y]_{\theta_I} \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

(1) If $\underline{\theta_I}(\mathfrak{S}) = \overline{\theta_I}(\mathfrak{S})$, $\mathfrak{S}$ is called definable;

(2) If $\underline{\theta_I}(\mathfrak{S}) \neq \overline{\theta_I}(\mathfrak{S})$, $\underline{\theta_I}(\mathfrak{S})$ $(\overline{\theta_I}(\mathfrak{S}))$ is called a lower (upper) rough soft set. Moreover, $\mathfrak{S}$ is called a rough soft set.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ with $0 \leq a, b \leq c \leq 1$. Obviously, L is a distributive lattice. Let $I = \{0, a\}$ be an ideal of L. Then $[1]_{\theta_I} = \{1\}$, $[c]_{\theta_I} = \{b, c\}$, $[a]_{\theta_I} = \{0, a\}$. Define a soft set $\mathfrak{S} = (F, A)$ over L, $A = \{\alpha, \beta\}$, where $F(\alpha) = \{b, 1\}$, $F(\beta) = \{0, b, c\}$. It follows from Definition 3.1 that $\underline{F}_I(\alpha) = \{1\}$, $\underline{F}_I(\beta) = \{b, c\}$, $\overline{F}_I(\alpha) = \{1, b, c\}$, $\overline{F}_I(\beta) = \{0, a, b, c\}$.

Definition 3.3. In Definition 3.1, if $\underline{\theta_I}(\mathfrak{S}) \neq \overline{\theta_I}(\mathfrak{S})$, then $\mathfrak{S}$ is called a rough soft lattice (ideal, filter) if $\underline{F}_I(x)$ and $\overline{F}_I(x)$ are sublattices (ideals, filters) of L, for all $x \in A$.

Example 3.4. Consider Example 3.2. We define a soft set $\mathfrak{S} = (F, A)$ over L, $A = \{\alpha\}$, where $F(\alpha) = \{0, a\}$. It follows from Definition 3.1 that $\underline{F}_I(\alpha) = \{0, a\}$, $\underline{F}_I(\beta) = \{0, a\}$. It follows from Definition 2.2 that $\underline{F}_I(\alpha) = \{0, a\}$ and $\underline{F}_I(\beta) = \{0, a\}$ are ideals of L, for all $x \in A$. It follows from Definition 3.3 that $\mathfrak{S} = (F, A)$ is a rough soft ideal of L.

Proposition 3.5. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L. Then:

(1) $\underline{\theta_I}(\mathfrak{S}) \subseteq \mathfrak{S} \subseteq \overline{\theta_I}(\mathfrak{S})$;
(2) $\underline{\theta_I}((N, A)) = (N, A) = \overline{\theta_I}((N, A))$;
(3) $\underline{\theta_I}((W, A)) = (W, A) = \overline{\theta_I}((N, A))$;
(4) $\underline{\theta_I}(\underline{\theta_I}(\mathfrak{S})) = \underline{\theta_I}(\mathfrak{S})$;
(5) $\overline{\theta_I}(\overline{\theta_I}(\mathfrak{S})) = \overline{\theta_I}(\mathfrak{S})$;
(6) $\overline{\theta_I}(\underline{\theta_I}(\mathfrak{S})) = \underline{\theta_I}(\mathfrak{S})$;
(7) $\underline{\theta_I}(\overline{\theta_I}(\mathfrak{S})) = \overline{\theta_I}(\mathfrak{S})$;
(8) $\underline{\theta_I}(\mathfrak{S}) = (\overline{\theta_I}(\mathfrak{S})^r)^r$;
(9) $\overline{\theta_I}(\mathfrak{S}) = (\underline{\theta_I}(\mathfrak{S})^r)^r$.

Proposition 3.6. Let (L, θ_I) be a Pawlak approximation space, $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be soft lattices over L. Then:

(1) $\underline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I}) = \underline{\theta_I}(\mathfrak{S}) \Cap \underline{\theta_I}(\mathfrak{I})$;
(2) $\underline{\theta_I}(\mathfrak{S} \widetilde{\sqcap} \mathfrak{I}) = \underline{\theta_I}(\mathfrak{S}) \widetilde{\sqcap} \underline{\theta_I}(\mathfrak{I})$;
(3) $\overline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}) \Cap \underline{\theta_I}(\mathfrak{I})$;
(4) $\overline{\theta_I}(\mathfrak{S} \widetilde{\sqcap} \mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}) \widetilde{\sqcap} \overline{\theta_I}(\mathfrak{I})$;
(5) $\underline{\theta_I}(\mathfrak{S} \Cup \mathfrak{I}) \supseteq \underline{\theta_I}(\mathfrak{S}) \Cup \underline{\theta_I}(\mathfrak{I})$;
(6) $\underline{\theta_I}(\mathfrak{S} \widetilde{\sqcup} \mathfrak{I}) \supseteq \underline{\theta_I}(\mathfrak{S}) \widetilde{\sqcup} \underline{\theta_I}(\mathfrak{I})$;
(7) $\overline{\theta_I}(\mathfrak{S} \Cup \mathfrak{I}) = \overline{\theta_I}(\mathfrak{S}) \Cup \overline{\theta_I}(\mathfrak{I})$;
(8) $\overline{\theta_I}(\mathfrak{S} \widetilde{\sqcup} \mathfrak{I}) = \overline{\theta_I}(\mathfrak{S}) \widetilde{\sqcup} \overline{\theta_I}(\mathfrak{I})$;
(9) $\mathfrak{S} \subseteq \mathfrak{I} \Rightarrow \underline{\theta_I}(\mathfrak{S}) \subseteq \underline{\theta_I}(\mathfrak{I}), \overline{\theta_I}(\mathfrak{S}) \subseteq \overline{\theta_I}(\mathfrak{I})$;

Definition 3.7. Let $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be any two soft sets over L. Then:

(1) The $\widetilde{\vee}$ of $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$, denoted by $\mathfrak{S} \widetilde{\vee} \mathfrak{I} = (F \vee G, A \times B)$, where $(F \vee G)(x, y) = F(x) \vee G(y)$, for all $(x, y) \in A \times B$.

(2) The $\widetilde{\wedge}$ of $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$, denoted by $\mathfrak{S} \widetilde{\wedge} \mathfrak{I} = (F \wedge G, A \times B)$, where $(F \wedge G)(x, y) = F(x) \wedge G(y)$, for all $(x, y) \in A \times B$.

Theorem 3.8. Let I be an ideal of L, $\mathfrak{S} = (F, A)$ and $\mathfrak{I} = (G, B)$ be any two non-null soft sets over L. Then:
(1) $\overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\vee}\mathfrak{I})$
(2) $\overline{\theta_I}(\mathfrak{S})\widetilde{\wedge}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\wedge}\mathfrak{I})$

Proof. (1) For $m \in \mathrm{Supp}(F, A)$, $n \in \mathrm{Supp}(G, B)$. Let $t \in \overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I})$. Then $t = p \vee q$, where $p \in \overline{\theta_I}(\mathfrak{S})$ and $q \in \overline{\theta_I}(\mathfrak{I})$. Hence, there exist $a, b \in L$ such that $a \in [p]_{\theta_I} \cap F(m)$, $b \in [q]_{\theta_I} \cap G(n)$. Since θ_I is a congruence on L, we have $a \vee b \in [p]_{\theta_I} \vee [q]_{\theta_I} \subseteq [p \vee q]_{\theta_I}$. On the other hand, since $a \vee b \in F(m) \vee G(n)$, we have $a \vee b \in [p \vee q]_{\theta_I} \cap (F(m) \vee G(n))$. This implies $p \vee q \in \overline{\theta_I}(F \vee G, A \times B))$, i.e., $t \in \overline{\theta_I}(F \vee G, A \times B)$. Therefore, $\overline{\theta_I}(\mathfrak{S})\widetilde{\vee}\overline{\theta_I}(\mathfrak{I}) \subseteq \overline{\theta_I}(\mathfrak{S}\widetilde{\vee}\mathfrak{I})$.
(2) The proof is similar to that of (1). □

The following example shows that the containment in Theorem 3.8 (1) is proper.

Example 3.9. Consider Example 3.2. It is easy to check that $\overline{\theta_I}(F(\alpha))\widetilde{\vee}\overline{\theta_I}(F(\beta)) = \{1, b, c\}$ and $\overline{\theta_I}(F(\alpha)\widetilde{\vee}F(\beta)) = \{0, b, c, 1\}$. Thus, $\overline{\theta_I}(F(\alpha))\widetilde{\vee}\overline{\theta_I}(F(\beta)) \subsetneq \overline{\theta_I}(F(\alpha)\widetilde{\vee}F(\beta))$.

In a similar way, we can check that the containment in Theorem 3.8 (2) is proper.

4. Characterizations of rough soft lattices (ideals, filters) of lattices based on ideals

In this section, we investigate rough soft lattices (ideals, filters) of lattices based on ideals.

Theorem 4.1. Let (L, θ_I) be a Pawlak approximation space, $\underline{\theta_I}(\mathfrak{S})$ and $\underline{\theta_I}(\mathfrak{I})$ be lower rough soft lattices (ideals, filters) over L. If $\mathfrak{S} \Cap \mathfrak{I}$ and $\mathfrak{S}\widetilde{\sqcap}\mathfrak{I}$ are non-null soft sets, then $\underline{\theta_I}(\mathfrak{S} \Cap \mathfrak{I})$ and $\underline{\theta_I}(\mathfrak{S}\widetilde{\sqcap}\mathfrak{I})$ are lower rough soft lattices (ideals, filters) over L.

Proof. It follows from Proposition 3.6 (1) and (2). □

Theorem 4.2. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L. Then $\overline{\theta_I}(\mathfrak{I})$ is a soft lattice.

Proof. Let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft lattice over L, we have $F(x)$ is a sublattice of L for all $x \in$Supp(F, A). Thus, $a \vee b \in F(x)$ and $a \wedge b \in F(x)$. On the other hand, $a \vee b \in [m]_{\theta_I} \vee [n]_{\theta_I} \subseteq [m \vee n]_{\theta_I}$ and $a \wedge b \in [m]_{\theta_I} \wedge [n]_{\theta_I} \subseteq [m \wedge n]_{\theta_I}$. Hence, $a \vee b \in [m \vee n]_{\theta_I} \cap F(x)$ and $a \wedge b \in [m \wedge n]_{\theta_I} \cap F(x)$, which implies that $m \vee n \in \overline{\theta_I}(F(x))$ and $m \wedge n \in \overline{\theta_I}(F(x))$. Thus, for all $x \in$Supp(F, A), $\overline{\theta_I}(F(x))$ is a sublattice of L. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft lattice. □

Theorem 4.3. Let (L, θ_I) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft lattice over L.

(1) If $\mathfrak{S} = (F, A)$ is a soft ideal of L, then $\overline{\theta_I}(\mathfrak{I})$ is a soft ideal of L.

(2) If $\mathfrak{S} = (F, A)$ is a soft filter of L, then $\overline{\theta_I}(\mathfrak{I})$ is a soft filter of L.

Proof. (1) For all $x \in$Supp(F, A), let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft ideal of L, we have $F(x)$ is an ideal of L for all $x \in$Supp(F, A). Thus $a \vee b \in F(x)$. On the other hand, $a \vee b \in [m]_{\theta_I} \vee [n]_{\theta_I} \subseteq [m \vee n]_{\theta_I}$. Hence, $a \vee b \in [m \vee n]_{\theta_I} \cap F(x)$, which implies that $m \vee n \in \overline{\theta_I}(F(x))$. Let $m \in L, n \in \overline{\theta_I}(F(x))$ and $m \leq n$. Then there exist $a \in [n]_{\theta_I} \cap F(x)$. Let $b \in [m]_{\theta_I}$. Then $a \wedge b \in [n]_{\theta_I} \wedge [m]_{\theta_I} \subseteq [n \wedge m]_{\theta_I} = [m]_{\theta_I}$. Since $a \wedge b \leq b$, we have $a \wedge b \in F(x)$. Hence, $[m]_{\theta_I} \cap F(x) \neq \emptyset$, which implies $m \in \overline{\theta_I}(F(x))$. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft ideal of L.

(2) For all $x \in$Supp(F, A), let $m, n \in \overline{\theta_I}(F(x))$. Then $[m]_{\theta_I} \cap F(x) \neq \emptyset$ and $[n]_{\theta_I} \cap F(x) \neq \emptyset$. Hence, there exist $a \in [m]_{\theta_I} \cap F(x)$ and $b \in [n]_{\theta_I} \cap F(x)$. Since $\mathfrak{S} = (F, A)$ is a soft filter of L, we have $F(x)$ is a filter of L for all $x \in$Supp(F, A). Thus $a \wedge b \in F(x)$. On the other hand, $a \wedge b \in [m]_{\theta_I} \wedge [n]_{\theta_I} \subseteq [m \wedge n]_{\theta_I}$. Hence, $a \wedge b \in [m \wedge n]_{\theta_I} \cap F(x)$, which implies that $m \wedge n \in \overline{\theta_I}(F(x))$. Let $m \in L, n \in \overline{\theta_I}(F(x))$ and $m \geq n$. Then there exist $a \in [n]_{\theta_I} \cap F(x)$. Let $b \in [m]_{\theta_I}$. Then $a \vee b \in [n]_{\theta_I} \vee [m]_{\theta_I} \subseteq [n \vee m]_{\theta_I} = [m]_{\theta_I}$. Since $a \vee b \geq b$, we have $a \vee b \in F(x)$. Hence, $[m]_{\theta_I} \cap F(x) \neq \emptyset$, which implies $m \in \overline{\theta_I}(F(x))$. Therefore, $\overline{\theta_I}(\mathfrak{I})$ is a soft filter of L. □

5. Conclusion

In this paper, we built up a connection between rough sets, soft sets and lattices. We introduced a new congruence relation induced by an ideal of a lattice, by using a novel congruence relation, we investigate the roughness of soft lattices under this special ideal of L, such as rough soft sublattices, rough soft ideals and rough soft filters.

As an extension of this work, the following problems maybe considered:

(I) Roughness of lattices based on fuzzy ideals;

(II) Rough prime ideals and rough fuzzy prime ideals based on fuzzy ideals;

(III) Rough sets induced by fuzzy ideal in lattices.

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