Legendre-adomian-homotopy analysis method for solving multi-term nonlinear differential equations of fractional order

O.H. Mohammed
Department of Mathematices
Al-Nahrain University
Baghdad
Iraq
dr_usama79@yahoo.com
uhm@sc.nahrainuniv.edu.iq

DH.A. Jaleel*
Department of Mathematices
Al-Nahrain University
Baghdad
Iraq
duha2015555@gmail.com

Abstract. In this article a modified approach using shifted Legendre-Adomian Homotopy analysis method is applied in order to solve multi-term fractional order nonlinear differential equations. This approach enable us to integrate the deformation equation in an easily manner, moreover the computations connected with the algorithm are so elementary than the standard Homotopy analysis method. The fractional derivatives are described in the Caputo's sense; the results of applying this procedure to the studied case showed that the proposed method is accurate and effective when it's compared with the usual Homotopy analysis method.

Keywords: fractional order differential equation (FDEs), shifted Legendre polynomials, homotopy analysis method (HAM).

1. Introduction

Differential equations stand out dramatically in the applications of physics, chemistry and even mathematical models of biological, social and even economic process. However, differential equations fail some times to formulate or describe like phenomena,so, to get best models, FDEs are appointed in state integer order ones [1], [2], [3], [4]. Fractional differential equations have been utilized in different area of science, engineering. applied mathematics et al. they are popularization of differential equations that exchange integral order derivatives by fractional order derivatives [5], [6]. Since the exact solutions of nonlinear FDE's are difficult to get it, therefore numerical and approximate approaches must be used, such as Adomian decomposition method [7], fractional differential

∗. Corresponding author

transformation method [8], homotopy analysis method [9], polynomials method [10], variational iteration method [11], wavelet method [12], and B-Spline operational matrix method [13]. The feasible interest of the HAM has aroused of many authors in recent years. it has been successfully performed to handle different kinds of non-linear, non-homogenous equations and system of equations as well as problems in science and engineering, see for instance, [14], [15], [16], [17], [18], [19] , [20], [21], [22], [23], etc. Legendre polynomials (LPs) have a wide implementations to integer and fractional order ordinary, partial and integral equations such as [24], [25], [26], [27], [28], [29], [30]. In this paper a modified approach which combined the HAM and the orthogonal shifted Legendre polynomials which is so-called Legendre-Homotopy analysis method (LHAM) have been introduced for solving the following class of multi-term fractional order differential equations:

$$ {}^{c}D_a^{\alpha_M} u(x) = f(x, u(x), D_a^{\alpha_1} u(x), ..., D_a^{\alpha_{M-1}} u(x)) \tag{1} $$

We assume that $\alpha_M > \alpha_{M-1} > ... > \alpha_1 > 0, \alpha_i - \alpha_{i-1} \leq 1,\ M \in N$, and $\alpha_i \in Q$ for all i, subject to initial conditions:

$$ u^{(k)}(0) = u_0{}^{(k)} \quad k = 0, 1, ..., \lceil \alpha_M \rceil - 1, \tag{2} $$

where $\lceil \alpha_M \rceil$ is the least integer greater than or equal to α_M.

2. Fractional order derivatives and integrals

Here we review basic definitions and properties of derivatives and integrals of fractional order [31].

Definition 1. *The fractional integral operator in the Riemann-Liouville sense of order $\alpha > 0$ of function y is given by:*

$$ I_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1}\, y(\tau)\, d\tau, \alpha > 0, & \tau > 0, \\ y(t), & \alpha = 0, \end{cases} $$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2. *The fractional derivative operator in the Caputo sense of order $\alpha > 0$ of function y, is given as:*

$$ {}^{c}D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1}\, y^{(n)}(\tau)\, d\tau, & n-1 < \alpha \leq n, \\ \dfrac{d^n y(t)}{dt^n}, & \alpha = n, \end{cases} $$

where n is an integer.

Some properties of the fractional differential equations that we will use later.

2.1 Properties

(1) ${}^{\alpha}I_t t^{\beta} = (\Gamma(\beta+1)/\Gamma(\beta+\alpha+1))t^{\alpha+\beta}$, $\beta > -1, \alpha > 0$.

(2) ${}^{c}D_t^{\alpha} t^{\beta} = (\Gamma(\beta+1)/\Gamma(\beta-\alpha+1))t^{\beta-\alpha}, \beta > -1, \alpha > 0$.

(3) ${}^{c}D_t^{\alpha}(I_t^{\alpha} y(t)) = y(t)$.

(4) $I_t^{\alpha}({}^{c}D_t^{\alpha} y(t)){=}y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0^+)}{k!} t^k, t \geq 0, n-1 < \alpha \leq n$.

3. Shifted Legendre polynomials

LPs which is denoted by $L_j(x)$ are outlined through the interval [-1,1] and satisfy the following recurrence relations:

(3) $L_{j+1}(x) = ((2j+1)/(j+1))xL_j(x) - (j/(j+1))L_{j-1}(x), \quad j = 1, 2, 3, ...,$

where $L_0(x) = 1$ and $L_1(x) = x$.

The shifted Legendre polynomials (SLPs) over the interval [0,1] are defined as $L_j^*(x) = L_j(2x-1)$. And satisfy the recurrence relation

(4) $L_{j+1}^*(x){=}((2j+1)/(j+1))(2x-1)L_j^*(x) - (j/(j+1))L_{j-1}^*(x), j = 1, 2, 3, ...,$

where $L_0^*(x) = 1$ and $L_1^*(x) = 2x - 1$.

Since LPs are orthogonal, therefore any function $f(x) \in L_2[0,1]$ can be decomposed in terms of SLPs, as

$$f(x) = \sum_{k=0}^{\infty} c_k L_k^*(x), \tag{5}$$

where $c_k' s$ can be calculated, as

$$c_k = (2k+1)\int_0^1 f(x)L_k^*(x)dx. \tag{6}$$

4. The approach

In this section we shall construct a series solution corresponding to the multi-term FDEs of the form:

$${}^{c}D_a^{\alpha_M} u(x) = f(x, u(x), D_a^{\alpha_1} u(x), ..., D_a^{\alpha_{M-1}} u(x)), \quad x \in [a, b], \tag{7}$$

where $\alpha_M > \alpha_{M-1} > ... > \alpha_1 > 0, \alpha_i - \alpha_{i-1} \leq 1, M \in N$, and $\alpha_i \in Q$ for all i. Subject to the initial conditions:

$$u^{(k)}(0) = {u_0}^{(k)} \quad k = 0, 1, ..., \lceil \alpha_M \rceil - 1 \tag{8}$$

in vision of the homotopy technique, a zero order deformation equations can be defined using an embedding parameter $q \in [0,1]$ by $(1-q)\text{Ł}[\phi(x,q) - u_0(x)] = qhH(x)[{}^cD_a^{\alpha_M}\phi(x,q) - f(x,\phi(x,q),{}^cD_a^{\alpha_1}\phi(x,q),...,$

$$ {}^cD_a^{\alpha_{M-1}}\phi(x,q))], \tag{9} $$

where $h \neq 0$ and $H(x) \neq 0$ are an auxiliary parameter and function respectively, and Ł an auxiliary linear operator with property $\text{Ł}[u(x)] = 0$ when $u(x) = 0$. Its clear that when $q = 0$ and $q = 1$, we have $\phi(x,0) = u_0(x)$ and $\phi(x,1) = u(x)$.

In the proposed method an initial approximation $u_0(x)$ can be defined as:

$$ u_0(x) = \sum_{j=0}^{m} a_{0j} L_j^*(x), \tag{10} $$

$$ a_{0j} = (2j+1)\int_0^1 U_0(x) L_j^*(x)dx, \tag{11} $$

where $U_0(x)$ is the solution of the equation ${}^cD_a^{\alpha_M}u(x) = 0$, with respect to the conditions given by eq. (8).

The first order deformation equations can be expressed using shifted Legendre and Adomian polynomials, as:

$$ \begin{aligned} \text{Ł}[u_1(x)] &= hH(x)[{}^cD_a^{\alpha_M}(\sum_{j=0}^{m} a_{0j}L_j^*(x)) \\ &- A_0(x,(\sum_{j=0}^{m} a_{0j}L_j^*(x)),{}^cD_a^{\alpha_1}(\sum_{j=0}^{m} a_{0j}L_j^*(x)),...,{}^cD_a^{\alpha_{M-1}}(\sum_{j=0}^{m} a_{0j}L_j^*(x))] \end{aligned} \tag{12} $$

and for $i = 1,2,3,,$ the deformation equations can be utilized, as:

$$ \begin{aligned} \text{Ł}[u_{i+1}(x)] &= \text{Ł}[u_i(x)] + hH(x)[{}^cD_a^{\alpha_M}(\sum_{j=0}^{m} a_{ij}L_j^*(x)) \\ &- A_i(x,(\sum_{j=0}^{m} a_{ij}L_j^*(x)), D_a^{\alpha_1}(\sum_{j=0}^{m} a_{ij}L_j^*(x)),..., D_a^{\alpha_{M-1}}(\sum_{j=0}^{m} a_{ij}L_j^*(x))], \end{aligned} \tag{13} $$

where

$$ a_{ij} = (2j+1)\int_0^1 u_i(x)L_j^*(x)dx \quad i = 1,2,3,..., \quad j = 0,1,2,...,m \tag{14} $$

and A_i are the Adomian polynomials of $x, u(x),{}^cD_a^{\alpha_1}u(x),...,{}^cD_a^{\alpha_{M-1}}u(x)$ defined by

$$ \begin{aligned} &A_n(x,u(x),{}^cD_a^{\alpha_1}u(x),...,{}^cD_a^{\alpha_{M-1}}u(x)) \\ &= (1/n!)(d^n/d\lambda^n)[f(x,\sum_{k=0}^{n}\lambda^k u_k,\sum_{k=0}^{n}\lambda^k\,{}^cD_a^{\alpha_1}u_k,..., \\ &\sum_{k=0}^{n}\lambda^k\,{}^cD_a^{\alpha_{M-1}}u_k)]_{\lambda=0}, n = 0,1,2,... \end{aligned} \tag{15} $$

Finally, the exact solution $u(x)$ of problem (7)-(8) can be calculated as $u(x) = \sum_{i=0}^{\infty} u_i(x)$ and the n^{th} order analytical approximate solution can be calculated as:

$$u_n(x) = \sum_{i=0}^{n} u_i(x). \tag{16}$$

5. Illustrative examples

Example 1. Consider the following multi-term nonlinear higher-order non-homogenous FDEs,

$$\begin{aligned} &{}^cD_a^{2.2}u(x) +{}^c D_a^{1.25}u(x) +{}^c D_a^{0.75}u(x) + u(x)^3 \\ &= 2x^{0.8}/\Gamma(1.8) + 2x^{1.75}/\Gamma(2.75) + 2x^{2.25}/\Gamma(3.25) + (x^3/3)^3. \end{aligned} \tag{17}$$

Subject to:

$$u(0) = u^{(1)}(0) = u^{(2)}(0) = 0 \tag{18}$$

the closed form of problem (17)-(18) is $u(x) = x^3/3$.

The solution curves of problem (17)-(18) are plotted in Figure 1 using the proposed method, the exact solution and results were resulting from the usual HAM.

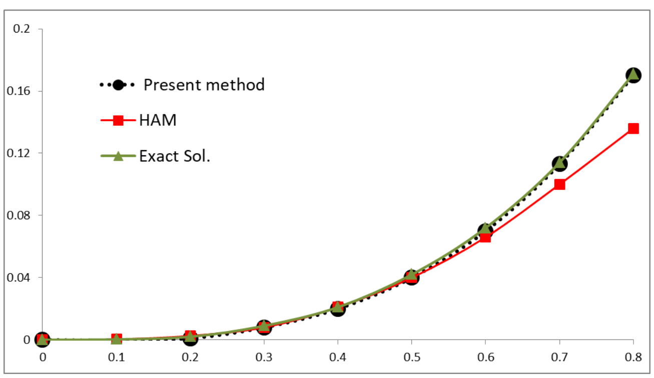

Figure 1: The proposed solution of problem (17)-(18)

Example 2. Consider the following linear nonhomogeneous FDE,

$${}^cD_a^2u(x) +{}^c D_a^{3/4}u(x) + u(x) = x^3 + 6x + 8.533x^{2.25}/\Gamma(0.25) \tag{19}$$

with

$$u(0) = u^{(1)}(0) = 0 \tag{20}$$

the closed form of problem (19)-(20) is $u(x) = x^3$.

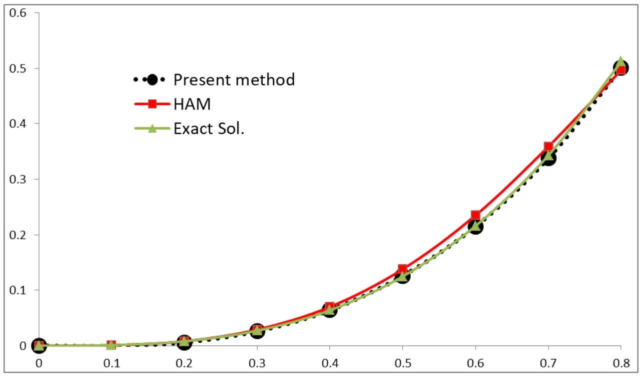

Figure 2: The proposed solution of problem (19)-(20)

The solution curves of problem (19)-(20) are plotted in Figure 2 using the proposed method, the exact solution and results were resulting from the usual HAM.

Example 3. Consider the following multi-term nonlinear non homogeneous FDE,

$$\begin{aligned} &{}^{c}D^{2}u(x) + 2\ D^{0.07621}u(x) + (1/2)\ D^{0.00196}u(x) + u(x)^{3} \\ &= 2x + 4x^{2.92379}/\Gamma(3.92379) + x^{2.99804}/\Gamma(3.99804) + x^{9}/27 \end{aligned} \tag{21}$$

with

$$u(0) = 0, \quad u^{(1)}(0) = 1 \tag{22}$$

the exact solution of problem (21)-(22) is $u(x) = (x^3)/3$.

The solution curves of problem (21)-(22) are plotted in Figure 3 using the proposed method, the exact solution and results were resulting from the usual HAM.

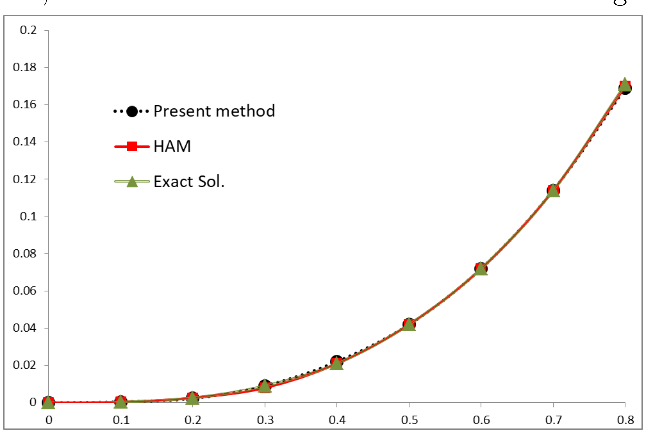

Figure 3: The proposed solution of problem (21)-(22)

6. Conclusions

In this work we have been used a modified approach consist of, the HAM, Adomian polynomials and the SLPs which is so called (SLAHAM) for solving a certain class of multi-term nonlinear FDEs. in the proposed method the deformation equations are easily integrable especially its decomposed the nonlinear term using Adomian polynomials therefore the computations connected with the algorithm are straightforward than the usual HAM. three examples are presented in order to illustrate the effectiveness and accuracy of the numerical scheme.

References

[1] A.M.A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal., 33 (1998), 181-186.

[2] H.G. Sun, W. Chen, H. Sheng, Y.Q. Chen, *On mean square displacement behaviors of anomalous diffusions with variable and random order*, Phys. Lett. A., 374 (2010), 906-910.

[3] I.L. El-Kalla, *Error estimate of the series solution to a class of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat., 16 (2011), 1408-1413.

[4] Z.D. Cen, A.B. Le, A.M. Xu, *A robust numerical method for a fractional differential equation*, Appl. Math. Comput., 315 (2017), 445-452.

[5] H. Sun, W. Chen, C. Li, *Fractional differential models for anomalous diffusion, Phys*, A-Stat. Mech. Appl., 389 (2012), 2719-2724.

[6] M.S. Semary, H.N. Hassan, A.G. Radwan, *Modified methods for solving two classes of distributed order linear fractional differential equations*, Appl. Math. Comput., 323 (2018), 106-119.

[7] M.M. Hosseini, *Adomian decomposition method for solution of nonlinear differential algebraic equations*, Appl. Math. Comput., 181 (2006), 1737-1744.

[8] S. Momani, Z. Odibat, *Generalized differential transform method for solving a space and time-fractional diffusion-wave equation*, Phys. Lett. A., 370 (2007), 379-387.

[9] S. Abbasbandy, A. Shirzadi, *The series solution of problems in the calculus of variations via the Homotopy analysis method*, Z Naturforsch A., 64 (2009), 30-6.

[10] A. Saadatmandi, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl., 59 (2010), 1326-1336.

[11] Z.M. Odibat, *A study on the convergence of variational iteration method*, Math. Comput. Model., 51 (2010), 1181-1192.

[12] Y.M. Chen, M.X. Yi, C.X. Yu, *Error analysis for numerical solution of fractional differential equation by Haar Wavelets method*, Comput. Sci., 5 (2012), 267-373.

[13] J. Hossein, T. Haleh, B. Dumitru, *A numerical approach for fractional order Riccati differential equation using B-Spline operational matrix*, Appl. Anal., 18 (2015), 387-399.

[14] H. Xu, J. Cang, *Analysis of a time fractional wave-like equation with the Homotopy analysis method*, phys. Lett. A., 372 (2008), 1250-5.

[15] M. Sajid, T. Hayat, *The application of Homotopy analysis method of thin film flows of a third order fluid*, Chaos Solitons Fractals, 38 (2008), 506-15.

[16] J. Cheng, SJ. Liao, RN. Mohapatra, K. Vajravelu, *Series solutions of nano boundary layer flows by means of Homotopy analysis method*,J. Math. Anal. Appl., 343 (2008), 233-45.

[17] AK. Alomari, MSM. Noorani, R. Nazar, *Solving the one-loop solution solution of the Vakhnenko equation by means of the Homopotopy analysis method*, Commun Nonlinear Sci Numer Simulat., 4 (2009), 2336-46.

[18] H. Jafari, S. Seifi, *Solving a system of nonlinear fractional partial differential equations using homotopy analysis method*, Commun Nonlinear Sci Numer Simulat., 14 (2009), 1962-9.

[19] S. Abbasbandy, E. Babolian, M. Ashtiani, *The series solution of problems in the calculus of variations via the Homotopy analysis method*, Commun Nonlinear Sci Numer Simulat., 14 (2009), 4114-21.

[20] A. Molabahrami, F. Khani, *The Homotopy analysis method to solve the Burgers-Hoxleyequation*, Nonlinear Anal Real World Appl., 10 (2009), 589-600.

[21] A.S. Bataineh, M.S.M. Noorani, I. Hashim, *Modified Homotopy analysis method for solving systems of second-order boundary value problems*, Commun. Nonlinear Sci.Numer simul., 14 (2009), 430-442.

[22] M. Zurigat, S. Momani, Z. Odibat, A. Al-Alawneh, *The Homotopy analysis method for handling systems of fractional differential equations*, Appl. Math.Modelling, 34 (2010), 24-35.

[23] O.H. Mohammed, *Solution of Fractional Integro-Differential equations by Homotopy analysis method*, Journal of Al-Nahrain University, 13 (2010), 149-155.

[24] S. Yalcinbas, M. Sezer, H.H. Sorkun, *Legendre polynomial solutions of high order linear fredholm integro-differential equations*, Applied Mathematics and Computation, 210 (2009), 334-349.

[25] Z. Odibat, *On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equation*, Journal of Computational and Applied Mathematics, 235 (2011), 2956-2968.

[26] A.H. Bhrawy, M.M. Al-Shomrani, *A shifted Legendre spectral method for fractional-order multi-point boundary value problems*, DOI: 10.1186/1687-1847-2012-8.

[27] N.H. Sweilam, M.M. Khader, W.Y. Kota, *On the numerical solution of Hammerstcin integral equations using Legendre approximation*, International Journal of Applied Mathematics Research, 1 (2012), 65-76.

[28] Y. Liu, *Application of Legendre polynomials in solving Volterra integral equations of the second kind*, Applied Mathematics, 3 (2013), 157-159.

[29] M.M. Khader, A.M.S. Mahdy, M.M. Shehata, *An integral collocation approach based on Legendre polynomials for solving Riccati, Logistic and Delay Differential Equations*, Applied Mathematics, 5 (2014), 2360-2369.

[30] J.E. Mamadu, I.N. Njoseh, *Numerical solutions of Volterra equations using Galerkin method with certain orthogonal polynomials*, Journal of Applied Mathematics and Physics, 4 (2016), 376-382.

[31] I. Podlubny, *Fractional differential equations*, New York, Academic Press, 1999.

Accepted: 18.06.2019