Dunkl-Williams inequality for operators associated with r-angular distance

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Abstract. In this paper, We present several operator versions of the Dunkl-Williams inequality with respect to the r-angular distance for operators. We obtain refinements of some operator inequalities presented by Jiang and Zou.
Keywords: Dunkl-Williams inequality, R-angular distance, operator inequalities.

1. Introduction

Let B(H) be the algebra of all bounded linear operators acting on a complex Hilbert space H. For $A \in B(H)$, we denote the absolute value operator of A by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint operator of A. Throughout this paper, we assume that $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In 1964, Dunkl and Williams [2] showed that for any nonzero elements x, y in a normed linear space X,

$$\left\| \frac{x}{\|x\|} \right\| - \left\| \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}. \tag{1.1}$$

Pečarić and Rajić [9] gave the following refinement of (1.1): For any nonzero elements x, y in a normed linear space X,

$$\left\| \frac{x}{\|x\|} \right\| - \left\| \frac{y}{\|y\|} \right\| \leq \frac{(2\|x-y\|^2 + 2(\|x\| - \|y\|)^2)^{\frac{1}{2}}}{max\{\|x\|, \|y\|\}}. \tag{1.2}$$

Also they introduced an operator version of (1.2) by estimating $|A|A|^{-1} - B|B|^{-1}|$,

$$|A|A|^{-1} - B|B|^{-1}|^2 \leq |A|^{-1}(p|A-B|^2 + q(|A| - |B|^2)|A|^{-1}, \tag{1.3}$$

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where $A, B \in B(H)$ such that $|A|$ and $B|$ are invertible, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Saito and Tominaga [10] gave a generalization of the inequality(1.3) without additional assumptions related to inverse conditions. We would like to refer the reader to [4, 5, 6, 7, 8] and references therein for more information.

$$|(U-V)|A||^2 \leq p|A-B|^2 + q(|A| - |B|^2),$$

where $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Dadipour, Fujii and Moslehian [1] introduced operator versions of the Dunkl-Williams inequality with respect to the r-angular distance as a generalization of both the main result of Saito and Tominaga.

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \leq p|A-B|^2 + q||B|^r|A|^{1-r} - |B||^2, \tag{1.4}$$

where $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$, $0 < r \leq 1$,$p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Recently, Jiang and Zou [11] gave refinements of some operator inequalities presented by Zou, He and Qaisar [12] as follows:

Theorem A. Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$. If $q > 0$, then

$$|(U-V)|A||^2 + \frac{q}{p}|(1-p)(A-B) - V(|A|-|B|)|^2 \leq p|A-B|^2 + q||A|-|B||^2. \tag{1.5}$$

If $q < 0$, then

$$|(U-V)|A||^2 + \frac{q}{p}|(1-p)V(|A|-|B|) - (A-B)|^2 \leq q|A-B|^2 + p||A|-|B||^2. \tag{1.6}$$

Theorem B. Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$. If $p < 0$, then

$$p(|A|-|B|)^2 + q|A-B|^2 - \frac{q}{p}|(1-p)V(|A|-|B|) - (A-B)|^2 \leq |(U-V)|A||^2. \tag{1.7}$$

If $0 < p < 1$, then

$$p|A-B|^2 + q||A|-|B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|A|-|B|)|^2 \leq |(U-V)|A||^2. \tag{1.8}$$

If $1 < p \leq 2$, then

$$\begin{aligned} &p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{p+2}{2q}|(A-B) - (1-q)V(|A|-|B|)|^2 \\ &\leq |(U-V)|A||^2. \end{aligned} \tag{1.9}$$

If $p > 2$, then

$$p|A-B|^2 + q|V(|A|-|B|)|^2 - \frac{q+2}{2p}|(1-p)(A-B)-V(|A|-|B|)|^2$$
$$\leq |(U-V)|A||^2. \tag{1.10}$$

In this paper, We present several operator versions of the Dunkl-Williams inequality with respect to the r-angular distance for operators. We obtain refinements of some operator inequalities presented by Jiang and Zou [11].

2. Main results

In this section, we consider Dunkl-Williams inequality for operators as an application of the generalized parallelogram law of operators [3]:

$$|T_1 - T_2|^2 + \frac{1}{t}|tT_1 + T_2|^2 = (1+t)|T_1|^2 + (1+\frac{1}{t})|T_2|^2. \tag{2.1}$$

Applying the above generalized parallelogram law of operators, we can get the following lemma:

Lemma 2.1. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $0 \leq r \leq 1$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \tag{2.2}$$

and

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2$$
$$= q|A-B|^2 + p|V(|B|^r|A|^{1-r} - |B|)|^2 \tag{2.3}$$

Proof. Putting $T_1 = A - B, T_2 = V(|B|^r|A|^{1-r} - |B|), p = 1 + t$ and $q = 1 + \frac{1}{t}$ in (2.1), then we have

$$|(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2.$$

Note that

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 = |U|A| - V|B|^r|A|^{1-r}|^2 = |A - V|B|^r|A|^{1-r}|^2$$
$$= |(A-B) - V|B|^r|A|^{1-r} + B|^2 = |(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

The combination of above two equalities gives $|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 = p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2$.

By putting $T_1 = V(|B|^r|A|^{1-r} - |B|), T_2 = A - B, p = 1 + t$ and $q = 1 + \frac{1}{t}$ in (2.1), we can obtain the equality (2.3) in a similar way.

This completes the proof. □

Remark 2.2. Note that $p > 1$ implies $q > 1$, so the identity (2.2) gives immediately the inequality (1.4):

$$|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2.$$

Theorem 2.3. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, $0 \leq r \leq 1$. If $q > 0$, then*

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.4}$$

If $q < 0$, then

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\ &\leq q|A - B|^2 + p||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.5}$$

Proof. If $q > 0$, since $V^*V \leq I$, we get

$$\begin{aligned} |V(|B|^r|A|^{1-r} - |B|)|^2 &= (|B|^r|A|^{1-r} - |B|)V^*V(|B|^r|A|^{1-r} - |B|) \\ &\leq ||B|^r|A|^{1-r} - |B||^2. \end{aligned} \tag{2.6}$$

Combining with (2.2) and (2.6), we obtain

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2. \end{aligned}$$

Because $q < 0$ implies $0 < p < 1$, Combining with (2.3) and (2.6), we have

$$\begin{aligned} &|(U|A|^r - V|B|^r)|A|^{1-r}|^2 + \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\ &\leq q|A - B|^2 + p||B|^r|A|^{1-r} - |B||^2. \end{aligned}$$

Remark 2.4. Obviously, the inequalities (1.5) and (1.6) are special cases of the Theorem 2.2 when $r = 0$. Furthermore, inequality (2.4) is a refinement of inequality (1.4) when $p > 1$.

Next, we present some lower bounds for$|(U|A|^r - V|B|^r)|A|^{1-r}|^2$.

Theorem 2.5. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, $0 \leq r \leq 1$. Then the following statements hold:*

(i) *If* $p < 0$, *then*

$$p||B|^r|A|^{1-r} - |B||^2 + q|A - B|^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \tag{2.7}$$

(ii) *If* $0 < p < 1$, *then*

$$q||B|^r|A|^{1-r} - |B||^2 + p|A-B|^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \tag{2.8}$$

(iii) *If* $1 < p \leq 2$ *and* $0 < \lambda < 1$, *then*

$$\begin{aligned} &q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2 \\ &- \frac{(p-2)\lambda + 2}{q}|(A - B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.9}$$

(iv) *If* $p > 2$ *and* $0 < \lambda < 1$, *then*

$$\begin{aligned} &q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2 \\ &- \frac{(2-q)\lambda + q}{p}|(1-p)(A - B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned} \tag{2.10}$$

Proof. (i) Since $V^*V \leq I$ and $p < 0$, we have

$$p|V(|B|^r|A|^{1-r} - |B|)|^2 \geq p||B|^r|A|^{1-r} - |B||^2$$

Combining with the above equality and (2.3), we obtain

$$\begin{aligned} &p||B|^r|A|^{1-r} - |B||^2 + q|A - B|^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\ &\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned}$$

(ii) If $0 < p < 1$, then $q < 0$. So we have

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 \geq q||B|^r|A|^{1-r} - |B||^2.$$

Combining with the above equality and (2.2), we obtain

$$\begin{aligned} &q||B|^r|A|^{1-r} - |B||^2 + p|A - B|^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\ &\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2. \end{aligned}$$

(iii) Note that

$$\frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 = \frac{p}{q}|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2$$

So, the equality (2.2) implies

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 = p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{q}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{p}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}\tag{2.11}
$$

If $1 < p \leq 2$ and $0 < \lambda < 1$, then we have

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \geq p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{q}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2 \\
&= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 - \frac{(p-2)\lambda + 2}{q}|(A-B) \\
&- (1-q)V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

(iv) If $p > 2$ and $0 < \lambda < 1$, then $1 < q < 2$. By equality (2.11), we also have

$$
\begin{aligned}
&|(U|A|^r - V|B|^r)|A|^{1-r}|^2 \geq p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{p}\lambda|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{2}{q}(1-\lambda)|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2 \\
&= p|A-B|^2 + q|V(|B|^r|A|^{1-r} - |B|)|^2 \\
&- \frac{(2-q)\lambda + q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

Remark 2.6. It is easy to see that the Theorem B is a special case of the Theorem 2.5 when $r = 0$ and $\lambda = \frac{1}{2}$. In addition, inequalities (2.9) and (2.10) are refinements of inequalities (1.9) and (1.10) respectively when $r = 0$ and $\lambda > \frac{1}{2}$.

Remark 2.7. Since $p < 0$ implies $0 < q < 1$, $0 < p < 1$ implies $q < 0$, $1 < p \leq 2$ implies $q \geq 2$, and $p > 2$ implies $0 < q < 2$, combining with Theorems 2.3 and 2.5, we have the following results.

If $p < 0$, then

$$
\begin{aligned}
&p||B|^r|A|^{1-r} - |B||^2 + q|A-B|^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2 \\
&\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2 \\
&\leq p|A-B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.
\end{aligned}
$$

If $0 < p < 1$, then

$$q||B|^r|A|^{1-r} - |B||^2 + p|A - B|^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq q|A - B|^2 + p||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)V(|B|^r|A|^{1-r} - |B|) - (A-B)|^2.$$

If $1 < p \leq 2$ and $0 < \lambda < 1$, then

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2$$
$$- \frac{(p-2)\lambda + 2}{q}|(A-B) - (1-q)V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

If $p > 2$ and $0 < \lambda < 1$, then

$$q|V(|B|^r|A|^{1-r} - |B|)|^2 + p|A - B|^2$$
$$- \frac{(2-q)\lambda + q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2$$
$$\leq |(U|A|^r - V|B|^r)|A|^{1-r}|^2$$
$$\leq p|A - B|^2 + q||B|^r|A|^{1-r} - |B||^2 - \frac{q}{p}|(1-p)(A-B) - V(|B|^r|A|^{1-r} - |B|)|^2.$$

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