On skew GQC and skew QC codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$

Negin Karimi
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil
Iran
neginkarimi@uma.ac.ir

Ahmad Yousefian Darani*
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil
Iran
yousefian@uma.ac.ir

Abstract. In this paper, we study some algebraic structural properties of skew quasi cyclic codes and skew generalized quasi-cyclic codes over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. We discuss on Hermitian dual of these classes of codes over R. Then, we investigate on the generator polynomials and the parity-check polynomials of 1-generator skew QC codes and 1-generator skew GQC codes. Finally, we show that the Gray image of a skew quasi-cyclic code over R is a skew l-quasi-cyclic code of index 4 and the Gray image of skew GQC code over R is a skew GQC code of index 4.

Keywords: skew polynomial ring, skew QC code, skew GQC code, gray map, Hermitian dual.

1. Introduction

Algebraic coding theory is an area of discrete applied mathematics that is concerned with developing error-control codes and encoding/decoding procedures. Many areas of mathematics are used in coding theory, and we focus on the interplay between algebra and coding theory. Cyclic codes over finite fields have been studied since the late 1950 and play a significant role in the coding theory. In 1994 Hammons et al. [12] studied codes over $\mathbb{Z}_4$, then a lot of researches went towards studying codes over $\mathbb{Z}_4$. Bonnecaze and Udaya [5] have studied cyclic codes over the ring $F_2 + uF_2$; $u^2 = 0$. This ring is useful because it shares many properties of $\mathbb{Z}_4$. A complete structure of cyclic codes over $\mathbb{Z}_4$ of odd length has been given in [16].

∗. Corresponding author

Recently, it has been shown that codes over finite rings are an important class of codes. Quasi-cyclic (QC) codes are a natural and remarkable generalization of cyclic codes. QC code of index l over finite ring, defined by the property that a cyclic shift of a codeword by l places is another codeword, generalize the class of cyclic codes ($l = 1$). Skew polynomial rings form an important class of non-commutative rings. Boucher, Geiselmann and Ulmer in [4] and [10] studied linear and cyclic codes over skew polynomial rings. Also skew QC codes are constructed in [1] with the property that $| < \theta > | = m$. The algebraic structure of generalized quasi-cyclic (GQC) codes over finite fields were introduced by Siap and Kulhan in [18]. Since then, some properties of these codes were studies by Esmaeili and Yari in [9]. In [7], Cao studied GQC code of arbitrary length over finite fields. Cao investigated structural properties of 1-generator GQC code. Also GQC codes over Galois rings were introduced by Cao.

In this paper we firstly study some basic properties of the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. Also we organize basic notations of skew QC and skew GQC codes. In Section 3 we discuss on the dual of skew QC and skew GQC codes. Also we obtain a necessary and sufficient condition on self-dual Hermitian skew QC codes and self-dual Hermitian skew GQC codes. Finitely we investigate the Gray image of skew QC codes and skew GQC codes over the ring R in Section 4.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of cardinality q. A k-dimensional vector subspace C of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$ is called a linear (n, k)-code over $\mathbb{F}_q$. A linear code C is called a cyclic code if whenever $(a_0, a_1, \ldots, a_{n-1}) \in C$, then $(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in C$. In order to convert the combinatorial structure of cyclic codes into an algebraic one, we consider the following correspondence:

$$\begin{aligned}
&\pi : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q[x]/(x^n - 1),\\
&(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1x + \ldots + a_{n-1}x^{n-1}.
\end{aligned}$$

The mapping π is a linear transformation of vector spaces over $\mathbb{F}_q$. Then a nonempty subset C of F_q^n is a cyclic code if and only if $\pi(C)$ is an ideal of $F_q[x]/(x^n-1)$. We will sometimes identify $\mathbb{F}_q^n$ with $\mathbb{F}q[x]/(x^n-1)$, and a vector $u = (u_0, u_1, \ldots, u_{n-1})$ with the polynomial $u(x) = \sum_{i=0}^{n-1} u_i x^i$.

The notion of skew cyclic codes introduced in [4] as a generalization of cyclic codes. Let θ be an automorphism of $\mathbb{F}_q$. A θ-cyclic code (or skew cyclic code) is a linear code C with the property that

$$(a_0, a_1, \ldots, a_{n-1}) \in C \Rightarrow (\theta(a_{n-1}), \theta(a_0), \ldots, \theta(a_{n-2})) \in C.$$

It is easy to check that cyclic codes correspond to the case where θ is the identity mapping.

Throughout this paper we consider the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ where $u^2 = v^2 = 0$ and $uv = vu$. This ring is a Frobenius ring of characteristic 2. Since

$$u(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2) \not\subset v(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)$$

and

$$v(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2) \not\subset u(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)$$

so R is not a finite chained ring. Note that R is a local ring with the maximal ideal $I_{u,v} = \{0, u, v, u+v, uv, u+uv, v+uv, u+v+uv\}$, since $\overline{R} = \frac{R}{I_{u,v}} \cong \mathbb{F}_2$ and $R = \{0 + I_{u,v}\} \cup \{1 + I_{u,v}\}$. If we define the non-trivial ring homomorphism θ from R to R with $\theta(0) = 0, \theta(u) = v, \theta(1) = 1, \theta(uv) = vu, \theta(v) = u$, then $\theta^2(X) = X$ for any $X \in R$. This implies that θ is a ring automorphism of order 2. The Skew polynomial ring of R by θ is defined by

$$R[x;\theta] = \{a_0 + a_1x + \ldots + a_nx^n | a_i \in R, \forall i = 0, 1, 2, \ldots, n\},$$

where the coefficients are written on the left side of the variable x. The multiplication is defined by the basic rule $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$, and the addition is defined to be the usual addition rule of polynomials.

We can consider R as a natural extension of the ring $\mathbb{F}_2 + u\mathbb{F}_2$, so we can extend the definition of the Lee weight and Hamming weight from $\mathbb{F}_2 + u\mathbb{F}_2$ to the ring R.

Let w_l denote the Lee weight and w_H denote the Hamming weight for the binary codes. We set

$$w_L(x + uy + vz + uvt) = w_H(x + y + z + t, z + t, y + t, t) \quad \forall x, y, z, t \in \mathbb{F}_2.$$

The definition of the Lee weight and Hamming weight lead to a Gray map. By [20] we consider the following Gray map

$$\phi' : \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2 \to \mathbb{F}_2^4,$$
$$\phi'(x + uy + vz + uvt) = (x + y + z + t, z + t, y + t, t).$$

This map can be extended to R^n as follows:

$$\phi : ((\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2)^n, Lee\ weight) \to (\mathbb{F}_2^{4n}, \text{Hamming weight}).$$

Now, let $R_m = \frac{R[x;\theta]}{<x^m-1>}$ and let l be a positive integer. Then, the ring R_m^l is a left R_m-module where the scalar multiplication from left is defined by

$$f(x)(g_1(x), g_2(x), ..., g_l(x)) = (f(x)g_1(x), f(x)g_2(x), ..., f(x)g_l(x)),$$

for every $f(x) \in R[x;\theta]$ and $g_i(x) \in R_m$ for $i = 1, 2, \ldots, l$. Let C be a subset of R^{ml}. Consider the map ψ from R^{ml} to R_m^l given by $\psi(C) = (X_0(x), ..., X_{l-1}(x))$ where

$$X_j(x) = \sum_{i=0}^{m-1} r_{j,i}x^i \in \frac{R[x,\theta]}{< x^m - 1 >}, \qquad \forall j = 0, 1, 2, \ldots, l-1.$$

The map ψ is a one-to-one correspondence between the ring R^{ml} and R_m^l.

Definition 2.1. *A nonempty subset C of R^n is called a skew cyclic code of length n, if C satisfies the following conditions:*

1. *C is submodule of R^n; and*
2. *if $X = (r_0, r_1, ..., r_{n-1}) \in C$, then the skew cyclic shift $\sigma(X)$ is in C, that is $\sigma(X) = (\theta(r_{n-1}), \theta(r_0), ..., \theta(r_{n-2})) \in C$.*

Definition 2.2. *A subset C of R^n, where $n = ml$, is called a skew quasi-cyclic code of length n and index l (or a skew l-QC code) if C satisfies the following conditions:*

1. *C is a subspace of R^n; and*
2. *if $X = (r_{0,0}, r_{0.1}, \dots, r_{0,l-1}, r_{1,0}, r_{1,1}, \dots, r_{1,l-1}, \dots, r_{m-1,0}, r_{m-1,1}, \dots, r_{m-1,l-1})$ is a codeword in C, then $\tau_{\theta,m,l}(X) = (\theta(r_{m-1,0}), \theta(r_{m-1,1}), \dots, \theta(r_{m-1,l-1}), \theta(r_{0,0}), \theta(r_{0,1}), \dots, \theta(r_{0,l-1}), \dots, \theta(r_{m-2,0}), \theta(r_{m-2,1}), \dots, \theta(r_{m-2,l-1})) \in C$.*

Theorem 2.1. *A subset C of R^n, where $n = ml$, is a skew l-QC code of length n if and only if $\psi(C)$ is a left submodule of the ring R_m^l.*

Proof. Let C be a skew l-QC code and $c = (r_{0,0}, r_{0,1}, \dots, r_{0,l-1}, r_{1,0}, r_{1,1}, \dots, r_{1,l-1}, \dots, r_{m-1,0}, r_{m-1,1}, \dots, r_{m-1,l-1}) \in C$. Since $x^m = 1$ in R_m, we have

$$xX_j(x) = \sum_{i=0}^{m-1} \theta(r_{j,i})x^{i+1} = \sum_{i=0}^{m-1} \theta(r_{j,i-1})x^i \in R_m, \tag{1}$$

for any $j = 0, 1, \dots, l-1$ and $i - 1 \in \{0, 1, \dots, m-1\}$ by taking modulo m. Hence $(xX_0(x), xX_1(x), \dots, xX_{l-1}(x)) \in \psi(C)$. Then, it follows from linearity that $f(x)\psi(c) \in \psi(C)$ for any $f(x) \in R_m$. Therefore, $\psi(C)$ is a left submodule of R_m^l.

Conversely, suppose that Y is a left R_m-submodule of R_m^l. We claim that $X = \psi^{-1}(Y) \in R^{ml}$ is a skew l-QC code. It is enough to show that C is closed under skew cyclic shift. Let $c = (r_{0,0}, r_{0.1}, \dots, r_{0,l-1}, r_{1,0}, x_{1,1}, \dots, r_{1,l-1}, \dots, r_{m-1,0}, r_{m-1,1}, \dots, r_{m-1,l-1})) \in X$. Then $\psi(c) \in Y$. By a similar argument like in (2.1), we see that $\psi(\tau_{\theta,m,l}(c)) = x\psi(c) \in Y$, so $\tau_{\theta,m,l} \in X$. Hence X is a skew l-QC code. □

Definition 2.3. *Let $m_1, m_2, \dots, m_t$ be positive integers and $m = \sum_{i=1}^{t} m_i$. A subset C of R^m is called a skew generalized quasi-cyclic (skew GQC) code of block length $(m_1, m_2, \dots, m_t)$ and length m if C satisfies in the following conditions:*

1. *C is a subspace of R^m,*
2. *if $X = (r_{0,0}, \dots, r_{0,m_1-1}, r_{1,0}, \dots, r_{1,m_2-1}, \dots, r_{t-1,0}, \dots, r_{t-1,m_t-1}) \in C$, then $\tau_{\theta,m}(X) = (\theta(r_{t-1,0}), \dots, \theta(r_{t-1,m_t-1}), \theta(r_{0,0}), \dots, \theta(r_{0,m_1-1}), \dots, \theta(r_{t-2,0}), \dots, \theta(r_{t-2,m_{t-1}-1})) \in C$.*

If m_1=m_2=$\ldots = m_t = m$, then a GQC code of block length $(m_1, m_2, \ldots, m_t)$ is a skew t-QC code of length mt over the ring R. Let m_i be as in Definition 2.3 and $R_i = \frac{R[x;\theta]}{<x^{m_i}-1>}$ for any $i = 1, 2, \ldots, t$. Assume that the multiplication is defined by

$$f'(x)(g'_1(x), g'_2(x), \ldots, g'_t(x)) = (f'(x){g'}_1(x), f'(x){g'}_2(x), \ldots, f'(x){g'}_t(x))$$

in which every $f'(x)g'_i(x)$ is considered $mod(x^{m_i}-1)$, $f'(x) \in R[x;\theta]$ and $g'_i(x) \in R_i$ for $i = 1, 2, \ldots, t$. In this case, the ring $R' := R_1 \times R_2 \times \ldots \times R_t$ is an $R[x;\theta]$-submodule.

Consider the map ψ' from R^m to R' defined by $\psi'(C) = (Y_0(x), \ldots, Y_{t-1}(x))$ where C is a subset of R^m and

$$Y_j(x) = \sum_{i=0}^{m_{j+1}-1} r'_{j,i}x^i \in \frac{R[x;\theta]}{< x^{m_{j+1}} - 1 >},$$

for any $j = 0, 1, ..., t-1$.

The following is a similar result to Theorem 2.1 for skew GQC codes.

Theorem 2.2. *Let $m_1, m_2, \ldots, m_t$ be positive integers and $m = \sum_{i=1}^{t} m_i$. A subset C of R^m is a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ and length m if and only if $\psi'(C)$ is a left submodule of the ring R'.*

Proof. The proof is similar to that of Theorem 2.1. □

3. Dual of SKEW QC codes and skew GQC codes

Assume that C is a skew l-QC code of length $n(n = ml)$. Let $X = (x_{0,0}, \ldots, x_{0,l-1}, x_{1,0}, \ldots, x_{1,l-1}, \ldots, x_{m-1,0}, \ldots, x_{m-1,l-1})$ and $Y = (y_{0,0}, \ldots, y_{0,l-1}, y_{1,0}, \ldots, y_{1,l-1}, \ldots, y_{m-1,0}, \ldots, y_{m-1,l-1})$ be codewords in C. The Hermitian inner product is defined by

$$< X, Y >_H = \sum_{j=0}^{l-1} \sum_{i=0}^{s-1} x_{i,j}\theta(y_{i,j}),$$

where $x_{i,j} \in X$ and $y_{i,j} \in Y$. The codewords X, Y are called orthogonal Hermitian with respect to the Hermitian inner product if $< X, Y >_H = 0$. The dual code $C^{\perp_H}$ with respect to Hermitian inner product of C is defined by

$$C^{\perp_H} = \{X \in R^n \mid < X, Y >_H = 0 \ , \forall Y \in C\}.$$

A skew l-QC code is called Hermitian self-dual if $C = C^{\perp_H}$ and whenever $C \subseteq C^{\perp_H}$, we say that C is Hermitian self-orthogonal.

Theorem 3.1. *Let C be a skew l-QC code of length $n(n = ml)$ over R. Then, the Hermitian dual of C is also a skew l-QC code.*

Proof. Let $X = (x_{0,0}, \ldots, x_{0,l-1}, x_{1,0}, \ldots, x_{1,l-1}, \ldots, x_{m-1,0}, \ldots, x_{m-1,l-1}) \in C$ and $Y = (y_{0,0}, \ldots, y_{0,l-1}, y_{1,0}, \ldots, y_{1,l-1}, \ldots, y_{m-1,0}, \ldots, y_{m-1,l-1}) \in C^{\perp_H}$. Since C is skew l-QC code, so $(\theta(x_{m-1,0}), \ldots, \theta(x_{m-1,l-1}), \theta(x_{0,0}), \ldots, \theta(x_{0,l-1}), \ldots, \theta(x_{m-2,0}), \ldots, \theta(x_{m-2,l-1})) \in C$.

Continuing this way, we have $X^i = (\theta^i(x_{m-i,0}), \ldots, \theta^i(x_{m-i,l-1}), \theta^i(x_{m-i+1,0}), \ldots, \theta^i(x_{s-i+1,l-1}), \ldots, \theta^i(x_{s-i-1,0}), \ldots, \theta^i(x_{s-i-1,l-1})) \in C$, for any i=$1, 2, \ldots, m$. Hence $< X^i, Y >_H = 0$. If $i = m-1$, then $\theta^{m-1}(x_{i',j'}) = \theta(x_{i',j'})$ for any $x_{i',j'} \in X$. Therefore, $\theta(x_{1,0})y_{0,0} + \ldots + \theta(x_{1,l-1})y_{0,l-1} + \theta(x_{2,0})y_{1,0} + \ldots + \theta(x_{2,l-1})y_{1,l-1} + \ldots + \theta(x_{0,0})y_{s-1,0} + \ldots + \theta(x_{0,l-1})y_{m-1,l-1} = 0$. Note that θ is a ring automorphism of order 2. Hence, $x_{0,0}\theta(y_{m-1,0}) + \ldots + x_{0,l-1}\theta(y_{m-1,l-1}) + x_{1,0}\theta(y_{0,0}) + \ldots + x_{1,l-1}\theta(y_{0,l-1}) + \ldots + x_{m-1,0}\theta(y_{m-2,0}) + \ldots + x_{m-1,l-1}\theta(y_{m-2,l-1}) = 0$. This gives $(\theta(y_{m-1,0}), \ldots, \theta(y_{m-1,l-1}), \theta(y_{0,0}), \ldots, \theta(y_{0,l-1}), \ldots, \theta(y_{m-2,0}), \ldots, \theta(y_{m-2,l-1})) \in C^{\perp_H}$. Hence $C^{\perp_H}$ is also a skew QC code of index l and length n. □

We can define the Hermitian dual skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ in a similar way. Let C be a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$. Suppose that $X = (x_{0,0}, \ldots, x_{0,m_1-1}, x_{1,0}, \ldots, x_{1,m_2-1}, \ldots, x_{t-1,0}, \ldots, x_{t-1,m_t-1})$ and $Y = (y_{0,0}, \ldots, y_{0,m_1-1}, y_{1,0}, \ldots, y_{1,m_2-1}, \ldots, y_{t-1,0}, \ldots, y_{t-1,m_t-1})$ are two codewords in C. The Hermitian inner product for X and Y is defined as

$$< X, Y >_H = \sum_{j=0}^{m_1-1} x_{0,j}\theta(y_{0,j}) + \sum_{j=0}^{m_2-1} x_{1,j}\theta(y_{1,j}) + \ldots + \sum_{j=0}^{m_t-1} x_{t-1,j}\theta(y_{t-1,j}),$$

where $x_{i,j} \in X$, $y_{i,j} \in Y$ for each $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, m_{i+1} - 1$.

Theorem 3.2. *The Hermitian dual of a skew GQC code of block length $(m_1, m_2, ..., m_t)$ is a skew GQC code of block length $(m_1, m_2, ..., m_t)$.*

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.1. *Let $h(x) = \sum_{i=0}^{n} h_i x^i$ be a polynomial in the ring $R[x;\theta]$. The skew reciprocal polynomial of h is defined by*

$$h^*(x) = \sum_{i=0}^{n} x^{n-i} h_i = \sum_{i=0}^{n} \theta^i(h_{n-i}) x^i.$$

If $h(x) = h^*(x)$, then $h(x)$ is called a self-reciprocal polynomial.

In order to describe some properties of the skew reciprocal polynomial, we need the following morphism of rings:

$$\Theta : R[x;\theta] \longrightarrow R[x;\theta],$$
$$\sum_{i=0}^{n} a_i X^i \mapsto \sum_{i=0}^{n} \theta(a_i) X^i.$$

We recall the following useful Lemma from [3, Lemma 1].

Lemma 3.1. *Let f and g be skew polynomial in R and $n = deg(f)$. Then:*
1. $(fg)^* = \Theta^n(g^*)f^*$.
2. $(f^*)^* = \Theta^n(f)$.

We briefly state some facts regarding skew cyclic codes. We know that there exists a one-to-one correspondence between the codewords $(c_0, c_1, ..., c_{m-1})$ and polynomials $c(x) = c_0 + c_x + ... + c_{m-1}x^{m-1}$ in $R[x;\theta]$. Under this correspondence a skew cyclic code C of length n over the ring R can be considered as a principal ideal in R_m. Among all the generators of the ideal C, there is a unique monic one with minimal degree that divides $x^m - 1$. This polynomial is called the generator polynomial of the skew cyclic code C and we display it by $G(x)$. A polynomial $H(x)$ which satisfy in $H(x)G(x) = x^m - 1$ is called the check polynomial of C.

If C is a skew cyclic code with generator polynomial $G(x)$, then $C^{\perp_H}$ is a skew cyclic code with generator polynomial $H^*(x)$, where $H^*(x)$ is skew reciprocal polynomial of the check polynomial $H(x)$. Thus we obtain the following result.

Proposition 3.1. *A skew cyclic code C of length n is a skew cyclic Hermition self-dual if and only if $G(x) = H^*(x)$ where $G(x)$ is the generator polynomial of C, $H(x)$ is the check polynomial and $H^*(x)$ is the skew reciprocal polynomial of $H(x)$.*

By [2] we know that when n is an odd integer, $x^n - 1$ factors over $\mathbb{F}_2$ into pairwise coprime irreducible factors. We can consider the map $R[x;\theta] \to \mathbb{F}_2[x;\theta]$ and apply Hensel's Lemma. The factorization $x^n - 1$ in $\mathbb{F}_2[x;\theta]$ can be uniquely lifted to a factorization of $x^n - 1$ over R into pairwise coprime basic irreducible factors. Also the factorization of $x^n - 1$ over $\mathbb{F}_2$ is still valid over R. Therefore, if n is odd, then all factors of $x^n - 1$ in $R[x;\theta]$ are just its factors in $\mathbb{F}_2[x;\theta]$.

Let C be a skew l-QC code of length $n (n = ml)$ where $m = 2^a m'$ such that $(m', 2) = 1$ and a is a integer which dependent on m. By [17], one can write

$$x^{m'} - 1 = f_1 \dots f_s h_1 h_1^* \dots h_t h_t^*,$$

where h_j^* is the skew reciprocal polynomial of h_j for $j = 1, 2, \dots, t$ and f_i is skew self-reciprocal polynomial for any $i = 1, 2, \dots, s$. Also

$$(*) \qquad x^m - 1 = (x^{m'} - 1)^{2^a} = f_1^{2^a} \dots f_s^{2^a} h_1^{2^a} h_1^{*2^a} \dots h_t^{2^a} h_t^{*2^a}.$$

We continue this section to discuss on some structural properties of 1-generator skew QC codes and 1-generator skew GQC codes. Consider a skew l-QC code C of length $n (n = ml)$ over the ring R. Let $F(x) = (F_1(x), \dots, F_l(x)) \in (\frac{R[x;\theta]}{<x^m-1>})^l$ where $F_i(x) \in \frac{R[x;\theta]}{<x^m-1>}$. Then $R[x;\theta]F(x) := \{\alpha(x)F(x) | \alpha(x) \in R[x;\theta]\} = \{(\alpha(x)F_1(x), \alpha(x)F_2(x), \dots, \alpha(x)F_l(x)) | \alpha(x) \in R[x;\theta]\}$. is called a 1-generator skew l-QC code with generator $F(x)$. Note that for $l = 1$, a skew l-QC code over R is a skew cyclic code of length m over R. Define a well defined R-homomorphism ϕ_j from R^n onto R_m such that $\phi_j(F_1(x), \dots, F_l(x)) = F_j(x)$. Then $\phi_j(C)$ is a skew cyclic code of length m over R.

Theorem 3.3. *Suppose that C is a 1-generator skew l-QC code of length $n(n = ml)$ which is generated by $F(x) = (F_1(x), \dots, F_l(x))$ such that $\phi_j(C)$ is a Hermitian self-dual code over the ring $\frac{R[x;\theta]}{<x^m-1>}$ for any $j = 1, 2, \dots, l$. Then the generator polynomial of C is*

$$K(x).f_1^{(2^{a-1})} \dots f_s^{(2^{a-1})} h_1^{l_1} h_1^{*2^a - l_1} \dots h_t^{l_t} h_t^{*2^a - l_t},$$

where $K(x) \in R[x;\theta]$, $l_r = lcm\{\beta_{jr}\}$ and $0 \leq \beta_{jr} \leq 2^a$ for any $j = 1, 2, \dots, l$ and $r = 1, 2, \dots, t$.

Proof. Consider the homomorphism ϕ_j defined in the preceding paragraph. We know that $\phi_j(C)$ is a skew cyclic code of length m over the ring R_m for any $j = 1, 2, \dots, l$. So $\phi_j(C)$ has a generator polynomial such as $G_j(x)$ such that $G_j(x)|x^m - 1$. By factorization $(*)$ for $x^m - 1$ we can write

$$G_j(x) = {f_1}^{\alpha_{j1}} \dots {f_s}^{\alpha_{js}} {h_1}^{\beta_{j1}} h_1^{*\eta_{j1}} \dots {h_t}^{\beta_{jt}} h_t^{*\eta_{jt}},$$

where $0 \leq \alpha_{ji} \leq 2^a$ and $0 \leq \beta_{jk}, \eta_{jk} \leq 2^a$ for any $i = 1, 2, \dots, s$ and $k = 1, 2, \dots, t$. The following polynomial is the check polynomial of $\phi_j(C)$: $H_j(x) = {f_1}^{2^a - \alpha_{j1}} \dots {f_s}^{2^a - \alpha_{js}} h_1^{2^a - \beta_{j1}} h_1^{*2^a - \eta_{j1}} \dots h_t^{2^a - \beta_{jt}} h_t^{*2^a - \eta_{jt}}$.

Assume that $deg(f_i^{2^a - \alpha_{ji}}) = n_i$, $deg(h_k^{2^a - \beta_{jk}}) = m_k$ and $deg(h_k^{*2^a - \eta_{jk}}) = m'_k$. We use Lemma 3.1 to obtain $H_j^*(x)$. Thus

$$\begin{aligned} H_j^*(x) = {} & \Theta^{\sum_{i=1}^{s} n_i + \sum_{j=1}^{t} m_j + \sum_{j=1}^{t-1} m_j{}'} ({h_{jt}^{*}}^{2^a - \eta_{jt}})^* \times \\ & \Theta^{\sum_{i=1}^{t} n_i + \sum_{j=1}^{t-1} m_j + \sum_{j=1}^{t-1} m'_j} (h_{jt}^{2^a - \beta_{jt}})^* \times \\ & \Theta^{\sum_{i=1}^{t} n_i + \sum_{j=1}^{t-1} m_j + \sum_{j=1}^{t-2} m'_j} ({h_{j,t-1}^{*}}^{2^a - \eta_{j,t-1}}) \times \\ & \vdots \\ & \Theta^{\eta_1} (f_{j1}^{2^a - \alpha_{j1}})^*. \end{aligned}$$

By proposition 3.1, a skew cyclic code generated by the polynomial generator $G_j(x)$ is skew Hermitian self-dual if and only if $G_j(x) = H_j^*(x)$. Therefore, $\alpha_{ji} = 2^a - \alpha_{ji}$ and $\eta_{jk} = 2^a - \beta_{jk}$. Thus

$$(**) \qquad G_j(x) = {f_1}^{2^{a-1}} \dots {f_s}^{2^{a-1}} {h_1}^{\beta_{j1}} h_1^{*2^a - \beta_{j1}} \dots {h_t}^{\beta_{jt}} h_t^{*2^a - \beta_{jt}}.$$

We know that $H_j(x)$ and $G_j(x)$ are the check polynomial and the generator polynomial for the skew cyclic code $\phi_j(C)$, respectively. Thus $H_j(x).G_j(x) = x^m - 1$. This implies that $H(x) = lcm\{H_1(x), H_2(x), \dots, H_l(x)\}$ is the check polynomial of C. Let $G' = lcm\{G_j(x)\}_{j=1}^{l}$. Therefore,

$$(1) \qquad G'(x) = {f_1}^{2^{a-1}} \dots {f_s}^{2^{a-1}} {h_1}^{l_1} h_1^{*2^a - l_1} \dots {h_t}^{l_t} h_t^{*2^a - l_t},$$

where $l_r = lcm\{\beta_{jr}\}$ for each $r = 1, 2, \dots, t$. If $H(x)$ and $G(x)$ are check polynomial and generator polynomial for C respectively, then, $G(x).H(x) = x^m - 1$. It is clear that $G(x)|G'(x)$. By (1), there exists a polynomial such as $K(x) \in R[x;\theta]$ and $G(x) = K(x).{f_1}^{2^{a-1}} \dots {f_s}^{2^{a-1}} {h_1}^{l_1} h_1^{*2^a - l_1} \dots {h_t}^{l_t} h_t^{*2^a - l_t}$. $\square$

Consequently, by the factorization $(*)$, we can write

$$\frac{R[x;\theta]}{< x^m - 1 >} = (\bigoplus_{i=1}^{s} \frac{R[x;\theta]}{< f_i >}) \oplus (\bigoplus_{j=1}^{t} (\frac{R[x;\theta]}{< h_j >} \oplus \frac{R[x;\theta]}{< h_j^* >})). \tag{2}$$

Consider the following notations

$$G_i = \frac{R[x;\theta]}{< f_i >}, \quad H'_j = \frac{R[x;\theta]}{< h_j >}, \quad H''_j = \frac{R[x;\theta]}{< h_j^* >}.$$

Let $X, Y \in R_m^l$ where $X = (X_1, \ldots, X_l)$ and $Y = (Y_1, \ldots, Y_l)$. By (2), we can write $X_i = (x_{i1}, \ldots, x_{is}, x'_{i1}, x''_{i1}, \ldots, x'_{it}, x''_{it})$ and $Y_i = (y_{i1}, \ldots, y_{it}, y'_{i1}, y''_{i1}, \ldots, y'_{it}, y''_{it})$ where $x_{ij}, y_{ij} \in G_j$, $x'_{ik}, y'_{ik} \in H'_k$ and $x''_{ik}, y''_{ik} \in H''_k$ for any $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, s$ and $k = 1, 2, \ldots, t$. Therefore,

$$< X, Y >_H = \sum_{j=0}^{l-1} \sum_{i=1}^{s} x_{ji}\theta(y_{ji}) + \sum_{j=0}^{l-1} \sum_{k=1}^{t} x'_{jk}\theta(y'_{jk}) + \sum_{j=0}^{l-1} \sum_{k=1}^{t} x''_{jk}\theta(y''_{jk}).$$

Clearly $< X, Y >_H = 0$ if and only if

$$\sum_{j=0}^{l-1} \sum_{i=1}^{s} x_{ji}\theta(y_{ji}) = 0, \quad \sum_{j=0}^{l-1} \sum_{k=1}^{t} x'_{jk}\theta(y'_{jk}) = 0, \quad \sum_{k=1}^{t} x''_{jk}\theta(y''_{jk}) = 0.$$

Thus we have the following result about characterization of skew Hermitian self-dual codes over $R[x;\theta]$.

Corollary 3.1. *A skew l-QC code of length $n(n = ml)$ is Hermitian self-dual if and only if $C = (\bigoplus_{i=1}^{s} C_i) \oplus (\bigoplus_{j=1}^{t} (C'_j \oplus C'^{\perp}_j))$ where C_i is a skew Hermitian self-dual code over G_i for any $i = 1, 2, \ldots, s$, C'_j is a linear code over H'_j and $C'^{\perp}_j$ is its Hermitian dual.*

Our next step is to introduce the generator polynomial and the check polynomial for skew GQC codes. Assume that C is a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$ and length $m = \sum_{i=1}^{t}$. A 1-generator skew GQC code over R generated by $F(x) = (F_1(x), F_2(x), \ldots, F_t(x))$ where $F_i(x) \in \frac{R[x;\theta]}{<x^{m_i}-1>}$ is defined as

$$R[x;\theta]F(x) = \{\alpha(x)F_1(x), \ldots, \alpha(x)F_t(x) | \alpha(x) \in R[x;\theta]\}.$$

In a similar way for skew l-QC codes we can consider a well defined R-homomorphism ψ_i from R' to R_i given by $\psi_i(F(x)) = F_i(x)$. Then $\psi_i(C)$ is a skew cyclic code of length m_i and generated by $F_i(x)$ in R_i. Therefore, $\psi_i(x)$ has a generator polynomial such as $G_i(x)$ where $G_i(x)$ is a right divisor of $x^{m_i} - 1$. Therefore, $H_i(x) = \frac{x^{m_i}-1}{G_i(x)}$ is the check polynomial of $\psi_i(C)$. This implies that $H(x) = lcm\{H_i(x)\}_{i=1}^{t}$ is the check polynomial for C. Let C be a skew GQC code of block length $(m_1, m_2, \ldots, m_t)$. We know that

$$x^{m_i} - 1 = f_{i1} \ldots f_{is} h_{i1} h_{i1}^* \ldots h_{ik} h_{ik}^*,$$

where f_{ir} is a skew self-reciprocal polynomial and $h^*_{ir'}$ is the skew reciprocal polynomial of $h_{ir'}$ for any $r = 1, 2, \dots, s$ and $r' = 1, 2, \dots, k$. In a similar way for skew l-QC codes we can write the following factorization for R_i.

$$R_i = \frac{R[x;\theta]}{< x^{m_i} - 1 >} = (\bigoplus_{r=1}^{s} \frac{R[x;\theta]}{< f_{ir} >}) \oplus (\bigoplus_{r'=1}^{k} (\frac{R[x;\theta]}{< h_{ir'} >} \oplus \frac{R[x;\theta]}{< h^*_{ir'} >})). \tag{3}$$

Let $X, Y \in R'$ where $X = (X_1, X_2, \dots, X_t)$ and $Y = (Y_1, Y_2, \dots, Y_t)$ such that $X_i, Y_i \in \frac{R[x;\theta]}{<x^{m_i}-1>}$ for each $i = 1, 2, \dots, t$. By (3), X_i and Y_i are as follows:

$$X_i = (x_{i1}, \dots, x_{is}, x'_{i1}, x''_{i1}, \dots, x'_{it}, x''_{it})$$
$$Y_i = (y_{i1}, \dots, y_{is}, y'_{i1}, y''_{i1}, \dots, y'_{it}, y''_{it}),$$

where $x_{ir}, y_{ir} \in \frac{R[x;\theta]}{<f_{ir}>}$, $x'_{ir'}, y'_{ir'} \in \frac{R[x;\theta]}{<h_{ir'}>}$ and $x''_{ir'}, y''_{ir'} \in \frac{R[x;\theta]}{<h^*_{ir'}>}$. Therefore,

$$< X, Y >_H = \sum_{i=1}^{t} \sum_{r=1}^{s} x_{i,r}\theta(y_{ir}) + \sum_{i=1}^{t} \sum_{r'=1}^{k} x'_{ir'}\theta(y'_{ir'}) + \sum_{i=1}^{t} \sum_{r'=1}^{k} x''_{ir'}\theta(y''_{ir'}).$$

Clearly $< X, Y >_H =0$ if and only if

$$\sum_{i=1}^{t} \sum_{r=1}^{s} x_{i,r}\theta(y_{ir}) = 0, \sum_{i=1}^{t} \sum_{r'=1}^{k} x'_{ir'}\theta(y'_{ir'}) = 0$$

and $\sum_{i=1}^{t} \sum_{r'=1}^{k} x''_{ir'}\theta(y''_{ir'}) = 0$. Hence we have the following result.

Corollary 3.2. *A skew GQC code of block length $(m_1, m_2, \dots, m_t)$ is a self-dual Hermitian if and only if*

$$C = (\bigoplus_{r=1}^{s} C_r) \oplus (\bigoplus_{r'=1}^{k} (C'_{r'} \oplus C'^{\perp_H}_{r'})),$$

where C_i is a skew Hermitian self-dual over $\frac{R[x;\theta]}{<x^{m_i-1}>}$ for any $r = 1, 2, \dots, s$ and $C'_{r'}$ is a linear code over $\frac{R[x;\theta]}{<h_{ir'}>}$ and $C'^{\perp_H}_{r'}$ is its dual Hermitian over $\frac{R[x;\theta]}{h^*_{ir'}}$.

4. Gray image of skew GQC codes and skew QC codes

Let $X = (X_0, X_1, \dots, X_{4n-1}) = (X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) \in \mathbb{F}_2^{4n}$ where $X_{(i)} \in \mathbb{F}_2^n$ for all $i = 0, 1, 2, 3$. We define the following map

$$\Upsilon : \mathbb{F}_2^{4n} \to \mathbb{F}_2^{4n},$$
$$\Upsilon(X) = (\eta(X_{(0)}), \eta(X_{(1)}), \eta(X_{(2)}), \eta(X_{(3)})),$$

where η is a map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ with the property that

$$\eta(X_{(i)}) = (X_{i,n-1}, X_{i,0}, \dots, X_{i,n-2}),$$

for every $X_{(i)} = (X_{i,0}, X_{i,1}, X_{i,n-1})$ where $X_{i,j} \in \mathbb{F}_2$ for $j = 0, 1, \ldots, n-1$. Also we consider the map λ from $\mathbb{F}^{4n}$ to $\mathbb{F}^{4n}$ given by $\lambda(X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) = (X_{(0)}, X_{(2)}, X_{(1)}, X_{(3)})$.

Definition 4.1. *A code C of length $4n$ over $\mathbb{F}_2$ is said l-QC code of index 4 if $\Upsilon(C) = C$.*

Proposition 4.1. *Assume σ_θ is the skew cyclic shift on R^n, Υ is as above and ϕ is the Gray map from R^n to $\mathbb{F}_2^{4n}$. Then $\phi\sigma_\theta = \lambda\Upsilon\phi$.*

Proof. Let $X_i = x_i + uy_i + vz_i + uvt_i$ be elements of R for each $i = 0, 1, 2, \ldots, n-1$. By definition of the Gray map we have $\phi(X_0, X_1, ..., X_{n-1}) = (x_0 + y_0 + z_0 + t_0, x_1 + y_1 + z_1 + t_1, \ldots, x_{n-1} + y_{n-1}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-1} + t_{n-1}, t_0, \ldots, t_{n-1})$. We apply Υ, so $\Upsilon(\phi(X_0, X_1, \cdot, X_{n-1})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$. If we apply λ, then $\lambda(\Upsilon(\phi(X_0, X_1, \ldots, X_{n-1})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, + t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$.

On the other hand, we have $\sigma_\theta(X_0, X_1, \ldots, X_{n-1}) = (\theta(X_{n-1}), \theta(X_0), \ldots, \theta(X_{n-2}))$. We apply ϕ, so $\phi(\sigma_\theta(X_0, X_1, \ldots, X_{n-1})) = \phi(\theta(X_{n-1}), \theta(X_0), \ldots, \theta(X_{n-2})) = (x_{n-1} + y_{n-1} + z_{n-1} + t_{n-1}, x_0 + y_0 + z_0 + t_0, \ldots, x_{n-2} + y_{n-2} + z_{n-2} + t_{n-2}, \ldots, y_{n-1} + t_{n-1}, y_0 + t_0, \ldots, y_{n-2} + t_{n-2} + t_{n-2}, z_{n-1} + t_{n-1}, z_0 + t_0, \ldots, z_{n-2} + t_{n-2}, t_{n-1}, t_0, \ldots, t_{n-2})$. □

Theorem 4.1. *Let C be a skew cyclic code of length n over the ring R. Then the Gray image of C is permutation equivalent to skew 4-QC code of index 4 and length $4n$ over $\mathbb{F}_2$.*

Proof. We know that $\sigma_\theta(C) = C$ and $\phi(\sigma_\theta(C)) = C$. By Proposition 4.1, $\phi(\sigma_\theta(C)) = \phi(C) = \lambda(\Upsilon(\phi(C)))$. So we can say that $\phi(C)$ is permutation equivalent to l-QC code of index 4 and length $4n$ over $\mathbb{F}_2$. □

Let Γ be a map from $\mathbb{F}_2^{4n}$ to $\mathbb{F}_2^{4n}$ given by $\Gamma(X) = (\xi(X_{(0)}), \xi(X_{(1)}), \xi(X_{(2)}), \xi(X_{(3)}))$, where ξ is the map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ given by

$$\xi(X_{(i)}) = ((X_{(i,m-1)}), (X_{(i,0)}), \ldots, (X_{(i,m-2)})),$$

for every $X_{(i)} = (X_{(i,0)}, \ldots, X_{(i,m-1)})$ where $X_{(i,j)} \in \mathbb{F}_2^l$ for all $j = 0, 1, \ldots, m-1$. A code of length $4n$ over $\mathbb{F}_2$ is called l-QC code of index 4 if $\Gamma(C) = C$.

Proposition 4.2. *Let $\tau_{\theta,m,l}$ be a skew QC shift on R^n and ϕ be the Gray map over R^n. Then $\phi\tau_{\theta,m,l} = \lambda\Gamma\phi$.*

Proof. The proof is similar to the proof of the Proposition 4.1. □

Theorem 4.2. *Let C be a skew l-QC cyclic code of length $n(n = ml)$. Then, C is permutation equivalent to l-QC code of index 4 and with length $4n$ over $\mathbb{F}_2$.*

Proof. Since C is a skew QC code, so $\tau_{\theta,m,l}(C) = C$. Also $\phi(\tau_{\theta,m,l}(C)) = \phi(C)$ and by the previous Proposition $\phi((\tau_{\theta,m,l}(C)) = \lambda\Upsilon\phi(C)$. This implies that $\phi(C)$ is permutation equivalent to l-QC code of index 4 and with length $4n$ over $\mathbb{F}_2$. □

We finish by investigating the above results for skew GQC codes. Suppose that $m = \sum_{k=1}^{t} m_k$ and $X = (X_0, X_1, \ldots, X_{4m-1}) = (X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) \in F_2^{4m}$ for $i = 0, 1, 2, 3$. Consider the map Γ' from $\mathbb{F}_2^{4m}$ to $\mathbb{F}_2^{4m}$ given by $\Gamma'(X) = (\xi'(X_{(0)}), \xi'(X_{(1)}), \xi'(X_{(2)}), \xi'(X_{(3)}))$, where ξ' is the map from $\mathbb{F}_2^{m}$ to $\mathbb{F}_2^{m}$ given by $\xi'(X_{(i)}) = (X_{(i,m-1)}, X_{(i,0)}, \ldots, X_{(i,m-2)})$, for any $X_{(i)} = (X_{(i,0)}, ..., X_{(i,m-1)})$ such that $X_{(i,j)} \in \mathbb{F}_2^{m_k}$ for $j = 0, 1, \ldots, m-1$. Also let λ' be a map from $4m$ to $4m$ given by $\lambda'(X_{(0)}, X_{(1)}, X_{(2)}, X_{(3)}) = (X_{(0)}, X_{(2)}, X_{(1)}, X_{(3)})$.

Definition 4.2. *A code C of length $4m$ over F_2 is called GQC code of index 4 if $\Gamma'(C) = C$.*

By introduce the above maps and in similar ways of Proposition 4.1 and Theorem 4.1 we have the following results.

Theorem 4.3. *Let $\tau_{\theta,m}$ be the skew GQC shift on R^m where $m = \sum_{k=1}^{t} m_k$ and ϕ be the Gray map from R^m to F_2^{4m}. Then $\phi\tau_{\theta,m} = \lambda'\Gamma'\phi$.*

Theorem 4.4. *The Gray image a skew GQC code over R of length m and block length $(m_1, m_2, ..., m_t)$ is permutation equivalent to skew GQC code of index 4 and length $4m$ over F_2.*

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