On k-special R-implications

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Abstract. Triangular norm based implications play a significant role in many fields of mathematics and computer science. In this paper, the notion of k-special fuzzy implications is introduced for all $k \in [1, \infty[$. Then, it is shown that all k-special R-implications can be characterized by the k-Lipschitz continuity of the corresponding t-norms.

Keywords: fuzzy logics, fuzzy implications, triangular norms, k-Lipschitz continuity, k-special R-implications.

1. Introduction

Triangular norms (t-norms for short) and its related operations have various applications in many fields of mathematics, computer science and decision making [2,4,12,14,15]. Many existing results show that the connectives in fuzzy logic closely link to their counterparts in applications. In 1996, Hájek and Kohout uncovered the connections between implication connectives in fuzzy logic and implication quantifiers in the literature of GUHA data mining method in [3], the main result is that each special implication quantifier determines a special implication connective, and conversely each special implications connective is given by a special implicational quantifier. Recently, Emilia et. al. in [13] characterized all special R-implication connectives by 1-Lipschitz property of their corresponding t-norms. Inspired by those above, we introduce the k-speciality of R-implications, which can be characterized by k-Lipschitz continuity of the corresponding t-norms with $k \in [1, +\infty[$, which have been studied extensively in the framework of the controlling the output stability in the case of noisy inputs[5, 8, 9, 10]. By this mean, we can add some new insights into the fuzzy implications and investigate some properties of a subclass of BL-implications, which can be used to generate implication quantifiers in data mining in virtue of the close relationship between implication connectives and implication quantifiers.

1.1 Continuous Archimedean t-norms and their additive generators

For the basic notions and results on t-norms, we recommend references [6,7].

Lemma 1.1. *A t-norm T is continuous and Archimedean if and only if there is continuous strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$, and for all $(x,y) \in [0,1]^2, T(x,y) = f^{-1}(\min(f(x)+f(y), f(0)))$.*

Usually, the function f in Lemma 1.1 is called the additive generator of the t-norm T. It is known that all continuous Archimedean t-norms can be represented by their corresponding additive generators. Namely, a continuous t-norm T is characterized to be an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$, i.e.,

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}); & (x,y) \in]a_i, b_i[^2; \\ \min(x,y), & \text{else.} \end{cases} \tag{1}$$

with the index set $\mathcal{I}$ being countable and each T_i being a continuous Archimedean t-norm, which is isomorphic to Łukasiewicz or Product t-norm. According to [1,6,11], for any continuous Archimedean t-norm T with additive generator f, the R-implication I_T can also be represented by its additive generator f, i.e.,

$$I_T(x,y) = f^{-1}(\max(0, f(y) - f(x))). \tag{2}$$

Moreover, the R-implication of a continuous t-norm has related ordinal sum structure [1,2], i.e.,

$$I_T(x,y) = \begin{cases} g_i^{-1}(g_i(y) - g_i(x) + a_i); & (x,y) \in]a_i, b_i[^2, y < x; \\ I_{T_M}(x,y), & \text{else.} \end{cases} \tag{3}$$

where $T = (]a_i, b_i[, T_i)_{i\in\mathcal{I}}$ with the additive generator of T_i being f_i, and $g_i(x) = (b_i - a_i)f(\frac{x-a_i}{b_i-a_i}) + a_i$.

1.2 k-Lipschitz continuity of t-norms and k-convexity of its additive generators

Definition 1.1 ([8]). (i) *Let $f : [0,1] \to [0,+\infty[$ be a strictly decreasing function and $k \in [1,+\infty]$ be a real constant, then f is called k-convex, if*

$$f(x + k\varepsilon) - f(x) \le f(y+\varepsilon) - f(y)$$

holds for all $x \in [0,1[, y \in]0,1[$, with $x \le y$ and $\varepsilon \in]0, \min(1-y, \frac{1-x}{k})]$.

(ii) *A t-norm is called k-Lipschitz, if there is a real constant $k \in [1,+\infty[$, such that*

$$|T(x_1,y_1) - T(x_2,y_2)| \le k(|x_1-x_2| + |y_1-y_2|),$$

for all $(x_1,y_1),(x_2,y_2) \in [0,1]^2$.

The k-Lipschitz continuity [5, 8, 9] of t-norms was proposed in the text of stability of t-norms. Trivially, a t-norm is necessarily continuous if it is k-Lipschitz. In order to keep the paper self-contained, we list some related results here.

Lemma 1.2 ([8, 9]). *(i) A continuous Archimedean t-norm is k-Lipschitz if and only if its additive generator is k-convex;*

(ii) A continuous t-norm is k-Lipschitz if and only if its summands (continuous Archimedean t-norms) are all k-Lipschitz.

2. k-special R-implications

Usually, a binary operation $I : [0,1]^2 \to [0,1]$ is called fuzzy implication, if it holds that:

(i) $I(0,0) = I(0,1) = I(1,1) = 1, I(1,0) = 0$;

(ii) I is non-increasing w.r.t. the first argument and non-decreasing w.r.t. the second argument.

The prototypical example of fuzzy implication is the residual implication I_T (R-implication for short) of a left-continuous t-norm T, defined by $I_T(x,y) = \max\{t \in [0,1] | T(x,t) \leq y\}$. By the axioms of t-norms it can be concluded that $I_T(x,y) = 1$ for all $x \leq y$.

Definition 2.1. *Let $k \in [1,\infty[$, then a fuzzy implication I is called k-special if it follows that:$I(x,y) \leq I(x+\varepsilon, y+k\varepsilon)$, for all $(x,y) \in [0,1]^2, \varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$.*

Remark 2.1. If $k = 1$, then it is just the case (special fuzzy implications) discussed in [12], where a fuzzy implication I is called special if $I(x,y) \leq I(x+\varepsilon, y+\varepsilon)$for all $\varepsilon > 0$ with $\max(x+\varepsilon, y+\varepsilon) \leq 1$. Therefore, in the following we only focus on the cases $k \in]1,\infty[$.

Lemma 2.1. *If a fuzzy implication I is k-special, then I is l-special for all $l \geq k$.*

Proof. It is enough to consider the cases $l > k$. For all $\varepsilon > 0$, assume $(x,y) \in [0,1]^2$ and $x + \varepsilon \leq 1, y + k\varepsilon < y + l\varepsilon \leq 1$. Since I is k-special, then

$$I(x,y) \leq I(x+\varepsilon, y+k\varepsilon). \tag{4}$$

Since I is non-increasing w.r.t. the second argument, then

$$I(x+\varepsilon, y+k\varepsilon) \leq I(x+\varepsilon, y+l\varepsilon). \tag{5}$$

Therefore,$I(x,y) \leq I(x+\varepsilon, y+l\varepsilon)$ by the combination of (4) and (5). Thus I is l-special for all $l \geq k$. □

However, the converse is not true. That is to say, there may be a k-special fuzzy implication being not l-special, where $k > 1$ and $1 \leq l < k$(See Example 2.1). Moreover, by direct calculation, the Łuksiewicz implication I_{T_L} and Gödel implication I_{T_M}, defined by $I_{T_L}(x,y) = \max(1, 1-x+y)$ and

$$I_{T_M}(x,y) = \begin{cases} 1; & x \leq y \\ y, & \text{else.} \end{cases}$$

respectively, is k-special for all $k \geq 1$.

Theorem 2.1. *For any left-continuous t-norm T, if the R-implication I_T is k-special, then T is continuous.*

Proof. It is enough to check the right-continuity of T w.r.t. the first argument due to its commutativity. Assume that T is not right-continuous at a point $x \in [0,1[$. Trivially, the function $T(x,0) = 0$ is continuous. Thus there must be $z \in]0,1]$ and $d > 0$, such that $T(x,z) = y$, and for all $u > x, T(u,z) \geq T(x,z) + d = y + d$. In particular, $T(1,z) = z \geq y + d$.

In the following, we assume $z = I_T(x,y)$. In fact, $T(x,z) = y$ ensures $z \leq I_T(x,y)$, then $y = T(x,z) \leq T(x, I_T(x,y)) \leq y$. Thus $T(x, I_T(x,y)) = y$. Further, for all $u > x, T(u, I_T(x,y)) \geq T(u,z) \geq y + d$. Therefore, we can replace z with $I_T(x,y)$ whenever $z < I_T(x,y)$. Now let $\varepsilon = \min\{1-x, \frac{d}{2k}\}$, it follows that $T(x+\varepsilon, z) \geq y+d > y+k\varepsilon$, then $T(x+\varepsilon, z) > y+k\varepsilon$, which means that $I_T(x+\varepsilon, y+k\varepsilon) = \sup\{t \in [0,1] | T(x+\varepsilon, t) \leq y+k\varepsilon\} < z = I_T(x,y)$, which contradicts the k-speciality of I_T. □

Corollary 2.1. *k-special R-implications are BL-implications, i.e., R-implications of continuous t-norms.*

It must be mentioned that, not all BL-implications are special [13]. Furthermore, not all BL-implications are k-special. It can be checked similarly as [13] that, the R-implications of Schweizer-Sklar t-norms, i.e.,

$$T(x,y) = \begin{cases} \sqrt[p]{x^p + y^p - 1}, & x^p + y^p \geq 1; \\ 0, & \text{else.} \end{cases}$$

are not k-special whenever $p \neq 1$. Hence, k-special implications of left-continuous t-norms are a subclass of BL-implications. In other words, by Theorem 2.1, it holds that a left-continuous t-norm with its R-implication being k-special is necessarily continuous.

Theorem 2.2. *The R-implication I_T of a continuous Archimedean t-norm T is k-special if and only if the additive generator f of T is k-convex.*

Proof. "$\Rightarrow$": For all $(x,y) \in [0,1]^2$ and $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, there are the following cases to be discussed.

(i) If $x \leq y$, then it follows that $I_T(x,y) = I_T(x+\varepsilon, y+k\varepsilon) = 1$.

(ii) If $x > y$ with $x+\varepsilon \leq y+k\varepsilon$, then $f(y) - f(x) \geq 0 \geq f(y+k\varepsilon) - f(x+\varepsilon)$ by the non-increasingness of f. Thus $f(y+k\varepsilon) - f(y) \leq f(x+\varepsilon) - f(x)$.

(iii) If $x > y$ with $x+\varepsilon > y+k\varepsilon$, then $I_T(x,y) = f^{-1}(f(y) - f(x))$, $I_T(x+\varepsilon, y+k\varepsilon) = f^{-1}(f(y+k\varepsilon) - f(x+\varepsilon))$ by (2). With the k-speciality of I_T, $f^{-1}(f(y) - f(x)) \leq f^{-1}(f(y+k\varepsilon) - f(x+\varepsilon))$. It follows that $f(y) - f(x) \geq f(y+k\varepsilon) - f(x+\varepsilon)$, which means $f(y+k\varepsilon) - f(y) \leq f(x+\varepsilon) - f(x)$.

By all of the above, f is k-convex.

"$\Leftarrow$": For all $(x,y) \in [0,1]^2$ and $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, it can be gotten that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon) = 1$ for the case $x \leq y$ and the case $x > y$ with $x+\varepsilon \leq$

$y+k\varepsilon$. So it is enough to consider the case $x > y$ with $x+\varepsilon < y+k\varepsilon$, for which we have $I_T(x,y) = f^{-1}(f(y)-f(x))$, $I_T(x+\varepsilon, y+k\varepsilon) = f^{-1}(f(y+k\varepsilon)-f(x+\varepsilon))$. Moreover, the k-convexity of f ensures that $f(y)-f(x) \geq f(y+k\varepsilon)-f(x+\varepsilon)$, then $f^{-1}(f(y)-f(x)) \leq f^{-1}(f(y+k\varepsilon)-f(x+\varepsilon))$, i.e., $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$.

To sum up, I_T is k-special. □

By Lemma 1.2(i), we have the following result.

Corollary 2.2. *The R-implication of a continuous Archimedean t-norms T is k-special if and only if T is k-Lipschitz.*

Since the R-implications of continuous t-norms have ordinal sum structure, then we can characterize all k-special R-implications in the following.

Theorem 2.3. *The R-implication I_T of a continuous t-norm T is k-special if and only if the corresponding t-norm T is an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$, where each t-norm T_i is generated by a k-convex additive generator f_i. (Equivalently, each t-norm T_i is k-Lipschitz by Lemma 1.1(i).)*

Proof. "⇒": By the continuity of T, T can be represented as an ordinal sum of $(]a_i, b_i[, T_i)_{i\in\mathcal{I}}$ as (1), hence the R-implication I_T has the structure as (3). For each $i \in \mathcal{I}$ and $(x,y) \in [0,1]^2$ with all $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$, assume $x > y$, then it can be classified into the following two cases.

(i) If $x+\varepsilon \leq y+k\varepsilon$, then $f_i(y) - f_i(x) \geq 0 \geq f_i(y+k\varepsilon) - f_i(x+\varepsilon)$ by the non-increasingness of f_i which follows that $f_i(y+k\varepsilon) - f_i(y) \leq f_i(x+\varepsilon) - f_i(x)$.

(ii) If $x+\varepsilon > y+k\varepsilon$, then $I_T(x(b_i-a_i)+a_i, y(b_i-a_i)+a_i) = g_i^{-1}(g_i(y(b_i-a_i)+a_i) - g_i(x(b_i-a_i)+a_i)+a_i) \leq I_T((x+\varepsilon)(b_i-a_i)+a_i, (y+k\varepsilon)(b_i-a_i)+a_i) = g_i^{-1}(g_i((y+k\varepsilon)(b_i-a_i)+a_i) - g_i((x+\varepsilon)(b_i-a_i)+a_i)+a_i)$ by (3) and the k-speciality of I_T. Thus we have $g_i(y(b_i-a_i)+a_i) - g_i(x(b_i-a_i)+a_i) \geq g_i((y+k\varepsilon)(b_i-a_i)+a_i) - g_i((x+\varepsilon)(b_i-a_i)+a_i)$, which implies $f_i(y)-f_i(x) \geq f_i(y+k\varepsilon) - f_i(x+\varepsilon)$, whence $f_i(y+k\varepsilon) - f_i(y) \leq f_i(x+\varepsilon) - f_i(x)$.

Then we get the k-convexity of each f_i.

"⇐": For the case $x \leq y$ and the case $x > y$ with $x+\varepsilon \leq y+k\varepsilon$ it holds that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon) = 1$, where $\varepsilon \in]0, \min(1-x, \frac{1-y}{k})]$. For the case $x > y$ with $x+\varepsilon > y+k\varepsilon$, the proof can be divided into the following cases.

(i) If for each $i \in \mathcal{I}$, $(x,y) \notin]a_i, b_i[^2$ and $(x+\varepsilon, y+k\varepsilon) \notin]a_i, b_i[^2$, then $I_T(x,y) = I_{T_M}(x,y) = y \leq I_T(x+\varepsilon, y+k\varepsilon) = I_{T_M}(x+\varepsilon, y+k\varepsilon) = y+k\varepsilon$ by (3).

(ii) If $(x,y) \in]a_l, b_l[^2$ for some $l \in \mathcal{I}$ and $(x+\varepsilon, y+k\varepsilon) \notin]a_i, b_i[^2$ for each $i \in \mathcal{I}$, $I_T(x,y) \leq b_l$ and $I_T(x+\varepsilon, y+k\varepsilon) = I_{T_M}(x+\varepsilon, y+k\varepsilon) = y+k\varepsilon$. Since $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$ for $y+k\varepsilon \geq b_l$, it is enough to consider the case $y+k\varepsilon \in]a_l, b_l[$, for which it holds that $I_T(x,y) \leq I_T(x+\varepsilon, y+k\varepsilon)$. If not, there must be some x_0, y_0, ε_0 with $I_T(x_0,y_0) > I_T(x_0+\varepsilon_0, y_0+k\varepsilon_0) = y_0+k\varepsilon_0$, hence $T(x_0, y_0+k\varepsilon_0) \leq y_0$. Let $z_0 = I_T(x_0,y_0) = \max\{t \in [0,1] \mid T(x_0,t) \leq y_0\} > y_0+k\varepsilon_0$, then $T(x_0,z_0) \leq y_0$.

Moreover, $T(x_0, z_0) < y_0$. Indeed, if $T(x_0, z_0) = y_0$, then $y_0 \geq T(x_0, z_0) \geq T(x_0, y_0 + k\varepsilon_0) = y_0 \in]a_l, b_l[$, i.e., $T(x_0, z_0) = T(x_0, y_0 + k\varepsilon_0) = y_0 \in]a_l, b_l[$, which implies $g_l^{-1}(g_l(x_0) + g_l(z_0)) = g_l^{-1}(g_l(x_0) + g_l(y_0 + k\varepsilon_0)) = y_0$ with the fact that $g_l(x_0) + g_l(z_0), g_l(x_0) + g_l(y_0 + k\varepsilon_0) < g_l(a_l)$, thus $g_l(z_0) = g_l(y_0 + k\varepsilon_0)$, which contradicts the strictly decreasingness of g_l. So we have $T(x_0, y_0 + k\varepsilon_0) < y_0 < y_0 + k\varepsilon_0 = \min(x_0 + \varepsilon_0, y_0 + k\varepsilon_0) = T(x_0 + \varepsilon_0, y_0 + k\varepsilon_0)$. It follows that $T(x_0 + \varepsilon_0, y_0 + k\varepsilon_0) - T(x_0, y_0 + k\varepsilon_0) > k\varepsilon_0$ which contradicts the k-Lipschitz continuity of T implied by the k-convexity of each additive generator f_i with Lemma1.2(ii). To sum up, we have $I_T(x, y) \leq I_T(x + \varepsilon, y + k\varepsilon)$

(iii) If $(x, y) \in]a_i, b_i[^2$ and $(x + \varepsilon, y + k\varepsilon) \in]a_l, b_l[^2$for some $l, i \in \mathcal{I}$ with $i \neq l$, then $b_i \leq a_l$, whence $I_T(x, y) \leq b_i \leq a_l \leq I_T(x + \varepsilon, y + k\varepsilon)$.

(iv) If $(x, y) \in]a_i, b_i[^2$ and $(x + \varepsilon, y + k\varepsilon) \in]a_i, b_i[^2$ for some $i \in \mathcal{I}$, the k-speciality can be checked by the k-convexity of f_i.

(v) If $(x, y) \notin]a_i, b_i[^2$ for each $i \in \mathcal{I}$ and $(x + \varepsilon, y + k\varepsilon) \in]a_l, b_l[^2$for some $l \in \mathcal{I}$, then $y \leq a_l$, hence $I_T(x, y) = y \leq a_l \leq I_T(x + \varepsilon, y + k\varepsilon)$. □

Corollary 2.3. *An R-implication I_T of a left-continuous t-norm T is k-special if and only if T is k-Lipschitz.*

Proof. It can be easily checked by Theorem 2.1,Theorem 2.3 and Lemma 1.2. □

It is shown that if a t-norm T is k-Lipschitz, then T is l-Lipschitz for all $l \geq k$. Therefore Corollary 2.3 is consistent to Lemma 2.1.

There exists families of k-convex additive generators (see [5,8]). For a better understanding of k-speciality, we present an example of k-special R-implications.

Example 2.1. In virtue of the k-convex additive generators $t_k^{\frac{1}{2}}$ in [8], i.e., $t_k^{\frac{1}{2}}(x) = \min(1 - x, \frac{1}{2} + \frac{\frac{1}{2} - x}{k})$, with its corresponding t-norm $T_k^{\frac{1}{2}}$ being defined by

$$T_k^{\frac{1}{2}}(x, y) = \begin{cases} \max(0, x + y - k - \frac{1}{2}(k - 1)), & (x, y) \in [0, \frac{1}{2}]^2; \\ \max(0, x + ky - k), & (x, y) \in [0, \frac{1}{2}] \times]\frac{1}{2}, 1]; \\ \max(0, y + kx - k), & (x, y) \in]\frac{1}{2}, 1] \times [0, \frac{1}{2}]; \\ \max(0, k(x + y) - k - \frac{1}{2}(kn - 1)), & (x, y) \in]\frac{1}{2}, 1]^2, x + y \leq \frac{3}{2}; \\ x + y - 1, & (x, y) \in]\frac{1}{2}, 1]^2, x + y > \frac{3}{2} \end{cases}$$

where $k \in [1, +\infty[$. Then by direct calculation we can have a family of k-special R-implications I_k of the t-norms $T_k^{\frac{1}{2}}$, defined by

$$I_k^{\frac{1}{2}}(x, y) = \begin{cases} 1, & x \leq y; \\ 1 - \frac{x - y}{k}, & (x, y) \in [0, \frac{1}{2}[^2, x > y; \\ 1 - x + y, & (x, y) \in [\frac{1}{2}, 1]^2, x > y \\ \frac{y}{k} - x - \frac{1}{2k} + \frac{3}{2}, & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}[, x - \frac{y}{k} < 1 - \frac{1}{2k}, x > y; \\ y - kx + k, & (x, y) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}[, x - \frac{y}{k} \geq 1 - \frac{1}{2k}, x > y; \end{cases}$$

Let $k = 2$, since $t_2^{\frac{1}{2}}$ is 2-convex, then $I_2^{\frac{1}{2}}$ is 2-special. However, $I_2^{\frac{1}{2}}$ is not 1-special (special). Indeed, let $x_0 = 0.3, y_0 = 0.1, \varepsilon_0 = 0.3$, then $I_2^{\frac{1}{2}}(x_0, y_0) = 0.9$ and $I_2^{\frac{1}{2}}(x_0+\varepsilon_0, y_0+\varepsilon_0) = 0.85$, hence $I_2^{\frac{1}{2}}(x_0, y_0) \le I_2^{\frac{1}{2}}(x_0+\varepsilon_0, y_0+\varepsilon_0)$ does not hold.

3. Conclusion

The notion of k-special fuzzy implications is introduced and the k-special R-implications are characterized by k-Lipschitz continuity of the corresponding t-norms, also the k-convexity of the corresponding additive generators. We would expect the k-special implication connectives be applicable in generating new tools in (fuzzy) data analysis similar to that in [3].

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