Complete moment convergence for weighted sums of negatively orthant dependent random variables

Wenhua Lv*
School of Mathematical and Finance
Chuzhou University
Chuzhou 239000, China
qdlwhat2005@126.com

Yongfeng Wu
College of Mathematics and Computer Science
Tongling University
Tongling 244000
China
wyfwyf@126.com

Abstract. The authors study the complete moment convergence for weighted sums of negatively orthant dependent random variables and obtain some new results. These results extend and improve the corresponding theorems of Chen [P. Chen., 2016, Complete Convergence and Strong Laws of Large Numbers For Weighted Sums of Negatively Orthant Dependent Random Variables, Acta Math. Hungar., 148 (1), 83-95], complete moment convergence, weighted sums, negatively orthant dependent random variables.
Keywords: complete moment convergence, weighted sums, L^p convergence, negatively orthant dependent.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. The limiting behavior of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ play important roles in some useful linear statistics, many authors studied the strong convergence for the weighted sums. We refer the reader to Cuzick (1995), Wu (1999), Bai and Cheng (2000), Sung (2001), Chen and Gan (2007), Cai (2008), Wu (2011), Zarei and Jabbari (2011), Sung (2011, 2012), Shen (2012), Chen and Sung (2014).

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

The concept of the complete convergence was introduced by Hsu and Robbins (1947).

*. Corresponding author

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty, \quad \text{for some or all } \varepsilon > 0,$$

then the above result was called complete moment convergence by Chow (1988), which is the more general version of the complete convergence.

Joag-Dev and Proschan (1983) introduced the following negatively associated concept. A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinate-wise nondecreasing functions f on R^A and g on R^B,

$$\mathrm{Cov}(f(X_i, i \in A), g(Y_j, j \in B)) \leq 0$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The following concept was introduced by Lehmann (1966).

Definition 1.1. A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively orthant dependent if the following two inequalities hold,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i),$$

for all real numbers $x_1, \ldots, x_n$. An infinite family of random variables is negatively orthant dependent if every finite subfamily is negatively orthant dependent.

Note that negative association implies negative orthant dependence, but the converse does not hold.

Cai (2008) studied the complete convergence for weighted sums of identically distributed NA random variables, and obtained the results under the stronger moment condition $E\exp(h|X|^\gamma) < \infty$ for some $h > 0$. Sung (2011) obtained the following theorem, which improved the main results of Cai (2008) by replacing some much weaker moment conditions.

Theorem A. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n), \tag{1.1}$$

for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \le 2$. Then the following statements hold:

(i) If $\alpha > \gamma$, $E|X|^\alpha < \infty$, then

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le m\le n}\Big|\sum_{i=1}^{m} a_{ni}X_i\Big| > b_n\varepsilon\Big) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.2}$$

(ii) If $\alpha = \gamma$, then $E|X|^\alpha \log(1 + |X|) < \infty$ implies (1.2).

(iii) If $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies (1.2).

For the negatively orthant dependent random variables, under the same moment condition as (1.2), there are some results about

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\Big|\sum_{i=1}^{n} a_{ni}X_i\Big| > b_n\varepsilon\Big) < \infty, \quad \text{for all } \varepsilon > 0, \tag{1.3}$$

(see Huang and Wang (2012), Shen (2013) and Zhang and Wang (2014)). However, (1.2) is stronger than (1.3). Recently, Chen and Sung (2016) proved (1.3) for the case $\alpha < \gamma$.

The main purpose of this article is to study the complete moment convergence, we shall extend and improve the main results of Chen (2016) by obtaining a much stronger conclusion under the same conditions. Using our method, we can easily get L^p convergence for negatively orthant dependent random variables.

Throughout this paper, the symbol C represents positive constants whose values may change from one place to another. For a finite set A the symbol $\sharp(A)$ denotes the number of elements in the set A.

2. Preliminaries

To prove our main result, the following lemmas are needed, Lemma 2.1 is the Rosenthal-type inequality for the sum of negatively orthant dependent random variables, and Lemma 2.2 is a Rosenthal-type inequality for maximum partial sums of negatively orthant dependent random variables, which are all crucial for the proof of our main results.

Lemma 2.1 (Asadian et al. 2006)**.** Let $\{X_n, n \ge 1\}$ be a sequence of negatively orthant dependent random variables with mean zero, and $E|X_n|^q < \infty$ for some $q \ge 2$ and all $n \ge 1$. Let $S_n = \sum_{i=1}^n X_i$, then,

$$|S_n|^q \le C\Big\{\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{q/2}\Big\} \tag{2.1}$$

where the constant C depends only on q.

Using Lemma (2.1) and Theorem 3 in Móricz (1976), it can be get the following Rosenthal type inequality for the maximum of partial sums of negatively orthant dependent random variables.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of negatively orthant dependent random variables with mean zero and $E|X_k|^q < \infty$ for some $q \geq 2$ and every $1 \leq k \leq n$. Then, there exists a positive constant C depends only on q such that

$$E \max_{1\leq m\leq n} |S_m|^q \leq C\left(\frac{\log 2n}{\log 2}\right)^q \left\{\sum_{k=1}^n E|X_k|^q + \Big(\sum_{k=1}^n EX_k^2\Big)^{q/2}\right\}. \tag{2.2}$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leq 2$. For all $n \geq 1$, let

$$A_n = \{1 \leq i \leq n : |a_{ni}| \leq n^{1/\alpha}(\log n)^{-t_1}\}, \tag{2.3}$$

where $t > 0$ will be specified later. Defined b_n as follows:

$$b_n = n^{1/\alpha}(\log n)^{1/\gamma}. \tag{2.4}$$

Lemma 2.3. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \leq 2$, and X be a random variable with $E|X|^\gamma < \infty$, for $\gamma > \alpha$. Then, for $\beta > 0$ we have,

$$H =: \sum_{n=2}^\infty n^{-1} b_n^{-\alpha} \sum_{i\in A_n} (\log n)^{q-\beta q+\alpha\beta} E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n(\log n)^{-\beta}) \leq CE|X|^\gamma,$$

where b_n is defined as (2.4).

Proof. Since $\alpha < \gamma$, we have,

$$\begin{aligned}
H &\leq \sum_{n=2}^\infty n^{-1} \sum_{i\in A_n} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\leq \sum_{n=2}^\infty n^{-1-\gamma/\alpha} \sum_{i\in A_n} (\log n)^{q-\beta q+\beta\gamma-1} E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta}) \\
&\leq \sum_{n=2}^\infty n^{-2} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} \sum_{i\in A_n} |a_{ni}|^\alpha E|X|^\gamma \\
&\leq \sum_{n=2}^\infty n^{-1} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} E|X|^\gamma.
\end{aligned}$$

Hence $H \leq CE|X|^\gamma$ if we take t_1 large enough such that $t_1(\gamma-\alpha)+\beta q-\beta\gamma-q > 0$.

Lemma 2.4 Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$, and X be a random variable with $E|X|^\gamma < \infty$ for $\gamma > \alpha$. Then, for $q > \max\{\alpha, \gamma\}$, we have

$$L =: \sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log n)^q \sum_{i \in A_n} \int_1^{\infty} t^{-q/\alpha} E|a_{ni}X|^q I(|a_{ni}X| \le b_n (\log n)^{-\beta}) \mathrm{d}t$$
$$\le CE|X|^\gamma,$$

where b_n is defined as (2.4).

Proof. Since $q > \max\{\alpha, \gamma\}$, we can get,

$$L \le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} (\log n)^q \sum_{i \in A_n} |a_{ni}|^q E|X|^q I(|a_{ni}X| \le b_n (\log n)^{-\beta})$$
$$\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-\gamma} (\log n)^{q-\beta q+\beta\gamma} \sum_{i \in A_n} |a_{ni}|^\gamma E|X|^\gamma I(|a_{ni}X| \le b_n (\log n)^{-\beta})$$
$$\le C \sum_{n=2}^{\infty} n^{\gamma/\alpha-2} b_n^{-\gamma} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma} \sum_{i \in A_n} |a_{ni}|^\alpha E|X|^\gamma I(|a_{ni}X| \le b_n (\log n)^{-\beta})$$
$$\le C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1} E|X|^\gamma.$$

Hence, $L \le CE|X|^\gamma$ if we take t_1 large enough such that $t_1(\gamma-\alpha)+\beta q-\beta\gamma-q > 0$.

3. Main result

In this section, we state our main results and their proofs.

Theorem 3.1. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively orthant dependent random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$ for some $\gamma > \alpha$. Furthermore, suppose that $EX_i = 0$ when $1 < \alpha < 2$. If $E|X|^\gamma < \infty$, then

$$\sum_{n=2}^{\infty} n^{-1} E\left\{ b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| - \varepsilon \right\}_+^\alpha < \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

Proof. We may assume that $a_{ni} > 0$ and $\sum_{i=1}^{n} |a_{ni}|^\alpha \le n$. For any given $\varepsilon > 0$, we have

$$\sum_{n=2}^{\infty} n^{-1} E\left\{ b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| - \varepsilon \right\}_+^\alpha$$
$$= \sum_{n=2}^{\infty} n^{-1} \int_0^{\infty} P\left(b_n^{-1} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) \mathrm{d}t$$

$$\leq \sum_{n=2}^{\infty} n^{-1} P\Big(\max_{1\leq m\leq n}\Big|\sum_{i=1}^{m} a_{ni}X_i\Big| > b_n\varepsilon\Big)$$
$$+ \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\leq m\leq n}\Big|\sum_{i=1}^{m} a_{ni}X_i\Big| > b_n t^{1/\alpha}\Big)\mathrm{d}t$$
$$=: I_1 + I_2.$$

Therefore, to prove (3.1), it needs only to prove that $I_1 < \infty$ and $I_2 < \infty$. By Theorem 2.1 and Theorem 2.2 of Chen. P. (2016), we get directly $I_1 < \infty$. For all $t \geq 1$, we denote

$$X_{ni}(1) = a_{ni}X_iI(|a_{ni}X_i| \leq b_n(\log n)^{-\beta}t^{1/\alpha})$$
$$+ b_n(\log n)^{-\beta}t^{1/\alpha}I(a_{ni}X_i > b_n(\log n)^{-\beta}t^{1/\alpha})$$
$$- b_n(\log n)^{-\beta}t^{1/\alpha}I(a_{ni}X_i < -b_n(\log n)^{-\beta}t^{1/\alpha})$$
$$X_{ni}(2) = \Big(a_{ni}X_i - b_n(\log n)^{-\beta}t^{1/\alpha}\Big)\, I(b_n(\log n)^{-\beta}t^{1/\alpha} < a_{ni}X_i) \leq b_n\varepsilon t^{1/\alpha}/4N)$$
$$X_{ni}(3) = \Big(a_{ni}X_i + b_n(\log n)^{-\beta}t^{1/\alpha}\Big)\, I(-b_n\varepsilon t^{1/\alpha}/4N \leq a_{ni}X_i < -b_n(\log n)^{-\beta}t^{1/\alpha})$$
$$X_{ni}(4) = \Big(a_{ni}X_i - b_n(\log n)^{-\beta}t^{1/\alpha}\Big)\, I(a_{ni}X_i > b_n\varepsilon t^{1/\alpha}/4N)$$
$$+ \Big(a_{ni}X_i + b_n(\log n)^{-\beta}t^{1/\alpha}\Big)\, I(a_{ni}X_i < -b_n\varepsilon t^{1/\alpha}/4N)$$

Then,

$$I_2 \leq \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\big(\max_{1\leq m\leq n}|\sum_{i=1}^{m} X_{ni}(1)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t$$
$$+ \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\big(\max_{1\leq m\leq n}|\sum_{i=1}^{m} X_{ni}(2)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t$$
$$+ \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\big(\max_{1\leq m\leq n}|\sum_{i=1}^{m} X_{ni}(3)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t$$
$$+ \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\big(\max_{1\leq m\leq n}|\sum_{i=1}^{m} X_{ni}(4)| > b_n\varepsilon t^{1/\alpha}/4\big)\mathrm{d}t$$
$$=: J_1 + J_2 + J_3 + J_4.$$

To prove $J_1 < \infty$, we first show

$$\sup_{t\geq 1} t^{-1/\alpha}b_n^{-1} \max_{1\leq m\leq n}\Big|\sum_{i=1}^{m} EX_{ni}(1)\Big| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

For $1 < \alpha \leq 2$, by $EX_i = 0$, $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$, $|X_{ni}(1)| \leq a_{ni}|X_i|$, we have

$$\sup_{t\geq 1} t^{-1/\alpha}b_n^{-1} \max_{1\leq m\leq n}\Big|\sum_{i=1}^{m} EX_{ni}(1)\Big|$$

$$
\begin{aligned}
&\le \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni} E X_i I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha})\right| \\
&+ \sup_{t\ge 1}(\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&= \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni} E X_i I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha})\right| \\
&+ \sup_{t\ge 1}(\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C(\log n)^{\beta(\alpha-1)} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| > b_n(\log n)^{-\beta}) \\
&\le CE|X|^{\alpha} (\log n)^{\beta(\alpha-1)-\alpha/\gamma} \to 0
\end{aligned}
$$

as $n \to \infty$, since $0 < \beta < 1/\gamma$ and $E|X|^{\alpha} \le (E|X|^{\gamma})^{\alpha/\gamma} < \infty$. For $0 < \alpha \le 1$, we have

$$
\begin{aligned}
&\sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} E X_{ni}(1)\right| \\
&\le \sup_{t\ge 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&+ \sup_{t\ge 1}(\log n)^{-\beta} \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le \sup_{t\ge 1} t^{-1} b_n^{-\alpha} (\log n)^{\beta(\alpha-1)} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| \le b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&+ \sup_{t\ge 1} t^{-1} b_n^{-\alpha} (\log n)^{\beta(\alpha-1)} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X_i| > b_n(\log n)^{-\beta} t^{1/\alpha}) \\
&\le C(\log n)^{\beta(\alpha-1)-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

From (3.2), we know that while n is sufficiently large,

$$
t^{-1/\alpha} b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} E X_{ni}(1)\right| \le b_n t^{1/\alpha}/8. \tag{3.3}
$$

holds uniformly for $t \ge 1$.

Suppose $q > \max\{2,\, 2\gamma/\alpha\}$, by (3.3) and the Markov inequality, we get

$$\begin{aligned} J_1 &= \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n}\Big|\sum_{i=1}^{m} X_{ni}(1)\Big| > b_n\varepsilon t^{1/\alpha}/4\Big)\mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\Big(\max_{1\le m\le n}\Big|\sum_{i=1}^{m} (X_{ni}(1)-EX_{ni}(1))\Big| > b_n\varepsilon t^{1/\alpha}/8\Big)\mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} E\max_{1\le m\le n}\Big|\sum_{i=1}^{m} (X_{ni}(1)-EX_{ni}(1))\Big|^q \mathrm{d}t \end{aligned}$$

For $n \ge 1$ and $t > 0$, let A_n is defined as (2.3), and $B_n = \{1 \le i \le n : |a_{ni}| > n^{1/\alpha}(\log n)^{-t_1}\}$. By the fact that $n \ge \sum_{i=1}^{n} |a_{ni}|^\alpha \ge \sum_{i\in B_n} |a_{ni}|^\alpha \ge n(\log n)^{-t_1\alpha}\sharp B_n$ then

$$\sharp B_n \le (\log n)^{t_1\alpha}. \tag{3.4}$$

Then, By Lemma 2.2, we get that

$$\begin{aligned} &J_1 \le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} E\max_{1\le m\le n}\Big|\sum_{i\le m, i\in A_n} (X_{ni}(1)-EX_{ni}(1))\Big|^q \\ &+ \Big|\sum_{i\le m, i\in B_n} (X_{ni}(1)-EX_{ni}(1))\Big|^q \mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} \left\{(\log n)^q \left(\sum_{i\in A_n} E|X_{ni}(1)|^q + \left(\sum_{i\in A_n} EX_{ni}^2(1)\right)^{q/2}\right)\right\}\mathrm{d}t \\ &+ C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} \left\{(\log\log n)^q \left(\sum_{i\in B_n} E|X_{ni}(1)|^q + \left(\sum_{i\in B_n} EX_{ni}^2(1)\right)^{q/2}\right)\right\}\mathrm{d}t \\ &=: J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

Firstly, we will prove $J_{12} < \infty$ in two cases ($\alpha < \gamma < 2$ and $\gamma \ge 2$). For the case $\alpha < \gamma < 2$, since $q > \max\{2, 2\gamma/\alpha\}$, $|X_{ni}(1)| \le b_n(\log n)^{-\beta}t^{1/\alpha}$ and $|a_{ni}| \le n^{1/\alpha}(\log n)^{-t_1}$ for $i \in A_n$, we have,

$$\begin{aligned} J_{12} &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \int_1^{\infty} t^{-q/\alpha} (\log n)^q \left(\sum_{i\in A_n} EX_{ni}^2(1)\right)^{q/2} \mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} \left(b_n^{-\gamma} t^{-\gamma/\alpha} (\log n)^{\beta\gamma+2-2\beta} \sum_{i\in A_n} |a_{ni}|^\gamma E|X|^\gamma\right)^{q/2} \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}\left(n^{-1}t^{-\gamma/\alpha}(\log n)^{-t_1(\gamma-\alpha)+\beta\gamma-2\beta+1}\sum_{i\in A_n}|a_{ni}|^{\alpha}E|X|^{\gamma}\right)^{q/2}\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)}(E|X|^{\gamma})^{q/2}\int_1^{\infty} t^{-\frac{q\gamma}{2\alpha}}\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)}(E|X|^{\gamma})^{q/2}.
\end{aligned}
$$

Since $\alpha<\gamma$ and $E|X|^{\gamma}<\infty$, hence $J_{12}<\infty$ if we take t_1 large enough such that $\frac{q}{2}(t_1(\gamma-\alpha)-\beta\gamma+2\beta-1)>1$. For the case $\gamma\ge 2$, since $|X_{ni}(1)|\le|a_{ni}X_i|$, we can similarly have,

$$
\begin{aligned}
J_{12} &= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log n)^{q}\left(\sum_{i\in A_n}E(X_{ni}^2(1))\right)^{q/2}\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}(\log n)^{q}\int_1^{\infty} t^{-q/\alpha}\left(\sum_{i\in A_n}a_{ni}^2EX_i^2\right)^{q/2}\mathrm{d}t\\
&\le C(EX^2)^{q/2}\sum_{n=2}^{\infty} n^{-1}b_n^{-q}(\log n)^{q}\\
&\cdot\int_1^{\infty} t^{-q/\alpha}\left(n^{-1+2/\alpha}(\log n)^{-t_1(2-\alpha)}\sum_{i\in A_n}|a_{ni}|^{\alpha}\right)^{q/2}\mathrm{d}t\\
&\le C(EX^{\gamma})^{q/\gamma}\sum_{n=2}^{\infty} n^{-1}(\log n)^{-t_1q(2-\alpha)/2-q/\gamma+q}.
\end{aligned}
$$

Hence, $J_{12}<\infty$ if we take t_1 large enough such that $t_1q(2-\alpha)/2+q/\gamma-q>1$. For J_{11}, we have,

$$
\begin{aligned}
J_{11} &= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}E|X_{ni}(1)|^{q}\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}|a_{ni}|^{q}E|X|^{q}I(|a_{ni}X|\\
&\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\
&+C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}(\log n)^{q-\beta q}\sum_{i\in A_n}P(|a_{ni}X|>b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t\\
&= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log n)^{q}\sum_{i\in A_n}|a_{ni}|^{q}E|X|^{q}I(|a_{ni}X|\\
&\le b_n(\log n)^{-\beta})\mathrm{d}t
\end{aligned}
$$

$$+ C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log n)^q \sum_{i\in A_n}|a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$+ C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}(\log n)^{q-\beta q}\sum_{i\in A_n}P(|a_{ni}X| > b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$=: J_{11}^{(1)} + J_{11}^{(2)} + J_{11}^{(3)}$$

For $J_{11}^{(3)}$, by Markov inequality, we have,

$$J_{11}^{(3)} \le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\gamma}(\log n)^{q-\beta q+\beta\gamma}\int_1^{\infty} t^{-\gamma/\alpha}$$
$$\times \sum_{i\in A_n}E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\gamma}(\log n)^{q-\beta q+\beta\gamma}\sum_{i\in A_n}E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta})\int_1^{\infty} t^{-\gamma/\alpha}\mathrm{d}t$$
$$\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\gamma}(\log n)^{q-\beta q+\beta\gamma}\sum_{i\in A_n}E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$= C\sum_{n=2}^{\infty} n^{-1-\gamma/\alpha}\sum_{i\in A_n}(\log n)^{q-\beta q+\beta\gamma-1}E|a_{ni}X|^\gamma I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-2}(\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1}\sum_{i\in A_n}|a_{ni}|^\alpha E|X|^\gamma$$
$$\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-t_1(\gamma-\alpha)+q-\beta q+\beta\gamma-1}E|X|^\gamma.$$

Hence, $J_{11}^{(3)} \le CE|X|^\gamma < \infty$ if t_1 is large enough such that $t_1(\gamma-\alpha)+\beta q-\beta\gamma-q > 0$. For $J_{11}^{(2)}$, by $q > 2 \ge \alpha$, we have

$$\sum_{i\in A_n}\int_1^{\infty} t^{-q/\alpha}(\log n)^q|a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$\le \sum_{i\in A_n}\sum_{m=1}^{\infty}\int_m^{m+1} t^{-q/\alpha}(\log n)^q|a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$\le \sum_{i\in A_n}\sum_{m=1}^{\infty} m^{-q/\alpha}(\log n)^q|a_{ni}|^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|$$

$$\le b_n(\log n)^{-\beta}(m+1)^{1/\alpha})$$
$$\le \sum_{i\in A_n}\sum_{m=1}^{\infty} m^{-q/\alpha}\sum_{s=1}^{m}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})$$
$$\le \sum_{i\in A_n}\sum_{s=1}^{\infty}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})\sum_{m=s}^{\infty} m^{-q/\alpha}$$
$$\le \sum_{i\in A_n}\sum_{s=1}^{\infty} s^{-q/\alpha+1}|a_{ni}|^q(\log n)^q E|X|^q I(b_n(\log n)^{-\beta}s^{1/\alpha} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}(s+1)^{1/\alpha}).$$

Since $q > \alpha$, then,

$$J_{11}^{(2)} \le C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in A_n}\sum_{s=1}^{\infty}(\log n)^{q+\beta\alpha-\beta q} b_n^{-\alpha}|a_{ni}|^{\alpha}$$
$$\times |E|X|^{\alpha} I(b_n(\log n)^{-\beta}s^{1/\alpha} < |a_{ni}X| \le b_n(\log n)^{-\beta}(s+1)^{1/\alpha})$$
$$\le C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in A_n}(\log n)^{q+\beta\alpha-\beta q} b_n^{-\alpha}|a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X| > b_n(\log n)^{-\beta}).$$

Then, by Lemma 2.3, we can get $J_{11}^{(2)} \le CE|X|^{\gamma} < \infty$. For $J_{11}^{(1)}$, by Lemma 2.4, we can get $J_{11}^{(1)} \le CE|X|^{\gamma} < \infty$. Consequently, we get $J_{11} < \infty$. For J_{13}, we have

$$J_{13} = C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q \sum_{i\in B_n} E|X_{ni}(1)|^q \mathrm{d}t$$
$$= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\sum_{i\in B_n}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q E|X|^q I(|a_{ni}X| \le b_n(\log n)^{-\beta})\mathrm{d}t$$
$$+ C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\sum_{i\in B_n}\int_1^{\infty} t^{-q/\alpha}(\log\log n)^q E|X|^q I(b_n(\log n)^{-\beta} < |a_{ni}X|$$
$$\le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$+ C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in B_n}\int_1^{\infty}(\log\log n)^q(\log n)^{-\beta q} P(|a_{ni}X| > b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$=: J_{13}^{(1)} + J_{13}^{(2)} + J_{13}^{(3)}.$$

For $J_{13}^{(3)}$, by $q > 2\gamma/\alpha$ and $\alpha \le 2$, we have $q > \gamma$, then,

$$J_{13}^{(3)} = C\sum_{n=2}^{\infty} n^{-1}\sum_{i\in B_n}\int_1^{\infty}(\log\log n)^q(\log n)^{-\beta q}P(|a_{ni}X| > b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\alpha}\sum_{i\in B_n}(\log\log n)^q(\log n)^{-\beta(q-\alpha)}E|a_{ni}X|^{\alpha}I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-1}(\log\log n)^q(\log n)^{-\beta(q-\gamma)}b_n^{-\gamma}\sum_{i\in B_n}E|a_{ni}X|^{\gamma}I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-1-\gamma/\alpha}(\log\log n)^q(\log n)^{-\beta(q-\gamma)-1}\sum_{i\in B_n}E|a_{ni}X|^{\gamma}I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-1}(\log\log n)^q(\log n)^{-\beta(q-\gamma)-1}E|X|^{\gamma} < \infty$$

For $J_{13}^{(1)}$, By $q > \gamma$, we have

$$J_{13}^{(1)} = C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i\in B_n}\int_1^{\infty}t^{-q/\alpha}(\log\log n)^qE|X|^qI(|a_{ni}X| \le b_n(\log n)^{-\beta})\mathrm{d}t$$
$$\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}(\log\log n)^q\sum_{i\in B_n}|a_{ni}|^qE|X|^qI(|a_{ni}X| \le b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\gamma}(\log\log n)^q(\log n)^{-\beta(q-\gamma)}\sum_{i\in B_n}|a_{ni}|^{\gamma}E|X|^{\gamma}I(|a_{ni}X|\le b_n(\log n)^{-\beta})$$
$$\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\beta(q-\gamma)-1}(\log\log n)^qE|X|^{\gamma} < \infty.$$

For $J_{13}^{(2)}$, similar to the proof of $J_{11}^{(2)}$, by $q > 2 \ge \alpha$ and Lemma 2.2, we have

$$J_{13}^{(2)} = C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i\in B_n}\int_1^{\infty}t^{-q/\alpha}(\log\log n)^q$$
$$\times E|X|^qI(b_n(\log n)^{-\beta} < |a_{ni}X| \le b_n(\log n)^{-\beta}t^{1/\alpha})\mathrm{d}t$$
$$\le Cn^{-1}b_n^{-\alpha}(\log n)^{-\beta(q-\alpha)}(\log\log n)^q\sum_{i\in B_n}|a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le Cn^{-1}b_n^{-\gamma}(\log n)^{-\beta(q-\gamma)}(\log\log n)^q\sum_{i\in B_n}|a_{ni}|^{\alpha}E|X|^{\gamma}I(|a_{ni}X| > b_n(\log n)^{-\beta})$$
$$\le Cn^{-1}(\log n)^{-\beta(q-\gamma)-1}(\log\log n)^qE|X|^{\gamma} < \infty.$$

For J_{14}, we have

$$\begin{aligned} J_{14} &= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}(\log\log n)^q \int_1^{\infty} t^{-q/\alpha}\left(\sum_{i\in B_n} EX_{ni}^2(1)\right)^{q/2} \mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1}(\log\log n)^q \int_1^{\infty}\left(b_n^{-\alpha}t^{-1}(\log n)^{-\beta(2-\alpha)}\sum_{i\in B_n}|a_{ni}|^{\alpha}E|X|^{\alpha}\right)^{q/2} \mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1}(\log\log n)^q \int_1^{\infty} t^{-q/2}\left((\log n)^{-\beta(2-\alpha)-\alpha/\gamma}E|X|^{\gamma}\right)^{q/2}\mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q\alpha/2\gamma-q\beta(2-\alpha)/2}(\log\log n)^q(E|X|^{\gamma})^{q\alpha/2\gamma}. \end{aligned}$$

Then, $J_{14} \le (E|X|^{\gamma})^{q\alpha/2\gamma} < \infty$ for $q > 2\gamma/\alpha$ and $\alpha \le 2$.Above all, we have $J_1 < \infty$. For J_2, first note that $0 \le X_{ni}(2) \le b_n\varepsilon t^{1/\alpha}/4N$, then

$$\begin{aligned} J_2 &= \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\left(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(2)| > b_n\varepsilon t^{1/\alpha}/4\right)\mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}\left(\sum_{i=1}^{n} P(a_{n,i}X_i > b_n(\log n)^{-\beta}t^{1/\alpha})\right)^N \mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}\left(b_n^{-\gamma}(\log n)^{\beta\gamma}t^{-\gamma/\alpha}E|X|^{\gamma}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\right)^N \mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1}(\log n)^{(\beta\gamma-1)N}\int_1^{\infty} t^{-N\gamma/\alpha}\mathrm{d}t. \end{aligned}$$

Hence, $J_2 < \infty$ if we take N large enough such that $(1-\beta\gamma)N > 1$. Similar to the proof of J_2, we can get that $J_3 < \infty$ if we take N large enough such that $(1-\beta\gamma)N > 1$. For J_4, it can be get that,

$$\begin{aligned} J_4 &= \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\left(\max_{1\le m\le n}|\sum_{i=1}^{m} X_{ni}(4)| > b_n\varepsilon t^{1/\alpha}/4\right)\mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\left(\sum_{i=1}^{n}|X_{ni}(4)| > b_n\varepsilon t^{1/\alpha}/4\right)\mathrm{d}t \\ &\le \sum_{n=2}^{\infty} n^{-1}\sum_{i=1}^{n} b_n^{-q}\varepsilon^{-q}\int_1^{\infty} t^{-q/\alpha}E|X_{ni}(4)|^q\mathrm{d}t \\ &\le C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| > b_n\varepsilon/4N). \end{aligned}$$

Hence, $J_4 \le CE|X|^{\gamma} < \infty$ by Lemma 2.3 of Wu et al. (2014). The proof is completed. $\square$

Remark 3.1. Noting that

$$\begin{aligned}&\sum_{n=2}^{\infty} n^{-1} E\left\{b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| - \varepsilon\right\}_+^{\alpha}\\&=\int_0^{\infty}\sum_{n=2}^{\infty} n^{-1} P\left(b_n^{-1} \max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right)\mathrm{d}t.\end{aligned}$$

Therefore, Theorem 3.1 extends and improves the main results of Chen (2016).

Using the arguments similar to the proof of Theorem 3.1, we can get the following results about L^p convergence.

Theorem 3.2. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed negatively orthant dependent random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1.1) for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > \alpha$. Furthermore, suppose that $EX_i = 0$ when $1 < \alpha < 2$. If $E|X|^\gamma < \infty$, then

$$\frac{1}{b_n}\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| \xrightarrow{L^\alpha} 0.$$

Proof. Following the notation and arguments as Theorem 3.1, we have

$$\begin{aligned}&E\left(\frac{1}{b_n}\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right|\right)^{\alpha}\\&= b_n^{-\alpha}\int_0^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > t^{1/\alpha}\right)\mathrm{d}t\\&\le \varepsilon + b_n^{-\alpha}\int_{b_n^\alpha\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > t^{1/\alpha}\right)\mathrm{d}t\\&\le \varepsilon + b_n^{-\alpha}\int_{b_n^\alpha\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(1)\right| > t^{1/\alpha}\right)\mathrm{d}t\\&+ b_n^{-\alpha}\int_{b_n^\alpha\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(2)\right| > t^{1/\alpha}\right)\mathrm{d}t\\&+ b_n^{-\alpha}\int_{b_n^\alpha\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(3)\right| > t^{1/\alpha}\right)\mathrm{d}t\\&+ b_n^{-\alpha}\int_{b_n^\alpha\varepsilon}^{\infty} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_{ni}(4)\right| > t^{1/\alpha}\right)\mathrm{d}t.\end{aligned}$$

Similarly, to the proof of Theorem 3.1, we can get the proof, so we omit it.

4. Conclusion

In this paper, we have studied the complete moment convergence for negatively orthant dependent random variables. Our results strengthen the main results of Chen (2016) by obtaining a much stronger conclusion under the same conditions. Based on our method, we also investigated L^p convergence for the NOD sequence.

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