Generalizations of Simpson-type inequalities for relative semi-(h, α, m)-logarithmically convex mappings

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Abstract. By discovering an integral equality defined on relative convex set, we prove some new Simpson-type inequalities for mappings which have absolute values of the first derivatives which are relative semi-(h, α, m)-logarithmically convex. Some special cases are also considered.
Keywords: relative convex set, relative semi-(h, α, m)-logarithmically convex, Simpson's inequality.

1. Introduction

If $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on $[r_1, r_2]$ with $r_1, r_2 \in \mathcal{I}$ and $r_1 < r_2$, then one has

$$f\Big(\frac{r_1+r_2}{2}\Big) \leq \frac{1}{r_2-r_1}\int_{r_1}^{r_2} f(x)\mathrm{d}x \leq \frac{f(r_1)+f(r_2)}{2}. \tag{1.1}$$

The following inequality is named in the literature as the Simpson's integral inequality:

$$\begin{aligned} &\left|\frac{1}{6}\left[f(r_1)+4f\Big(\frac{r_1+r_2}{2}\Big)+f(r_2)\right]-\frac{1}{r_2-r_1}\int_{r_1}^{r_2} f(t)\mathrm{d}t\right| \\ &\leq \frac{1}{2880}\big\|f^{(4)}\big\|_{\infty}(r_2-r_1)^4, \end{aligned} \tag{1.2}$$

where $f : [r_1, r_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (r_1, r_2) and $\big\|f^{(4)}\big\|_{\infty} = \sup_{t\in(r_1,r_2)}\big|f^{(4)}(t)\big| < \infty$.

For recent results with respect to (1.1) and (1.2), for example, see [1, 5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29] and the references mentioned in these papers.

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Let us consider a relative convex set M_φ. A set $M_\varphi \subseteq \mathbb{R}^n$ is named a relative convex set with respect to the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, if $t\varphi(x)+(1-t)\varphi(y) \in M_\varphi$ holds for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$. A function f is called relative convex on a relative convex set M_φ, if there exists a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq tf\big(\varphi(x)\big) + (1-t)f\big(\varphi(y)\big)$$

grips for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

A function $f : M_\varphi \to \mathbb{R}$ is called relative semi-convex on a relative convex set, if there exists a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

Let us now mention some definitions involving logarithmic convexity.

Definition 1.1 ([4])**.** *A function $f : M_\varphi \to \mathbb{R}^+$ is called relative semi-logarithmic convex on a relative convex set M_φ, if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq f(x)^t f(y)^{(1-t)}, \tag{1.3}$$

holds for every $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi$ and $t \in [0,1]$.

Definition 1.2 ([3])**.** *Let $h : K \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative mapping, $h \neq 0$. A nonnegative function $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is named $(h-(\alpha,m))$-logarithmically convex, if the following inequality:*

$$f\big(tx + m(1-t)y\big) \leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{m\left(1-h^\alpha(t)\right)} \tag{1.4}$$

grips for all $x, y \in \mathcal{I}$ and $t \in [0,1]$ with some fixed $(\alpha, m) \in (0,1] \times (0,1]$.

Definition 1.3 ([2])**.** *A function $f : [0,b] \longrightarrow (0,\infty)$ with $b > 0$ is called (α,m)-logarithmically convex if the inequality*

$$f\big(tx + m(1-t)y\big) \leq \big[f(x)\big]^{t^\alpha} \big[f(y)\big]^{m(1-t^\alpha)} \tag{1.5}$$

holds for all $x, y \in [0,b]$, $t \in [0,1]$ with some fixed $(\alpha, m) \in (0,1] \times (0,1]$.

For $(\alpha, m) \in \{(\alpha,1), (1,m), (1,1)\}$, one obtains the following classes of functions: α-logarithmically convex, m-logarithmically convex and logarithmically convex, respectively. Some recent interesting and important results related to logarithmically convexity can be found in [2, 15, 16, 28].

Very recently, Du et al. in [6] established the following lemma and developed some Simpson type inequalities via extended (s,m)-convex mappings.

Lemma 1.1. *Let $f : \mathcal{I} \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable on $\mathcal{I}^0$ (interior of $\mathcal{I}$), where $a, b \in \mathcal{I}^0$ such that $0 < a < b$. For some fixed $m \in (0, 1]$, if $f' \in L^1([a, b])$ along with $k, t \in \mathbb{R}$, then for each $x \in [ma, b]$ the following identity holds:*

$$\begin{aligned}
&tf(ma) + (1-k)f(b) + (k-t)f\left(\frac{b+ma}{2}\right) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \\
&= (b-ma)\left[\int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda\right].
\end{aligned} \tag{1.6}$$

Also, Zhou et al. [30] in 2017, introduced the concept of the relative semi-(α, m)-logarithmically convex functions and derived some Simpson-like type inequalities for such mappings by means of the following identity.

Lemma 1.2. *Let $f : \mathcal{I}^0 \subseteq \mathbb{R} \to (0, \infty)$ be a differentiable mapping on $\mathcal{I}^0$ and let $\varphi : \mathcal{I} \subseteq \mathbb{R} \to (0, \infty)$ be a continuous mapping along with $\varphi(a) < \varphi(b)$. If $f' \in L^1\big([\varphi(a), \varphi(b)]\big)$, then we have*

$$\begin{aligned}
&\frac{1}{8}\Big[f\big(\varphi(a)\big) + 6f\Big(\sqrt{\varphi(a)\varphi(b)}\Big) + f\big(\varphi(b)\big)\Big] \\
&\quad - \frac{1}{\ln\varphi(b) - \ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x) \\
&= \frac{\ln\varphi(b) - \ln\varphi(a)}{4}\sqrt{\varphi(b)} \\
&\quad \times \left\{\sqrt{\varphi(a)}\int_0^1 \left(\frac{3}{4} - t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{1+t}{2}}[\varphi(b)]^{\frac{1-t}{2}}\Big)\mathrm{d}t \right. \\
&\quad \left. + \sqrt{\varphi(b)}\int_0^1 \left(\frac{1}{4} - t\right)\left[\frac{\varphi(a)}{\varphi(b)}\right]^{\frac{t}{2}} f'\Big([\varphi(a)]^{\frac{t}{2}}[\varphi(b)]^{1-\frac{t}{2}}\Big)\mathrm{d}t\right\}.
\end{aligned} \tag{1.7}$$

Motivated by the inspiring idea in [3, 6] and [30], we introduce the class of relative semi-(h, α, m)-logarithmically convex functions defined on relative convex set, and we explore new Simpson-type inequalities for such mappings. Some interesting special cases of our principal results are also investigated.

2. New definitions and a lemma

For convenience, we next take the symbol $M_\varphi = [\varphi(a), \varphi(b)]$ with $0 < \varphi(a) < \varphi(b) < \infty$ be a relative convex set.

Definition 2.1. *Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. A function $f : M_\varphi \to \mathbb{R}^+$ is named relative semi-(h, α, m)-logarithmically convex on a relative convex set M_φ, if there exists a mapping $\varphi : \mathbb{R} \to \mathbb{R}$ such that,*

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \le \big[f(x)\big]^{h^\alpha(t)}\big[f(y)\big]^{m\left(1-h^\alpha(t)\right)}, \tag{2.1}$$

holds for every $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$, *along with some fixed* $(\alpha, m) \in (0,1] \times (0,1]$.

Clearly, when taking $h(t) = t$ in Definition 2.1, f becomes the relative semi-(α, m)-logarithmically convex function on M_φ, and when taking $h(t) = t^s$ in Definition 2.1, f becomes the relative semi-(s, α, m)-logarithmically convex function on M_φ.

Definition 2.2. *Let* $h : [0,1] \to \mathbb{R}$ *be a nonnegative function,* $h \neq 0$. *A function* $f : M_\varphi \to \mathbb{R}^+$ *is called relative semi-*(h, α)*-logarithmically convex on a relative convex set* M_φ, *if there exists a function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that,*

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{\left(1-h^\alpha(t)\right)}, \tag{2.2}$$

holds for all $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$, *together with some fixed* $\alpha \in (0,1]$.

If we take $t = \frac{1}{2}$, then the relative semi-(h, α)-logarithmically convex function reduces to

$$f\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \leq \big[f(x)\big]^{h^\alpha(\frac{1}{2})} \big[f(y)\big]^{\left(1-h^\alpha(\frac{1}{2})\right)}, \tag{2.3}$$

which is called Jensen relative semi-(h, α)-logarithmically convex function.

From Definition 2.2, it follows that

$$\begin{aligned} f\big(t\varphi(x) + (1-t)\varphi(y)\big) &\leq \big[f(x)\big]^{h^\alpha(t)} \big[f(y)\big]^{\left(1-h^\alpha(t)\right)} \\ &\leq h^\alpha(t) f(x) + \big(1 - h^\alpha(t)\big) f(y), \end{aligned} \tag{2.4}$$

which reveals that every relative semi-(h, α)-logarithmically convex function is a relative semi-(h, α)-convex mapping, but the converse is not true.

Definition 2.3. *Let* $h : [0,1] \to \mathbb{R}$ *be a nonnegative function,* $h \neq 0$. *A function* $f : M_\varphi \to \mathbb{R}^+$ *is named relative semi-*(h, m)*-logarithmically convex on a relative convex set* M_φ, *if there exists a function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that,*

$$f\big(t\varphi(x) + m(1-t)\varphi(y)\big) \leq \big[f(x)\big]^{h(t)} \big[f(y)\big]^{m\left(1-h(t)\right)}, \tag{2.5}$$

holds for all $x, y \in \mathbb{R} : \varphi(x), \varphi(y) \in M_\varphi, t \in (0,1)$, *along with some fixed* $m \in (0,1]$.

Clearly, when taking $h(t) = t$ in Definition 2.3, f becomes the relative semi-m-logarithmically convex function on M_φ. Here, we provide an example to show that a function can be relative semi-m-logarithmically convex on $\mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} 1, & x > 0, \\ e^{-|x|}, & x \leq 0, \end{cases}$$

and the function $\varphi:\mathbb{R}\to\mathbb{R}$, where $\varphi(x)=-|x|$, for some fixed $m\in(0,1]$.

However, it is neither convex nor logarithmically convex on $\mathbb{R}$. In fact, if we take $x=1$, $y=-1$ and $t=\frac{1}{2}$, then we have

$$\begin{aligned}f(tx+(1-t)y)&=f(0)=1>tf(x)+(1-t)f(y)\\&=\frac{1}{2}f(1)+\frac{1}{2}f(-1)=\frac{1}{2}\Big(1+\frac{1}{e}\Big)\end{aligned}$$

and

$$f(tx+(1-t)y)=f(0)=1>[f(x)]^t[f(y)]^{1-t}=[f(1)]^{\frac{1}{2}}[f(-1)]^{\frac{1}{2}}=\sqrt{\frac{1}{e}}.$$

Remark 2.1. In Definition 2.1, 2.2 and 2.3, if we take φ is an identity mapping, then we have the definition of (h,α,m)-, (h,α)- and (h,m)-logarithmically convex mappings, respectively.

To demonstrate new Simpson-type inequalities via the relative semi-(h,α,m)-logarithmically convexity, we need the following lemma.

Lemma 2.1. *Let $f:\mathcal{I}^0\subseteq\mathbb{R}\to(0,\infty)$ be a differentiable mapping on $\mathcal{I}^0$ and let $\varphi:\mathcal{I}\subseteq\mathbb{R}\to(0,\infty)$ be a continuous mapping with $\varphi(a)<\varphi(b)$. If $f'\in L^1\big([\varphi(a),\varphi(b)]\big)$ and $\lambda,k\in\mathbb{R}$, then for each $\varphi(x)\in[\varphi(a),\varphi(b)]$ we have*

$$\begin{aligned}&\triangle_{f,\varphi}(\lambda,k;a,b)\\&=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Bigg\{\int_0^{\frac{1}{2}}(t-\lambda)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t\\&\quad+\int_{\frac{1}{2}}^1(t-k)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t\Bigg\},\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}&\triangle_{f,\varphi}(\lambda,k;a,b)\\&:=\Big[\lambda f\big(\varphi(a)\big)+(k-\lambda)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)+(1-k)f\big(\varphi(b)\big)\Big]\\&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x).\end{aligned}\tag{2.7}$$

Proof. Integrating by parts, we have

$$\begin{aligned}&\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\int_0^{\frac{1}{2}}(t-\lambda)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t\\&=(t-\lambda)f\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}f\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\mathrm{d}t\\&=\lambda f(\varphi(a))+\Big(\frac{1}{2}-\lambda\Big)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)\\&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\sqrt{\varphi(a)\varphi(b)}}\frac{f(\varphi(x))}{\varphi(x)}\mathrm{d}\varphi(x).\end{aligned}\tag{2.8}$$

Similarly, we have

$$\begin{aligned}
&\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\int_{\frac{1}{2}}^{1}(t-k)\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^{t}\Big)\mathrm{d}t\\
&=(1-k)f(\varphi(b))+\Big(k-\frac{1}{2}\Big)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)\\
&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\sqrt{\varphi(a)\varphi(b)}}^{\varphi(b)}\frac{f(\varphi(x))}{\varphi(x)}\mathrm{d}\varphi(x).
\end{aligned} \tag{2.9}$$

Lemma 2.1 is thus proved. □

Example 2.1. If we take $\varphi(x)=e^{x}$ and $f(x)=x$, then all the assumptions in Lemma 2.1 are satisfied. Clearly, the first part of the right-sided term of the equation (2.8) is

$$\begin{aligned}
&\Big(b-a\Big)e^{a}\int_{0}^{\frac{1}{2}}(t-\lambda)\Big[\frac{e^{b}}{e^{a}}\Big]^{t}f'\Big([e^{a}]^{1-t}[e^{b}]^{t}\Big)\mathrm{d}t\\
&=(t-\lambda)\Big([e^{a}]^{1-t}[e^{b}]^{t}\Big)\Big|_{0}^{\frac{1}{2}}-\int_{0}^{\frac{1}{2}}[e^{a}]^{1-t}[e^{b}]^{t}\mathrm{d}t\\
&=\lambda e^{a}+\Big(\frac{1}{2}-\lambda\Big)\Big(\sqrt{e^{a}e^{b}}\Big)-\frac{1}{b-a}\int_{e^{a}}^{\sqrt{e^{a}e^{b}}}\mathrm{d}e^{x},
\end{aligned}$$

the second part of the right-sided term of the equation (2.9) is

$$\begin{aligned}
&(b-a)e^{a}\int_{\frac{1}{2}}^{1}(t-k)\Big[\frac{e^{b}}{e^{a}}\Big]^{t}f'\Big([e^{a}]^{1-t}[e^{b}]^{t}\Big)\mathrm{d}t\\
&=(1-k)e^{b}+\Big(k-\frac{1}{2}\Big)\Big(\sqrt{e^{a}e^{b}}\Big)-\frac{1}{b-a}\int_{\sqrt{e^{a}e^{b}}}^{e^{b}}\mathrm{d}e^{x},
\end{aligned}$$

and the left-side term of the equation (2.7) is

$$\begin{aligned}
&\Big[\lambda f(\varphi(a))+(k-\lambda)f\Big(\sqrt{\varphi(a)\varphi(b)}\Big)+(1-k)f(\varphi(b))\Big]\\
&\quad-\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\
&=\Big[\lambda e^{a}+(k-\lambda)\Big(\sqrt{e^{a}e^{b}}\Big)+(1-k)e^{b}\Big]-\frac{1}{b-a}\int_{e^{a}}^{e^{b}}\mathrm{d}e^{x}.
\end{aligned}$$

Corollary 2.1. *In Lemma 2.1, if we take φ is an identity mapping, $\lambda=\frac{1}{4}$ and $k=\frac{3}{4}$, then the equation (2.7) becomes to the following identity*

$$\frac{1}{4}\Big[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f(x)}{x}\mathrm{d}x$$

$$
\begin{aligned}
&= \Big(\ln b - \ln a\Big)\bigg\{ \int_0^{\frac{1}{2}} \Big(t - \frac{1}{4}\Big) a^{1-t} b^t f'\Big(a^{1-t}b^t\Big) \mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^{1} \Big(t - \frac{3}{4}\Big) a^{1-t} b^t f'\Big(a^{1-t}b^t\Big) \mathrm{d}t \bigg\},
\end{aligned}
$$

which is a new form of Lemma 2.1.

Corollary 2.2. *In Lemma 2.1, if we take φ is an identity mapping, $\lambda = \frac{1}{6}$ and $k = \frac{5}{6}$, then the equation (2.7) becomes to the following identity*

$$
\begin{aligned}
&\frac{1}{6}\Big[f(a) + 4f\Big(\sqrt{ab}\Big) + f(b)\Big] - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x \\
&= \Big(\ln b - \ln a\Big)\bigg\{ \int_0^{\frac{1}{2}} \Big(t - \frac{1}{6}\Big) a^{1-t} b^t f'\Big(a^{1-t}b^t\Big) \mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^{1} \Big(t - \frac{5}{6}\Big) a^{1-t} b^t f'\Big(a^{1-t}b^t\Big) \mathrm{d}t \bigg\},
\end{aligned}
$$

which is another new form of Lemma 2.1.

3. Main results

Throughout this paper, we shall present our main results by assuming $\varphi : [a,b] \subseteq \mathbb{R} \to (0,\infty)$ is a continuous positive homogeneous mapping with $\varphi(a) < \varphi(b)$, and $f : \mathcal{I}^0 \subseteq [0,\infty) \to (0,\infty)$ is a differentiable function on $\mathcal{I}^0$ such that $f' \in L^1\big([\varphi(a), \varphi(b)]\big)$.

We start the following results by using Lemma 2.1.

Theorem 3.1. *If $|f'(\varphi(x))|$ is increasing and relative semi-(h,α,m)-logarithmically convex function on $\big[0, \varphi\big(\frac{b}{m}\big)\big]$ with $(\alpha, m) \in (0,1] \times (0,1]$, $0 \le \lambda \le \frac{1}{2} \le k \le 1$, then the following inequality holds:*

$$
\begin{aligned}
&\Big| \triangle_{f,\varphi}(\lambda, k; a, b) \Big| \\
&\le \Big(\ln\varphi(b) - \ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\
&\quad \times \bigg\{ \int_0^{\frac{1}{2}} \big|t-\lambda\big| \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \Bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Bigg]^{h^\alpha(1-t)} \mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^{1} \big|t-k\big| \Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \Bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Bigg]^{h^\alpha(1-t)} \mathrm{d}t \bigg\}.
\end{aligned} \tag{3.1}
$$

Proof. By Lemma 2.1, Young inequality, monotonically increasing and relative semi-(h,α,m)-logarithmically convexity of $|f'(\varphi(x))|$, we get

$$
\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\left\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|\mathrm{d}t\right.\\
&\quad\left.+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\Big([\varphi(a)]^{1-t}[\varphi(b)]^t\Big)\Big|\mathrm{d}t\right\}\\
&\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\left\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\big((1-t)\varphi(a)+t\varphi(b)\big)\Big|\mathrm{d}t\right.\\
&\quad\left.+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'\big((1-t)\varphi(a)+t\varphi(b)\big)\Big|\mathrm{d}t\right\}\\
&\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\\
&\quad\times\left\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'(a)\Big|^{h^\alpha(1-t)}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m\left(1-h^\alpha(1-t)\right)}\mathrm{d}t\right.\\
&\quad\left.+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big|f'(a)\Big|^{h^\alpha(1-t)}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m\left(1-h^\alpha(1-t)\right)}\mathrm{d}t\right\}\\
&= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\
&\quad\times\left\{\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\right.\\
&\quad\left.+\int_{\frac12}^1\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{h^\alpha(1-t)}\mathrm{d}t\right\}.
\end{aligned}
\tag{3.2}
$$

The proof is completed. □

If $|f'(\varphi(x))|^q$ is relative semi-(h,α,m)-logarithmically convex, then we deduce the following result.

Theorem 3.2. *If $|f'(\varphi(x))|^q$ for $q>1$ is increasing and relative semi-(h,α,m)-logarithmically convex function on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac12\le k\le1$, then the following inequality holds:*

$$
\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\
&\times\left\{\vartheta_1^{1-\frac1q}\Big(\lambda,\varphi(a),\varphi(b)\Big)\Big(\int_0^{\frac12}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\Big]^{qh^\alpha(1-t)}\mathrm{d}t\Big)^{\frac1q}\right.
\end{aligned}
\tag{3.3}
$$

$$+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\bigg(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}\bigg]^{qh^{\alpha}(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\bigg\},$$

where

$$\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big):=\frac{\big(\frac{1}{2}-\lambda\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\lambda}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\lambda}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-1}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^{2}} \tag{3.4}$$

and

$$\vartheta_2\big(k,\varphi(a),\varphi(b)\big):=\frac{\big(\frac{1}{2}-k\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}+(1-k)\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{k}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^{2}}. \tag{3.5}$$

Proof. Continuing from inequality (3.2) in the proof of Theorem 3.1, using the power mean inequality, we get

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\le{}&\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}\\&\times\bigg\{\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\times\bigg(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}\bigg]^{qh^{\alpha}(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\\&+\bigg(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\times\bigg(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\bigg[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}\bigg]^{qh^{\alpha}(1-t)}\mathrm{d}t\bigg)^{\frac{1}{q}}\bigg\}.\end{aligned} \tag{3.6}$$

When $0\le\lambda\le\frac{1}{2}\le k\le1$, we have

$$\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\mathrm{d}t=\frac{\big(\frac{1}{2}-\lambda\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\lambda}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\lambda}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-1}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^{2}} \tag{3.7}$$

and

$$\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\mathrm{d}t=\frac{\big(\frac{1}{2}-k\big)\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}+(1-k)\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\ln\varphi(b)-\ln\varphi(a)}+\frac{2\big[\frac{\varphi(b)}{\varphi(a)}\big]^{k}-\big[\frac{\varphi(b)}{\varphi(a)}\big]^{\frac{1}{2}}-\big[\frac{\varphi(b)}{\varphi(a)}\big]}{\big(\ln\varphi(b)-\ln\varphi(a)\big)^{2}}. \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we obtain the desired inequality (3.3). The proof is completed. □

A different method leads to the following result.

Theorem 3.3. *If* $|f'(\varphi(x))|^q$ *for* $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *is increasing and relative semi-*(h,α,m)*-logarithmically convex function on* $\left[0,\varphi\left(\frac{b}{m}\right)\right]$ *along with* $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, *then the following inequality holds:*

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\chi_1^{\frac{1}{p}}(\lambda,p)\left(\int_0^{\frac{1}{2}}\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\left.+\chi_2^{\frac{1}{p}}(k,p)\left(\int_{\frac{1}{2}}^1\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{3.9}$$

where

$$\chi_1(\lambda,p):=\frac{\lambda^{p+1}+\left(\frac{1}{2}-\lambda\right)^{p+1}}{p+1}\tag{3.10}$$

and

$$\chi_2(k,p):=\frac{(1-k)^{p+1}+\left(k-\frac{1}{2}\right)^{p+1}}{p+1}.\tag{3.11}$$

Proof. Continuing from inequality (3.2) in the proof of Theorem 3.1, using Hölder's inequality, we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\left(\int_0^{\frac{1}{2}}|t-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{1}{2}}^1|t-k|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{qh^\alpha(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.12}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, we have

$$\int_0^{\frac{1}{2}}|t-\lambda|^p\mathrm{d}t=\frac{\lambda^{p+1}+\left(\frac{1}{2}-\lambda\right)^{p+1}}{p+1}$$

and

$$\int_{\frac{1}{2}}^1|t-k|^p\mathrm{d}t=\frac{(1-k)^{p+1}+\left(k-\frac{1}{2}\right)^{p+1}}{p+1}.$$

The proof is completed. □

Next, we obtain new results related to the relative semi-(s,α,m)-logarithmically convexity.

Theorem 3.4. *If $|f'(\varphi(x))|$ is increasing and relative semi-(s,α,m)-logarithmically convex function in the first sense on $\big[0,\varphi(\frac{b}{m})\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $\lambda+k=1$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, then the following inequality holds:*

$$\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}M(\alpha,s,k,\lambda),\tag{3.13}$$

where

$$\eta_1=\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}},\tag{3.14}$$

$$\eta_2=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha s}}\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}=\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{-\frac{1}{\alpha s}}\eta_1\tag{3.15}$$

and

$$M(\alpha,s,k,\lambda)=\begin{cases}\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big), & \eta_1=1,\\ \eta_1^{\alpha s}F_1(\eta_2,\alpha,s,k,\lambda), & 0<\eta_1<1,\\ \eta_1F_1(\eta_2,\alpha,s,k,\lambda), & \eta_1>1,\end{cases}$$

with

$$\begin{aligned}F_1(\eta_2,\alpha,s,k,\lambda):&=\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_2^{-\alpha st}\mathrm{d}t+\int_{\frac{1}{2}}^{1}\big|t-k\big|\eta_2^{-\alpha st}\mathrm{d}t\\&=\frac{(1-k)\eta_2^{-\alpha s}-\lambda}{-\alpha s\ln\eta_2}+\frac{2\eta_2^{-\alpha sk}+2\eta_2^{-\alpha s\lambda}-2\eta_2^{-\frac{\alpha s}{2}}-\eta_2^{-\alpha s}-1}{\big(\alpha s\ln\eta_2\big)^2}.\end{aligned}$$

Proof. For $h(t)=t^s$ in the inequality (3.2), we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}\\&\times\Bigg\{\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}\Big]^{(1-t)^{\alpha s}}\mathrm{d}t\\&+\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{t}\Big[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^{m}}\Big]^{(1-t)^{\alpha s}}\mathrm{d}t\Bigg\}.\end{aligned}\tag{3.16}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, let $\eta_1=\frac{|f'(a)|}{|f'(\frac{b}{m})|^{m}}$ in (3.16), we have three cases as follows.

Case 1: For $\eta_1 = 1$, the inequality (3.16) becomes the form

$$\begin{aligned}\Big| \triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}|t-\lambda|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \mathrm{d}t+\int_{\frac{1}{2}}^1 |t-k|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big)+\vartheta_2\big(k,\varphi(a),\varphi(b)\big)\right\}.\end{aligned}$$

Here, ϑ_1 and ϑ_2 are defined by (3.4) and (3.5), respectively.
Case 2: For $0 < \eta_1 < 1$, we have $\eta_1^{(1-t)^{\alpha s}} \le \eta_1^{\alpha s(1-t)}$. Thus the inequality (3.16) becomes

$$\begin{aligned}\Big| \triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}|t-\lambda|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \eta_1^{\alpha s(1-t)}\mathrm{d}t+\int_{\frac{1}{2}}^1 |t-k|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \eta_1^{\alpha s(1-t)}\mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \eta_1^{\alpha s}\cdot F_1(\eta_2,\alpha,s,k,\lambda).\end{aligned}$$

Case 3: For $\eta_1 > 1$, we have $\eta_1^{(1-t)^{\alpha s}} \le \eta_1{}^{(\alpha s(1-t)+1-\alpha s)} = \eta_1^{(1-\alpha st)}$. Thus the inequality (3.16) becomes

$$\begin{aligned}\Big| \triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\ &\quad\times\left\{\int_0^{\frac{1}{2}}|t-\lambda|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \eta_1^{(1-\alpha st)}\mathrm{d}t+\int_{\frac{1}{2}}^1 |t-k|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t \eta_1^{(1-\alpha st)}\mathrm{d}t\right\} \\ &= \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \eta_1\cdot F_1(\eta_2,\alpha,s,k,\lambda).\end{aligned}$$

The proof is completed. □

Corollary 3.1. *In Theorem 3.4, if φ is an identity mapping, then we obtain*

$$\begin{aligned}&\left|\Big[\lambda f(a)+(k-\lambda)f\Big(\sqrt{ab}\Big)+(1-k)f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right| \\ &\le a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M_1\big(\alpha,s,k,\lambda\big),\end{aligned}$$

where

$$\omega_1=\Big(\frac{b}{a}\Big)^{-\frac{1}{\alpha s}}\eta_1$$

and

$$M_1(\alpha,s,k,\lambda)=\begin{cases}\vartheta_1\Big(\lambda,a,b\Big)+\vartheta_2\Big(k,a,b\Big), & \eta_1=1,\\ \eta_1^{\alpha s}F_2(\omega_1,\alpha,s,k,\lambda), & 0<\eta_1<1,\\ \eta_1 F_2(\omega_1,\alpha,s,k,\lambda), & \eta_1>1,\end{cases}$$

with

$$F_2(\omega_1,\alpha,s,k,\lambda) \\ =\frac{(1-k)\omega_1^{-\alpha s}-\lambda}{-\alpha s\ln\omega_1}+\frac{2\omega_1^{-\alpha sk}+2\omega_1^{-\alpha s\lambda}-2\omega_1^{-\frac{\alpha s}{2}}-\omega_1^{-\alpha s}-1}{\big(\alpha s\ln\omega_1\big)^2}.$$

Remark 3.1. In Corollary 3.1,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\left|\frac{1}{4}\Big[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right| \\ \leq a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4}\Big),$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\left|\frac{1}{6}\Big[f(a)+4f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right| \\ \leq a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6}\Big),$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\leq a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m M_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2}\Big).$$

Theorem 3.5. *If $|f'(\varphi(x))|^q$ for $q>1$ is increasing and relative semi-(s,α,m)-logarithmically convex function on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\leq\lambda\leq\frac{1}{2}\leq k\leq 1$, then the following inequality holds:*

$$\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\leq\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}(\alpha,s,k,\lambda,q), \tag{3.17}$$

where

$$\mathcal{T}(\alpha,s,k,\lambda,q)=\begin{cases}\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big), & \eta_1=1,\\ \eta_1^{\alpha s}\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\mathcal{K}_1^{\frac{1}{q}}(\eta_3,\alpha,s,\lambda,q) & \\ \quad+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\mathcal{K}_2^{\frac{1}{q}}(\eta_3,\alpha,s,k,q)\Big], & 0<\eta_1<1,\\ \eta_1\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\mathcal{K}_1^{\frac{1}{q}}(\eta_3,\alpha,s,\lambda,q) & \\ \quad+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\mathcal{K}_2^{\frac{1}{q}}(\eta_3,\alpha,s,k,q)\Big], & \eta_1>1,\end{cases}$$

with

$$\mathcal{K}_1(\eta_3,\alpha,s,\lambda,q)=\frac{\left(\frac{1}{2}-\lambda\right)\eta_3^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sq\lambda}-\eta_3^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\eta_3)^2},$$

$$\mathcal{K}_2(\eta_3,\alpha,s,k,q)=\frac{\left(\frac{1}{2}-k\right)\eta_3^{-\frac{\alpha sq}{2}}+(1-k)\eta_3^{-\alpha sq}}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sqk}-\eta_3^{-\frac{\alpha sq}{2}}-\eta_3^{-\alpha sq}}{(\alpha sq\ln\eta_3)^2},$$

$$\eta_3=\left[\frac{\varphi(b)}{\varphi(a)}\right]^{-\frac{1}{\alpha sq}}\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}=\left[\frac{\varphi(b)}{\varphi(a)}\right]^{-\frac{1}{\alpha sq}}\eta_1,$$

and $\vartheta_1\big(\lambda,\varphi(a),\varphi(b)\big)$, $\vartheta_2\big(k,\varphi(a),\varphi(b)\big)$ *are defined by* (3.4) *and* (3.5), *respectively.*

Proof. For $h(t)=t^s$ in the inequality (3.6), we have

$$\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\left|f'\left(\frac{b}{m}\right)\right|^m\\
&\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\left(\int_0^{\frac{1}{2}}|t-\lambda|\left[\frac{\varphi(b)}{\varphi(a)}\right]^t\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\mathrm{d}t\right.\\
&\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\left(\int_{\frac{1}{2}}^1|t-k|\left[\frac{\varphi(b)}{\varphi(a)}\right]^t\left[\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\right\},
\end{aligned}\tag{3.18}$$

where ϑ_1 and ϑ_2 are defined by (3.4) and (3.5), respectively. When $0\le\lambda\le\frac{1}{2}\le k\le 1$, let $\eta_1=\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$ in (3.18), we have three cases as follows.

Case 1: For $\eta_1=1$, the inequality (3.18) becomes the form

$$\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\left|f'\left(\frac{b}{m}\right)\right|^m\\
&\times\left\{\vartheta_1\Big(\lambda,\varphi(a),\varphi(b)\Big)+\vartheta_2\Big(k,\varphi(a),\varphi(b)\Big)\right\}.
\end{aligned}$$

Case 2: For $0<\eta_1<1$, we have $\eta_1^{q(1-t)^{\alpha s}}\le\eta_1^{q\alpha s(1-t)}$. Thus the inequality (3.18) becomes

$$\begin{aligned}
\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\le \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\left|f'\left(\frac{b}{m}\right)\right|^m\\
&\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\left(\int_0^{\frac{1}{2}}|t-\lambda|\left[\frac{\varphi(b)}{\varphi(a)}\right]^t\eta_1^{\alpha sq(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\left(\int_{\frac{1}{2}}^1|t-k|\left[\frac{\varphi(b)}{\varphi(a)}\right]^t\eta_1^{\alpha sq(1-t)}\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
&=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\left|f'\left(\frac{b}{m}\right)\right|^m\eta_1^{\alpha s}
\end{aligned}$$

$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\Big(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t\Big)^{\frac{1}{q}}\right.$$
$$\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\Big(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t\Big)^{\frac{1}{q}}\right\}.$$

Here

$$\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t=\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_3^{-\alpha sqt}\mathrm{d}t$$
$$=\frac{(\frac{1}{2}-\lambda)\eta_3^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sq\lambda}-\eta_3^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\eta_3)^2}$$

and

$$\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{-\alpha sqt}\mathrm{d}t=\int_{\frac{1}{2}}^{1}\big|t-k\big|\eta_3^{-\alpha sqt}\mathrm{d}t$$
$$=\frac{(\frac{1}{2}-k)\eta_3^{-\frac{\alpha sq}{2}}+(1-k)\eta_3^{-\alpha sq}}{-\alpha sq\ln\eta_3}+\frac{2\eta_3^{-\alpha sqk}-\eta_3^{-\frac{\alpha sq}{2}}-\eta_3^{-\alpha sq}}{(\alpha sq\ln\eta_3)^2}.$$

Case 3: For $\eta_1>1$, we have $\eta_1^{q(1-t)^{\alpha s}}\leq\eta_1^{q(\alpha s(1-t)+1-\alpha s)}=\eta_1^{q(1-\alpha st)}$. Thus the inequality (3.18) becomes

$$\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\leq\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m$$
$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\Big(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{q(1-\alpha st)}\mathrm{d}t\Big)^{\frac{1}{q}}\right.$$
$$\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\Big(\int_{\frac{1}{2}}^{1}\big|t-k\big|\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^t\eta_1^{q(1-\alpha st)}\mathrm{d}t\Big)^{\frac{1}{q}}\right\}$$
$$=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1$$
$$\times\left\{\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,\varphi(a),\varphi(b)\Big)\Big(\int_0^{\frac{1}{2}}\big|t-\lambda\big|\eta_3^{-\alpha sqt}\mathrm{d}t\Big)^{\frac{1}{q}}\right.$$
$$\left.+\vartheta_2^{1-\frac{1}{q}}\Big(k,\varphi(a),\varphi(b)\Big)\Big(\int_{\frac{1}{2}}^{1}\big|t-k\big|\eta_3^{-\alpha sqt}\mathrm{d}t\Big)^{\frac{1}{q}}\right\}.$$

The proof is completed. □

Corollary 3.2. *In Theorem 3.5, if φ is an identity mapping, then we obtain*

$$\left|\Big[\lambda f(a)+(k-\lambda)f\Big(\sqrt{ab}\Big)+(1-k)f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\leq a\big(\ln b-\ln a\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{T}_1\big(\alpha,s,k,\lambda,q\big),$$

where

$$\mathcal{T}_1(\alpha,s,k,\lambda,q)=\begin{cases}\vartheta_1(\lambda,a,b)+\vartheta_2(k,a,b), & \eta_1=1,\\ \eta_1^{\alpha s}\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,a,b\Big)\mathcal{K}_3^{\frac{1}{q}}(\omega_2,\alpha,s,\lambda,q) & \\ +\vartheta_2^{1-\frac{1}{q}}\Big(k,a,b\Big)\mathcal{K}_4^{\frac{1}{q}}(\omega_2,\alpha,s,k,q)\Big], & 0<\eta_1<1,\\ \eta_1\Big[\vartheta_1^{1-\frac{1}{q}}\Big(\lambda,a,b\Big)\mathcal{K}_3^{\frac{1}{q}}(\omega_2,\alpha,s,\lambda,q) & \\ +\vartheta_2^{1-\frac{1}{q}}\Big(k,a,b\Big)\mathcal{K}_4^{\frac{1}{q}}(\omega_2,\alpha,s,k,q)\Big], & \eta_1>1,\end{cases}$$

with

$$\mathcal{K}_3(\omega_2,\alpha,s,\lambda,q)=\frac{\left(\frac{1}{2}-\lambda\right)\omega_2^{-\frac{\alpha sq}{2}}-\lambda}{-\alpha sq\ln\omega_2}+\frac{2\omega_2^{-\alpha sq\lambda}-\omega_2^{-\frac{\alpha sq}{2}}-1}{(\alpha sq\ln\omega_2)^2},$$

$$\mathcal{K}_4(\omega_2,\alpha,s,k,q)=\frac{\left(\frac{1}{2}-k\right)\omega_2^{-\frac{\alpha sq}{2}}+(1-k)\omega_2^{-\alpha sq}}{-\alpha sq\ln\omega_2}+\frac{2\omega_2^{-\alpha sqk}-\omega_2^{-\frac{\alpha sq}{2}}-\omega_2^{-\alpha sq}}{(\alpha sq\ln\omega_2)^2}$$

and

$$\omega_2=\left(\frac{b}{a}\right)^{-\frac{1}{\alpha sq}}\eta_1. \tag{3.19}$$

Remark 3.2. In Corollary 3.2,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\left|\frac{1}{4}\Big[f(a)+2f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{T}_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4},q\Big),$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\left|\frac{1}{6}\Big[f(a)+4f\Big(\sqrt{ab}\Big)+f(b)\Big]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{T}_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6},q\Big),$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|\le a\big(\ln b-\ln a\big)\left|f'\Big(\frac{b}{m}\Big)\right|^m\mathcal{T}_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2},q\Big).$$

We get the following result by another approach.

Theorem 3.6. *If $|f'(\varphi(x))|^q$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ is increasing and relative semi-(s,α,m)-logarithmically convex function on $\big[0,\varphi\big(\frac{b}{m}\big)\big]$ along with $(\alpha,m)\in(0,1]\times(0,1]$, $0\le\lambda\le\frac{1}{2}\le k\le 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|\\ &\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\mathcal{L}(\alpha,s,k,\lambda,p,q),\end{aligned}\tag{3.20}$$

where

$$\mathcal{L}(\alpha,s,k,\lambda,p,q)=\begin{cases}\chi_1^{\frac{1}{p}}(\lambda,p)\left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}-1}{q\big(\ln\varphi(b)-\ln\varphi(a)\big)}\right)^{\frac{1}{q}}\\ \quad+\chi_2^{\frac{1}{p}}(k,p)\left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{q}-\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}}{q\big(\ln\varphi(b)-\ln\varphi(a)\big)}\right)^{\frac{1}{q}}, & \eta_1=1,\\ \eta_1^{\alpha s}\mathcal{H}_1(\alpha,s,k,\lambda,p,q), & 0<\eta_1<1,\\ \eta_1\mathcal{H}_1(\alpha,s,k,\lambda,p,q), & \eta_1>1,\end{cases}$$

with

$$\mathcal{H}_1(\alpha,s,k,\lambda,p,q)=\chi_1^{\frac{1}{p}}(\lambda,p)\left(\frac{\eta_2^{-\frac{\alpha sq}{2}}-1}{-\alpha sq\ln\eta_2}\right)^{\frac{1}{q}}+\chi_2^{\frac{1}{p}}(k,p)\left(\frac{\eta_2^{-\alpha sq}-\eta_2^{-\frac{\alpha sq}{2}}}{-\alpha sq\ln\eta_2}\right)^{\frac{1}{q}}.$$

Here, $\chi_1(\lambda,p)$, $\chi_2(k,p)$, η_1 and η_2 are defined by (3.10), (3.11), (3.14) *and* (3.15), *respectively.*

Proof. For $h(t)=t^s$ in the inequality (3.12), we have

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\left(\frac{\lambda^{p+1}+\big(\frac{1}{2}-\lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\right]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &+\left.\left(\frac{(1-k)^{p+1}+\big(k-\frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\frac{\varphi(b)}{\varphi(a)}\right]^{qt}\left[\frac{|f'(a)|}{\big|f'\big(\frac{b}{m}\big)\big|^m}\right]^{q(1-t)^{\alpha s}}\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.21}$$

When $0\le\lambda\le\frac{1}{2}\le k\le 1$, let $\eta_1=\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}$ in (3.21), we have three cases as follows.

Case 1: For $\eta_1=1$, the inequality (3.21) becomes the form

$$\begin{aligned}\Big|\triangle_{f,\varphi}(\lambda,k;a,b)\Big|&\le\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\\ &\times\left\{\left(\frac{\lambda^{p+1}+\big(\frac{1}{2}-\lambda\big)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}-1}{q\big(\ln\varphi(b)-\ln\varphi(a)\big)}\right)^{\frac{1}{q}}\right.\\ &+\left.\left(\frac{(1-k)^{p+1}+\big(k-\frac{1}{2}\big)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left[\frac{\varphi(b)}{\varphi(a)}\right]^{q}-\left[\frac{\varphi(b)}{\varphi(a)}\right]^{\frac{q}{2}}}{q\big(\ln\varphi(b)-\ln\varphi(a)\big)}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Case 2: For $0 < \eta_1 < 1$, we have $\eta_1^{q(1-t)^{\alpha s}} \leq \eta_1^{q\alpha s(1-t)}$. Thus the inequality (3.21) becomes

$$\begin{aligned}
\Big| \triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\leq \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\
&\times\Bigg\{\Bigg(\frac{\lambda^{p+1}+\big(\frac12-\lambda\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_0^{\frac12}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{q\alpha s(1-t)}\mathrm{d}t\Bigg)^{\frac1q} \\
&+\Bigg(\frac{(1-k)^{p+1}+\big(k-\frac12\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_{\frac12}^{1}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{q\alpha s(1-t)}\mathrm{d}t\Bigg)^{\frac1q}\Bigg\} \\
&=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1^{\alpha s} \\
&\times\Bigg\{\Bigg(\frac{\lambda^{p+1}+\big(\frac12-\lambda\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_0^{\frac12}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{-q\alpha st}\mathrm{d}t\Bigg)^{\frac1q} \\
&+\Bigg(\frac{(1-k)^{p+1}+\big(k-\frac12\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_{\frac12}^{1}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{-q\alpha st}\mathrm{d}t\Bigg)^{\frac1q}\Bigg\}.
\end{aligned}$$

Here

$$\int_0^{\frac12}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{-q\alpha st}\mathrm{d}t=\int_0^{\frac12}\eta_2^{-q\alpha st}\mathrm{d}t=\frac{\eta_2^{-\frac{q\alpha s}{2}}-1}{-q\alpha s\ln\eta_2}$$

and

$$\int_{\frac12}^{1}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{-q\alpha st}\mathrm{d}t=\int_{\frac12}^{1}\eta_2^{-q\alpha st}\mathrm{d}t=\frac{\eta_2^{-q\alpha s}-\eta_2^{-\frac{q\alpha s}{2}}}{-q\alpha s\ln\eta_2}.$$

Case 3: For $\eta_1 < 1$, we have $\eta_1^{q(1-t)^{\alpha s}} \leq \eta_1^{q(\alpha s(1-t)+1-\alpha s)} = \eta_1^{q(1-\alpha st)}$. Thus the inequality (3.21) becomes

$$\begin{aligned}
\Big| \triangle_{f,\varphi}(\lambda,k;a,b)\Big| &\leq \Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m \\
&\times\Bigg\{\Bigg(\frac{\lambda^{p+1}+\big(\frac12-\lambda\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_0^{\frac12}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{q(1-\alpha st)}\mathrm{d}t\Bigg)^{\frac1q} \\
&+\Bigg(\frac{(1-k)^{p+1}+\big(k-\frac12\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_{\frac12}^{1}\Big[\frac{\varphi(b)}{\varphi(a)}\Big]^{qt}\eta_1^{q(1-\alpha st)}\mathrm{d}t\Bigg)^{\frac1q}\Bigg\} \\
&=\Big(\ln\varphi(b)-\ln\varphi(a)\Big)\varphi(a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^m\eta_1 \\
&\times\Bigg\{\Bigg(\frac{\lambda^{p+1}+\big(\frac12-\lambda\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_0^{\frac12}\eta_2^{-q\alpha st}\mathrm{d}t\Bigg)^{\frac1q} \\
&+\Bigg(\frac{(1-k)^{p+1}+\big(k-\frac12\big)^{p+1}}{p+1}\Bigg)^{\frac1p}\Bigg(\int_{\frac12}^{1}\eta_2^{-q\alpha st}\mathrm{d}t\Bigg)^{\frac1q}\Bigg\}.
\end{aligned}$$

The proof is completed. □

Corollary 3.3. *In Theorem 3.6, if φ is an identity mapping, then we obtain*

$$\left|\left[\lambda f(a)+(k-\lambda)f\left(\sqrt{ab}\right)+(1-k)f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\leq a\big(\ln b-\ln a\big)\left|f'\left(\frac{b}{m}\right)\right|^m\mathcal{L}_1\big(\alpha,s,k,\lambda,p,q\big),$$

where

$$\mathcal{L}_1(\alpha,s,k,\lambda,p,q)=\begin{cases}\chi_1^{\frac{1}{p}}(\lambda,p)\Big(\dfrac{[\frac{b}{a}]^{\frac{q}{2}}-1}{q(\ln b-\ln a)}\Big)^{\frac{1}{q}} \\ \quad +\chi_2^{\frac{1}{p}}(k,p)\Big(\dfrac{[\frac{b}{a}]^{q}-[\frac{b}{a}]^{\frac{q}{2}}}{q(\ln b-\ln a)}\Big)^{\frac{1}{q}}, & \eta_1=1,\\ \eta_1^{\alpha s}\mathcal{H}_2(\alpha,s,k,\lambda,p,q), & 0<\eta_1<1,\\ \eta_1\mathcal{H}_2(\alpha,s,k,\lambda,p,q), & \eta_1>1,\end{cases}$$

with

$$\mathcal{H}_2(\alpha,s,k,\lambda,p,q)=\chi_1^{\frac{1}{p}}(\lambda,p)\left(\frac{\omega_2^{-\frac{\alpha sq}{2}}-1}{-\alpha sq\ln\omega_2}\right)^{\frac{1}{q}}+\chi_2^{\frac{1}{p}}(k,p)\left(\frac{\omega_2^{-\alpha sq}-\omega_2^{-\frac{\alpha sq}{2}}}{-\alpha sq\ln\omega_2}\right)^{\frac{1}{q}}.$$

Here, η_1 and ω_2 are defined by (3.14) *and* (3.19)*, respectively.*

Remark 3.3. In Corollary 3.3,
(i) if we choose $\lambda=\frac{1}{4}$, $k=\frac{3}{4}$, then we obtain

$$\left|\frac{1}{4}\left[f(a)+2f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\leq a\big(\ln b-\ln a\big)\left|f'\left(\frac{b}{m}\right)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{3}{4},\frac{1}{4},p,q\Big),$$

(ii) if we choose $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, then we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\sqrt{ab}\right)+f(b)\right]-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\leq a\big(\ln b-\ln a\big)\left|f'\left(\frac{b}{m}\right)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{5}{6},\frac{1}{6},p,q\Big),$$

(iii) if we choose $\lambda=\frac{1}{2}$, $k=\frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x\right|$$
$$\leq a\big(\ln b-\ln a\big)\left|f'\left(\frac{b}{m}\right)\right|^m\mathcal{L}_1\Big(\alpha,s,\frac{1}{2},\frac{1}{2},p,q\Big).$$

Finally, we establish an inequality involving products of two relative semi-(h,α,m)-logarithmically convexity.

Theorem 3.7. *Assume that $f,g:\mathcal{I}^0\subseteq[0,\infty)\to(0,\infty)$ are differentiable mappings on $\mathcal{I}^0$ such that $f,g\in L^1\big([\varphi(a),\varphi(b)]\big)$. If $f(\varphi(x))$ is increasing and relative semi-(h,α,m_1)-logarithmically convex and $g(\varphi(x))$ is increasing and relative semi-(h,α,m_2)-logarithmically convex on $\big[0,\varphi(\frac{b}{m_i})\big]$ along with $(\alpha,m_i)\in(0,1]\times(0,1]$ and $i=1,2$, then the following inequality holds:*

$$\begin{aligned}&\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\ &\leq\left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1}\left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2}\\ &\quad\times\int_0^1\left\{f(a)g(a)\left[f\Big(\frac{b}{m_1}\Big)\right]^{-m_1}\left[g\Big(\frac{b}{m_2}\Big)\right]^{-m_2}\right\}^{h^\alpha(t)}\mathrm{d}t.\end{aligned}\tag{3.22}$$

Proof. Let $\varphi(x)=[\varphi(a)]^t[\varphi(b)]^{1-t}$ and using Young inequality, we get

$$\begin{aligned}&\frac{1}{\ln\varphi(b)-\ln\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)}\mathrm{d}\varphi(x)\\ &=\int_0^1 f\Big([\varphi(a)]^t[\varphi(b)]^{1-t}\Big)g\Big([\varphi(a)]^t[\varphi(b)]^{1-t}\Big)\mathrm{d}t\\ &\leq\int_0^1 f\Big(t\varphi(a)+(1-t)\varphi(b)\Big)g\Big(t\varphi(a)+(1-t)\varphi(b)\Big)\mathrm{d}t.\end{aligned}\tag{3.23}$$

Using the positive homogeneity of φ, the relative semi-(h,α,m)-logarithmically convexity of $f\big(\varphi(x)\big)$ and $g\big(\varphi(x)\big)$, we have

$$f\left(t\varphi(a)+m_1(1-t)\varphi\Big(\frac{b}{m_1}\Big)\right)\leq\big[f(a)\big]^{h^\alpha(t)}\left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1\big(1-h^\alpha(t)\big)}\tag{3.24}$$

and

$$g\left(t\varphi(a)+m_2(1-t)\varphi\Big(\frac{b}{m_2}\Big)\right)\leq\big[g(a)\big]^{h^\alpha(t)}\left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2\big(1-h^\alpha(t)\big)}.\tag{3.25}$$

Using (3.24) and (3.25) in (3.23), we get

$$\begin{aligned}&\int_0^1 f\Big(t\varphi(a)+(1-t)\varphi(b)\Big)g\Big(t\varphi(a)+(1-t)\varphi(b)\Big)\mathrm{d}t\\ &\leq\int_0^1\big[f(a)\big]^{h^\alpha(t)}\left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1\big(1-h^\alpha(t)\big)}\big[g(a)\big]^{h^\alpha(t)}\left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2\big(1-h^\alpha(t)\big)}\mathrm{d}t\\ &=\left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1}\left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2}\int_0^1\left\{f(a)g(a)\left[f\Big(\frac{b}{m_1}\Big)\right]^{-m_1}\left[g\Big(\frac{b}{m_2}\Big)\right]^{-m_2}\right\}^{h^\alpha(t)}\mathrm{d}t.\end{aligned}$$

□

Corollary 3.4. *If* $h(t) = t$ *in Theorem 3.7, then we have*

$$\frac{1}{\ln\varphi(b) - \ln\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} \frac{f\big(\varphi(x)\big)g\big(\varphi(x)\big)}{\varphi(x)} \mathrm{d}\varphi(x) \leq \left[f\Big(\frac{b}{m_1}\Big)\right]^{m_1} \left[g\Big(\frac{b}{m_2}\Big)\right]^{m_2} \Upsilon(\alpha),$$

where

$$\Upsilon(\alpha) = \begin{cases} 1, & \rho = 1, \\ \dfrac{\rho^\alpha - 1}{\alpha \ln \rho}, & 0 < \rho < 1, \\ \dfrac{\rho - \rho^{1-\alpha}}{\alpha \ln \rho}, & \rho > 1, \end{cases}$$

with

$$\rho = f(a)g(a) \left[f\Big(\frac{b}{m_1}\Big)\right]^{-m_1} \left[g\Big(\frac{b}{m_2}\Big)\right]^{-m_2},$$

which is the same result proved by Zhou et al. in [30].

4. Conclusion

Based on a new integral identity with multiple parameters, three main results of Simpson-type inequalities via relative semi-(h, α, m)-logarithmically convexity are established. For $h(t) = t^s$, more new multi-parameterized integral inequalities can be derived by means of relative semi-(s, α, m)-logarithmically convexity.

References

[1] M. U. Awan, G. Cristescu, M. A. Noor, L. Riahi, *Upper and lower bounds for Riemann type quantum integrals of preinvex and preinvex dominated functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 79 (2017), 33-44.

[2] R.-F. Bai, F. Qi, B.-Y. Xi, *Hermite-Hadamard type inequalities for the m- and* (α, m)*-logarithmically convex functions*, Filomat, 27 (2013), 1-7.

[3] J. H. Chen, X. J. Huang, *Some integral inequalities via* $(h - (\alpha, m))$*-logarithmically convexity*, J. Comput. Anal. Appl., 20 (2016), 374-380.

[4] X. S. Chen, *Some properties of semi-E-convex function and semi-E-convex programming*, The Eighth International Symposium on Operations Research and Its Applications, Zhangjiajie, China, September 20-22, (2009), 33-39.

[5] T. S. Du, J. G. Liao, Y. J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized* (s, m)*-preinvex functions*, J. Nonlinear Sci. Appl., 9 (2016), 3112-3126.

[6] T. S. Du, Y. J. Li, Z. Q. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions*, Appl. Math. Comput., 293 (2017), 358-369.

[7] S. Erden, M. Z. Sarikaya, *On generalized some inequalities for convex functions*, Ital. J. Pure Appl. Math., 38 (2017), 455-468.

[8] K.-C. Hsu, S.-R. Hwang, K.-L. Tseng, *Some extended Simpson-type inequalities and applications*, Bull. Iranian Math. Soc., 43 (2017), 409-425.

[9] S. Hussain, S. Qaisar, *Generalizations of Simpson's type inequalities through preinvexity and prequasiinvexity*, Punjab Univ. J. Math. (Lahore), 46 (2014), 1-9.

[10] İ. İşcan , S. Turhan, S. Maden, *Hermite-Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions*, Filomat, 31 (2017), 5945-5953.

[11] M. Jleli, D. O'Regan, B. Samet, *On Hermite-Hadamard type inequalities via generalized fractional integrals*, Turkish J. Math., 40 (2016), 1221-1230.

[12] A. Kashuri, R. Liko, *Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions*, Ital. J. Pure Appl. Math., 39 (2018), 683-700.

[13] M. A. Khan, Y. M. Chu, T. U. Khan, J. Khan, *Some new inequalities of Hermite-Hadamard type for s-convex functions with applications*, Open Math., 15 (2017), 1414-1430.

[14] M. A. Khan, Y. M. Chu, A. Kashuri, R. Liko, G. Ali, *Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations*, J. Funct. Spaces, 2018, Article ID 6928130, 9 pages.

[15] M. A. Latif, S. S. Dragomir, *New integral inequalities of Hermite-Hadamard type for n-times differentiable s-logarithmically convex functions with applications*, Miskolc Math. Notes, 16 (2015), 219-235.

[16] M. A. Latif, S. S. Dragomir, *Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions through preinvexity*, Georgian Math. J., 23 (2016), 97-104.

[17] Y. J. Li, T. S. Du, *A generalization of Simpson type inequality via differentiable functions using extended $(s, m)_\phi$-preinvex functions*, J. Comput. Anal. Appl., 22 (2017), 613-632.

[18] W. J. Liu, H. F. Zhuang, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, J. Appl. Anal. Comput., 7 (2017), 501-522.

[19] M. V. Mihai, M. A. Noor, K. I. Noor, M. U. Awan, *Some integral inequalities for harmonic h-convex functions involving hypergeometric functions*, Appl. Math. Comput., 252 (2015), 257-262.

[20] M. A. Noor, K. I. Noor, M. U. Awan, *Simpson-type inequalities for geometrically relative convex functions*, Ukrainian Math. J., 70 (2018), 1145-1154.

[21] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, *Some integral inequalities for harmonically h-convex functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 77 (2015), 5-16.

[22] S. Qaisar, C. J. He, S. Hussain, *A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications*, J. Inequal. Appl., 2013 (2013), 13 pages.

[23] F. Qi, B. Y. Xi, *Some integral inequalities of Simpson type for GA-ε-convex functions*, Georgian Math. J., 20 (2013), 775-788.

[24] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., 60 (2010), 2191-2199.

[25] E. Set, A. O. Akdemir, M. E. Özdemir, *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals*, Filomat, 31 (2017), 4415-4420.

[26] W. B. Sun, Q. Liu, *New Hermite-Hadamard type inequalities for (α, m)-convex functions and applications to special means*, J. Math. Inequal., 11 (2017), 383-397.

[27] M. Tunç, Ç. Yildiz, A. Ekinci, *On some inequalities of Simpson's type via h-convex functions*, Hacet. J. Math. Stat., 42 (2013), 309-317.

[28] Y. Wu, F. Qi, D. -W. Niu, *Integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex and other convex functions*, Maejo Int. J. Sci. Technol., 9 (2015), 394-402.

[29] Z. Q. Yang, Y. J. Li, T. S. Du, *A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions*, Ital. J. Pure Appl. Math., 35 (2015), 327-338.

[30] C. Zhou, C. Peng, T. S. Du, *Simpson-like type inequalities for relative semi-(α, m)-logarithmically convex functions*, J. Nonlinear Sci. Appl., 10 (2017), 4485-4498.

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