Generalization of locally cyclic and Condition (P) in Act-S

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Abstract. In this paper we introduce a new generalization of locally cyclic and Condition (P) in **Act**-S, called Condition (L_P). Then we give a classification of monoids over which Condition (L_P) implies other types of flatness and vice versa. Moreover we provide new equivalent conditions for (P) and strong flatness. This can help to have a better understanding of known conditions. Even more, it can help to provide a solution for open questions in the theory of acts over monoids in the future.
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1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to [11] for basic definitions and terminology relating to semigroups and acts over monoids. A right S-act, usually denoted by A_S, is a non-empty set A on which S acts unitarily from the right, that is to say $(as)t = a(st)$ and $a1 = a$ for every $a \in A, s, t \in S$, where 1 is the identity of S. The category of right S-acts with the mappings that preserve the S-action is denoted by $\mathbf{Act} - S$. Left S-act $_SA$ is defined dually. The study of flatness properties of S-acts began in the 1970s. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S, s, s' \in S$, $as = a's'$, implies that there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and

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$us = vs'$. Condition (P) plays an important role in the theory of S-acts. Many papers have been devoted to investigating this property. Normak [13] was the first to consider Condition (P) on its own. He proved that this condition does not imply pullback flatness, although the converse implication is true. We say A_S satisfies *Condition* (GP) if whenever $as = a't$ with $a, a' \in A_S, s, t \in S$, there exist $a'' \in A_S, u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u, a' = a''v$ and $us^n = vt^n$. A_S is said to satisfy *Condition* (E) if whenever $as = as'$ with $a \in A_S, s, s' \in S$, there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S is called *cyclic* if it is isomorphic to an act of the form S/ρ, where ρ is a right congruence on S. A_S is called *locally cyclic* if for all $a, a' \in A_S$ there exists $a'' \in A_S$ such that $a, a' \in a''S$. The monoid S is called right *reversible* if for all $s, t \in S$, there exist $u, v \in S$ such that $us = vt$. A right S-act A_S is called *decomposable* provided that there exist subacts $B_S, C_S \subset A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. Otherwise, A_S is called *indecomposable*.

2. Generalization of locally cyclic and Condition (P)

Definition 2.1. *The right S-act A_S satisfies Condition (L_P) if $aS \cap a'S \neq \emptyset$ for $a, a' \in A_S$, then there exists $a'' \in A_S$ such that $a, a' \in a''S$.*

Equivalently, we can say that A_S satisfies Condition (L_P) if $as = a't$ for $a, a' \in A_S, s, t \in S$, then there exist $a'' \in A_S, u, v \in S$ such that $a = a''u$ and $a' = a''v$.

Obviously in **Act**-S we have $(P) \Longrightarrow (GP) \Longrightarrow (L_P)$ and also *cyclic* $\Rightarrow$ *locally cyclic* $\Rightarrow$ *Condition*(L_P). The following examples show that these implications are strict.

Example 2.1. Let S be a monoid which is not right reversible. The one element right S-act Θ_S fails to satisfy Condition (GP) [1, Theorem 2.2], but evidently it satisfies Condition (L_P).

Example 2.2. Let $K \neq \{1\}$ be a proper right ideal of a monoid S. Then the right S-act S/K satisfies Condition (L_P) (because S/K is cyclic), but does not satisfy Condition (P) [11, III, 13.9].

Example 2.3. Let $\{P_i\}_{i\in I}$ be a family of acts so that each P_i satisfies Condition (L_P). Consider the right S-act $A_S = \coprod_{i\in I} P_i$ with $\mid I \mid> 1$. It can be easily checked that A_S is not locally cyclic, but it satisfies Condition (L_P).

Although it is clear that there are very many examples of acts which satisfy Condition (L_P), the following example shows that in general not all acts satisfy Condition (L_P).

Example 2.4. Let $S = (\mathbb{N}, .)$ and $A = \mathbb{Z} \setminus \{0, -1\}$. The equality $-2.3 = -3.2$ holds in A_S, but there is no $a \in A_S$ such that $-2, -3 \in aS$. So, A_S does not satisfy Condition (L_P).

Let I be a proper right ideal of a monoid S. Suppose that x, y and z are different symbols that do not belong to S. Consider the right S-act $A(I) = S\coprod^{I} S = ((S \setminus I) \times \{x,y\}) \cup (I \times \{z\})$ with the S-action defined by

$$(t,u)s = \begin{cases} (ts,u), & \text{if } ts \in S \setminus I, \\ (ts,z), & \text{if } ts \in I \end{cases}$$

where $u \in \{x, y\}$ and $(t, z)s = (ts, z)$.

Lemma 2.1. *For the right S-act $A(I)$ the following statements hold:*

1. *A(I) does not satisfy Condition (L_P).*
2. *A(I) is not locally cyclic.*
3. *A(I) satisfies Condition (E).*
4. *A(I) is indecomposable and is generated by exactly two elements.*

Proof. (1). The equality $(1,x)s = (1,y)s$ holds in $A(I)$ for $s \in I$. It can be easily checked that we cannot find $a \in A(I)$ and $s_1, s_2 \in S$ such that $(1,x) = as_1, (1,y) = as_2$. So $A(I)$ does not satisfy Condition (L_P).

(2). $(1,x)$ and $(1,y)$ are elements of A(I), but there is no cyclic subact of A(I) which contains these two elements.

(3). Note that $A(I) = (1,x)S \cup (1,y)S$. Obviously, $(1,x)S \cong S_S \cong (1,y)S$. So $A(I)$ is the union of two subacts both of which satisfy Condition (E). Then $A(I)$ satisfies Condition (E).

(4). If $A(I)$ is decomposable, then there exist $B_S, C_S \leq A(I)$ such that $A(I) = B_S \cup C_S, B_S \cap C_S = \emptyset$. Let $(1,x) \in B_S$. Since $A(I) = (1,x)S \cup (1,y)S$, hence $(1,y) \in C_S$. But $(1,x)S \cap (1,y)S = I \subseteq B_S \cap C_S$ which shows that $B_S \cap C_S \neq \emptyset$, a contradiction. Hence $A(I)$ is indecomposable. The equality $A(I) = (1,x)S \cup (1,y)S$ shows that $A(I)$ is generated by exactly two elements. □

Clearly every locally cyclic act is indecomposable, but by the previous lemma, we see that the converse is not true. Renshaw in [14] proved that a right S-act A_S satisfying Condition (P) is indecomposable if and only if it is locally cyclic. If one looks at the proof of this result, it is easy to see that it also holds for Condition (L_P). Indeed we have:

Proposition 2.1. *A right S-act A_S satisfying Condition (L_P) is indecomposable if and only if A_S is locally cyclic.*

Lemma 2.2. *Let A_S be a locally cyclic act. If $\{a_1, a_2, ..., a_n\}$ be a finite subset of A_S, then there exists $b \in A_S$ such that $a_i \in bS$ for each i $(1 \leq i \leq n)$.*

Proof. The result follows by induction. □

Corollary 2.1. *Every finitely generated right act satisfying Condition (L_P) is indecomposable if and only if A_S is cyclic.*

Lemma 2.3 ([11], I, 5.10). *Every S-act A_S has a unique decomposition into indecomposable subacts.*

Theorem 2.1. *Every right S-act A_S that satisfies Condition (L_P) is a coproduct of locally cyclic subacts.*

Proof. Let A_S be a right S-act which satisfies Condition (L_P). By Lemma 2.3, A_S is a coproduct of indecomposable subacts i.e. $A_S = \coprod_{i\in I} A_i$ where A_i is indecomposable for every $i \in I$. Since A_S satisfies Condition (L_P), each A_i should satisfy this condition. Then by Proposition 2.1, A_i is locally cyclic for every $i \in I$ and we are done. □

By Theorem 2.1 and Lemma 2.2 we have the following corollary:

Corollary 2.2. *Every finitely generated right S-act satisfying Condition (L_P) is a coproduct of cyclic right S-acts.*

The above corollary is also valid for conditions which are stronger than Condition (L_P).

Corollary 2.3. *The following statements are equivalent:*

1. *Every locally cyclic S-act is a coproduct of cyclic right S-acts.*
2. *Every right S-act satisfying Condition (L_P) is a coproduct of cyclic right S-acts.*

Recall from [7] that A_S satisfies *Condition* (EP) if for all $a \in A, s, s' \in S, as = as'$ implies that there exist $a' \in A, u, v \in S$ such that $a = a'u = a'v$ and $us = vs'$. It is clear that $(P) \Rightarrow (EP)$ and $(E) \Rightarrow (EP)$. In [1] it was shown that the combination of Conditions (GP) and (E) is equivalent to strong flatness.

Theorem 2.2. *Let A_S be a right S-act. Then A_S satisfies Condition (P) if and only if it satisfies Conditions (L_P) and (EP).*

Proof. Necessity. Evidently Condition (P) implies both Conditions (L_P) and (EP).

Sufficiency. Suppose that A_S satisfies Conditions (L_P) and (EP) and let $as = a't$ for $a, a' \in A_S$ and $s, t \in S$. Applying Condition (L_P) to the last equation, we get $a'' \in A_S, u_1, v_1 \in S$ such that $a = a''u_1, a' = a''v_1$. Then we have $a''(u_1 s) = a''(v_1 t)$ and so Condition (EP) implies that there exist $\overline{a} \in A_S$, $w, w' \in S$ such that $a'' = \overline{a}w = \overline{a}w'$ and $w(u_1 s) = w'(v_1 t)$. If $u = wu_1$ and $v = w'v_1$, then $a = \overline{a}u, a' = \overline{a}v$ and $us = vt$, that is A_S satisfies Condition (P). □

Recall from [15] that an act A_S is strongly flat if and only if it satisfies both Conditions (P) and (E) . Since equalizer flatness implies Condition (E) and strong flatness implies equalizer flatness, we can say that strong flatness is equivalent to the combination of Condition (P) and equalizer flatness. Using Theorem 2.2, we have the following corollary:

Corollary 2.4. *For any right S-act A_S, the following statements are equivalent:*

1. *A_S is strongly flat.*
2. *A_S satisfies Conditions (E) and (L_P).*
3. *A_S is equalizer flat and satisfies Condition (L_P).*

Note that Condition (E) and strong flatness are equivalent for cyclic acts [11, III, 16.7]. Renshaw in [14, Theorem 3.9] proved that this statement is also valid for locally cyclic acts. The last corollary indicates that even for the right S-acts satisfying Condition (L_P), strong flatness and Condition (E) are equivalent.

Recall that the right S-act A_S satisfies Condition (E') if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = ut$. In [12, Proposition 2.17] it was proved that weak pullback flatness is equivalent to Conditions (P) and (E'). Using Theorem 2.2, we have the following corollary:

Corollary 2.5. *The right S-act A_S is weakly pullback flat if and only if it satisfies Conditions (L_P), (EP) and (E').*

3. Classification of monoids by Condition (L_P) of their acts

Recall from [8] that an act A_S satisfies Condition $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. An act A_S is called *strongly faithful* if for $s, t \in S$, the equality $as = at$ for some $a \in A_S$, implies that $s = t$. A_S is called *faithful* if for $s, t \in S$, the equality $as = at$ for all $a \in A_S$, implies that $s = t$. The right S-act A_S which was defined in Example 2.4, is strongly faithful. Thus, strong faithfulness (and regularity as well as Conditions (E), (EP), (E'), $(E'P)$ which are weaker) does not imply Condition (L_P).

Theorem 3.1. *The following statements are equivalent:*

1. *All right S-acts satisfy Condition (L_P).*
2. *All right S-acts satisfying Condition $(E'P)$, satisfy Condition (L_P).*
3. *All right S-acts satisfying Condition (EP), satisfy Condition (L_P).*
4. *All right S-acts satisfying Condition (E'), satisfy Condition (L_P).*
5. *All right S-acts satisfying Condition (E), satisfy Condition (L_P).*
6. *All right S-acts satisfying Condition (E), are strongly flat.*
7. *All faithful right S-acts satisfy Condition (L_P).*
8. *All indecomposable right S-acts satisfy Condition (L_P).*

9. S is a group.

Proof. Since we have $(E)\Rightarrow(E')\Rightarrow(E'P)$ and $(E)\Rightarrow(EP)\Rightarrow(E'P)$, the implications (1)⇒(2)⇒(4)⇒(5), (2)⇒(3)⇒(5), (1)⇒(7) and (1)⇒(8) are obvious.

(5) ⇔ (6). This is obvious by Corollary 2.4.

(5)⇒(9). Take $s \in S$ such that $sS \neq S$. Lemma 2.1 implies that $A(sS)$ satisfies Condition (E). By assumption, $A(sS)$ satisfies Condition (L_P) which is impossible by Lemma 2.1. Thus $sS = S$ for any $s \in S$ and so S is a group.

(9) ⇒ (1). Suppose that S is a group and A_S is an arbitrary right S-act. If $as = a't$, for $a, a' \in A_S$ and $s, t \in S$, then $a = a'ts^{-1}$ and $a' = a'.1$. Thus A_S satisfies Condition (L_P).

(7) ⇒ (9). Take $s \in S$ such that $sS \neq S$. It was mentioned that $A(sS) = (1,x)S \cup (1,y)S$ and $(1,x)S \cong S_S \cong (1,y)S$. Since S_S is faithful, $A(sS)$ has a faithful subact. Therefore, $A(sS)$ is faithful and satisfies Condition (L_P) by assumption, which is impossible by Lemma 2.1. Then for any $s \in S, sS = S$ which means that S is a group.

(8) ⇒ (9). Let $s \in S$ be such that $sS \neq S$. The right S-act $A(sS)$ is indecomposable by Lemma 2.1. So $A(sS)$ should satisfy Condition (L_P) and this is impossible. Hence $sS = S$ for any $s \in S$ which shows that S is a group. □

Notice that if $sS \neq S$ for $s \in S$, then $A(sS) = S \coprod^{sS} S = (1,x)S \cup (1,y)S$, and so the above theorem is also valid for finitely generated right S-acts as well as for right S-acts generated by exactly two elements.

Recall that the right S-act A_S is called *torsion free* if, for every right cancellable element $c \in S$ and every $a, b \in A_S$, $ac = bc$ implies $a = b$. The following example shows that Condition (L_P) does not imply torsion freeness.

Example 3.1. Let $S = (\mathbb{N}, .)$ and $K = 2\mathbb{N}$ which is a right ideal of S. Since S/K_S is cyclic, it satisfies Condition (L_P). But S/K_S is not torsion free by [11, III, 8.10]. Indeed, the element $2 \in 2\mathbb{N}$ is right cancellabe and we have $3.2 \in 2\mathbb{N}$ but $3 \notin 2\mathbb{N}$.

Now it is a natural question to ask: When does Condition (L_P) imply torsion freeness?

Theorem 3.2. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are torsion free.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are torsion free.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are torsion free.*
4. *All right cancellable elements are right invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By assumption any cyclic right S-act satisfies Condition (L_P), is torsion free, and so by [11, IV, 6.1] every right cancellable element is right invertible.

(4) $\Rightarrow$ (1). If (4) holds, then by [11, IV, 6.1] all right S-acts are torsion free, and so any right S-act satisfying Condition (L_P) is torsion free. □

Recall from [12] that an act A_S satisfies Condition (PWP) if for all $a, a' \in A_S$ and $t \in S$, if $at = a't$, then there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and $ut = vt$. Condition (PWP_E) was defined in [4]. A right S-act A_S satisfies Condition (PWP_E) if whenever $a, a' \in A, s \in S$, and $as = a's$, there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue, a'f = a''vf, es = s = fs$ and $us = vs$.

Theorem 3.3. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (PWP_E).*
2. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (PWP_E).*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (PWP_E).*
4. *All right S-acts satisfying Condition (L_P) are principally weakly flat.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are principally weakly flat.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are principally weakly flat.*
7. *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. By [4, Theorem 2.3] Condition (PWP_E) implies principal weak flatness, and so we have the implications (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7). By assumption, all cyclic right S-acts satisfy Condition (L_P) and so are principally weakly flat. Then S is regular by [11, IV, 6.6].

(7) $\Rightarrow$ (1). By [4, Theorem 3.1], all right S-acts satisfy Condition (PWP_E) and consequently all right S-acts satisfying Condition (L_P) satisfy Condition (PWP_E). □

Recall that a right S-act A_S satisfies Condition (P_E) if and only if for $a, a' \in A, u, v \in S$ and $au = a'v$ there exist $a'' \in A, s, t, e = e^2, f = f^2 \in S$ such that $ae = a''se, a'f = a''tf, eu = u, fv = v$ and $su = tv$ (see [9]).

Theorem 3.4. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (P_E).*
2. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (P_E).*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (P_E).*
4. *All right S-acts satisfying Condition (L_P) are weakly flat.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are weakly flat.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are weakly flat.*
7. *S is regular and satisfies Condition:*

   (R): For any elements $s, t \in S$, there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$ ($\rho(s,t)$ is the smallest right congruence on S containing (s,t)).

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (5) \Rightarrow (6)$ are obvious. By [9, Theorem 2.3] Condition (P_E) implies weak flatness, and so we have the implications $(1) \Rightarrow (4)$ and $(3) \Rightarrow (6)$.

$(6) \Rightarrow (7)$. Since all cyclic right S-acts satisfy Condition (L_P), by assumption all cyclic right S-acts are weakly flat. Then by [11, IV, 7.5] S is regular and satisfies Condition (R).

$(7) \Rightarrow (1)$. By [11, IV, 7.5] all right S-acts are weakly flat. It follows from [9, Theorem 2.5] that all right S-acts satisfy Condition (P_E), and so all right S-acts satisfying Condition (L_P) satisfy Condition (P_E). □

Theorem 3.5. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (P).*
2. *All right S-acts satisfying Condition (L_P), satisfy Condition (EP).*
3. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (P).*
4. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (P).*
5. *S is a group or a group with a zero adjoined.*

Proof. $(1) \Leftrightarrow (2)$. Since $(P) \Leftrightarrow (L_P) \wedge (EP)$, it is clear.

The implications $(1) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. By assumption, all cyclic right S-acts satisfy Condition (L_P), and so satisfy Condition (P). Hence by [16, Theorem 2.1] $S = G^0$ or $S = G$, where G is a group.

(5) ⇒ (1). If $S = G$, then by [13, Theorem 3.10] all right S-acts satisfy Condition (P). Therefore, all right S-acts satisfying Condition (L_P) satisfy Condition (P). Suppose that $S = G^0$ and let A_S be a right S-act which satisfies Condition (L_P). We show that A_S satisfies Condition (P). Take $a, a' \in A_S$ and $s, t \in S$ such that $as = a't$. There are three cases as follows:

Case 1: If $s, t \in G$, then we have $a = a'ts^{-1}, a' = a'1$ and $(ts^{-1})s = 1t$.

Case 2: If $t = 0, s \in G$, then $as = a'0$ implies that $(as)s^{-1} = (a'0)s^{-1}$. So $a = a'0, a' = a'1$ and $0s = 10$.

Case 3: If $s = t = 0$, then $a0 = a'0$. Since A_S satisfies Condition (L_P), there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and clearly $u0 = v0$. □

We deduce from the above theorem that whenever $S \neq G^0$ and $S \neq G$ (G is a group), then there exists at least one (finitely generated) right S-act that satisfies Condition (L_P) but does not satisfy Condition (P).

Theorem 3.6. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (PWP).*
2. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (PWP).*
3. *S is regular and $(\forall x, y, t \in S), (\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y)$.*

Proof. (1) ⇒ (2). It is clear.

(2) ⇒ (3). By assumption all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts satisfy Condition (PWP). Hence by [12, Proposition 3.9], the result follows.

(3) ⇒ (1). If A_S is a finitely generated right S-act which satisfies Condition (L_P), then by Corollary 2.2, one can write $A_S = \coprod \alpha_i S$, where $1 \leq i \leq n (n \in \mathbb{N})$. On the other hand, by assumption and [12, Proposition 3.9], all cyclic right S-acts satisfy Condition (PWP). So each $\alpha_i S$ and consequently A_S satisfies Condition (PWP). □

Conditions (WP) was defined in [12]. A right S-act A_S satisfies Condition (WP) if and only if, for all elements $s, t \in S$, all homomorphisms $f :_S (Ss \cup St) \to_S S$, and all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'' \in A_S, u, v \in S, s', t' \in \{s, t\}$ such that $f(us') = f(vt'), a \otimes s = a'' \otimes us'$, and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes_S (Ss \cup St)$.

Theorem 3.7. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (WP).*
2. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (WP).*

3. *S is regular and* $(\forall x, y, t \in S), (\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y)$ *and for all* $s, t \in S$ *and all homomorphisms* $f : {}_S(Ss \cup St) \longrightarrow {}_SS$, *there exist* $u, v, z, w \in S$ *such that* $(uf(s) = vf(t) \wedge us\tau s \wedge vt\tau t) \vee (uf(s) = vf(s) \wedge u\tau 1 \wedge z\tau v \wedge zs = wt\tau t) \vee (uf(t) = vf(t) \wedge v\tau 1 \wedge w\tau u \wedge zs = wt\tau s)$, *where* $\tau = \rho(f(s), f(t))$.

Proof. By using of [12, Proposition 3.19], the proof is similar to Theorem 3.6. □

Recall from [5] that a right S-act A_S satisfies Condition (P') if for all $a, a' \in A_S, s, t, z \in S, as = a't$ and $sz = tz$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vt$.

Theorem 3.8. *The following statements are equivalent:*

1. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (P').*

2. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (P').*

3. *S is regular and* $(\forall x, y, z, t, t' \in S), (tz = t'z \Longrightarrow (\exists u, v \in S)(x\rho(xt, yt')u \wedge y\rho(xt, yt')v \wedge ut = vt'))$.

Proof. By using of [5, Theorem 3.3], the proof is similar to Theorem 3.6. □

Recall that a monoid S is called left collapsible if for every $s, s' \in S$, there exists $z \in S$ such that $zs = zs'$. S is called weakly left collapsible if for every $s, s', z \in S$, $sz = s'z$ implies the existence of $u \in S$ such that $us = us'$.

Example 3.2. Let S be a monoid which is not weakly left collapsible. Then Θ_S fails to satisfy Condition (E') but it satisfies Condition (L_P). So Condition (L_P) does not imply Condition (E') (as well as Condition (E), regularity, strong faithfulness , equalizer flatness which are stronger properties).

Theorem 3.9. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), satisfy Condition (E').*

2. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (E').*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (E').*

4. $(\forall s, t, z \in S), (sz = tz \Longrightarrow (\exists e \in E(S))(\rho(s, t) = ker\lambda_e))$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (4). By assumption all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts satisfy Condition (E'). Hence by [6, Theorem 2.5], the result follows.

(4) $\Rightarrow$ (1). By [6, Theorem 2.5], all right S-acts satisfy Condition (E'). Hence all right S-acts satisfying Condition (L_P), satisfy Condition (E'). □

Theorem 3.10. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are weakly pullback flat.*
2. *All right S-acts satisfying Condition (L_P), satisfy Conditions (EP) and (E').*
3. *All finitely generated right S-acts satisfying Condition (L_P) are weakly pullback flat.*
4. *All right S-acts generated by at most two elements satisfying Condition (L_P) are weakly pullback flat.*
5. *S is a group or $S = \{0, 1\}$.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4). It is obvious.

(1) $\Leftrightarrow$ (2). It is evident by Corollary 2.5.

(4) $\Rightarrow$ (5). By assumption, all cyclic right S-acts satisfy Condition (L_P), and so all cyclic right S-acts are weakly pullback flat. Hence the statement (5) holds by [2, Proposition 25].

(5) $\Rightarrow$ (1). If S is a group, then by [2, Proposition 9] all right S-acts are weakly pullback flat and we are done. Now suppose that $S = \{0, 1\}$. Let A_S be a right S-act that satisfies Condition (L_P). It suffices to show that A_S satisfies Conditions (P) and (E'). Let $a, a' \in A_S$ and $s, t \in S$ be such that $as = a't$. To show that A_S satisfies Condition (P), we consider three cases:

Case 1: $s = t = 1$. Then we have $a = a'$, and so we can write $a = a'.1, a' = a'.1$ and $1.s = 1.t$.

Case 2: $s = 1, t = 0$. Then we have $a = a'.0, a' = a'.1$ and $0.s = 1.t$.

Case 3: $s = t = 0$. Since A_S satisfies Condition (L_P), the equality $a0 = a'0$ implies that there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a = a''v$ and clearly $u0 = v0$.

So A_S satisfies Condition (P) in each case. It remains to show that A_S satisfies Condition (E'). Since $S = \{0, 1\}$, by [13, Theorem 2.4] all cyclic right S-acts satisfy Condition (E), and consequently satisfy Condition (E'). □

Theorem 3.11. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are strongly flat.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are strongly flat.*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are strongly flat.*

4. *All right S-acts satisfying Condition (L_P) are equalizer flat.*

5. *All finitely generated right S-acts satisfying Condition (L_P) are equalizer flat.*

6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are equalizer flat.*

7. *All right S-acts satisfying Condition (L_P), satisfy Condition (E).*

8. *All finitely generated right S-acts satisfying Condition (L_P), satisfy Condition (E).*

9. *All right S-acts generated by at most two elements satisfying Condition (L_P), satisfy Condition (E).*

10. $S = \{0, 1\}$ *or* $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3), (4) ⇒ (5) ⇒ (6) and (7) ⇒ (8) ⇒ (9) are obvious.

Since we have strongly flat ⇒ equalizer flat ⇒ (E), the implications (1) ⇒ (4) ⇒ (7) and (3) ⇒ (6) ⇒ (9) are valid.

(9) ⇒ (10). Since all cyclic right S-acts satisfy Condition (L_P), then by assumption all cyclic right S-acts satisfy Condition (E). By [13, Theorem 2.4] we are done.

(10) ⇒ (1). If $S = \{1\}$, then all right S-acts are strongly flat and the result follows. Suppose that $S = \{0, 1\}$. Then by [13, Theorem 2.4] all cyclic right S-acts satisfy Condition (E) which shows that all right S-acts satisfy Condition (E). By Corollary 2.4 all right S-acts satisfying Condition (L_P), are strongly flat. □

Recall that a monoid S satisfies Condition (K) if, every left collapsible submonoid of S contains a left zero. The monoid S satisfies Condition (A) if every right S-act satisfies the ascending chain condition for cyclic subacts.

Theorem 3.12. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are projective.*

2. *All finitely generated right S-acts satisfying Condition (L_P) are projective.*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are projective.*

4. $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since projectivity implies strong flatness, by assumption all right S-acts generated by at most two elements satisfying Condition (L_P), are strongly flat. So by [16, Corollary 2.2], $S = \{1\}$ or $S = \{0, 1\}$.

(4) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are projective. Now suppose that $S = \{0, 1\}$. First, we show that the monoid S satisfies Condition (A). Assume that A_S is a right S-act and $\alpha_1 S \subseteq \alpha_2 S \subseteq ...$ be an ascending chain of cyclic subacts of A_S. Since $S = \{0, 1\}$, we have for all $i \in \mathbb{N}, 1 \leq | \alpha_i S | \leq 2$. So $| \alpha_1 S | \leq | \alpha_2 S | \leq ...$ and for all $i \in \mathbb{N}, \alpha_i S \subseteq \alpha_{i+1} S$, which shows that the above chain terminates. Thus the monoid S satisfies Condition (A). Obviously, the monoid S satisfies Condition (K), and so by [10, Theorem 1.1] and [10, Theorem 2.1], strong flatness implies projectivity. Since all right S-acts satisfying Condition (L_P) are strongly flat by Theorem 3.11, the result follows. □

Theorem 3.13. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P) are free.*
2. *All finitely generated right S-acts satisfying Condition (L_P) are free.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P) are free.*
4. *All right S-acts satisfying Condition (L_P) are projective generator.*
5. *All finitely generated right S-acts satisfying Condition (L_P) are projective generator.*
6. *All right S-acts generated by at most two elements satisfying Condition (L_P) are projective generator.*
7. *All right S-acts satisfying Condition (L_P) are generator.*
8. *All finitely generated right S-acts satisfying Condition (L_P) are generator.*
9. *All right S-acts generated by at most two elements satisfying Condition (L_P) are generator.*
10. *All right S-acts satisfying Condition (L_P) are strongly faithful.*
11. *All finitely generated right S-acts satisfying Condition (L_P) are strongly faithful.*
12. *All right S-acts generated by at most two elements satisfying Condition (L_P) are strongly faithful.*
13. *$S = \{1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12), are obvious.

Since *free* $\Rightarrow$ *projective generator* $\Rightarrow$ *generator*, the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (7) and (3) $\Rightarrow$ (6) $\Rightarrow$ (9) are easily obtained.

(12) $\Rightarrow$ (13). Since the one element act Θ_S satisfy Condition (L_P), by assumption Θ_S is strongly faithful. So $S = \{1\}$.

(13) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are free.

(13) $\Rightarrow$ (10). If $S = \{1\}$, then all right S-acts are strongly faithful.

(9) $\Rightarrow$ (13). By assumption Θ_S is generator. So there exists an epimorphism $\pi : \Theta_S \to S_S$ and this implies that $S = \{1\}$. □

An act A_S is said to be divisible, if $Ac = A$ for any left cancellable element $c \in S$.

Theorem 3.14. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are divisible.*
2. *All finitely generated right S-acts satisfying Condition (L_P), are divisible.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are divisible.*
4. *All left cancellable elements of S are left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since S_S is generated by at most two elements and also satisfies Condition (L_P), by [11, III, 2.2] all left cancellable elements of S are left invertible.

(4) $\Rightarrow$ (1). By [11, III, 2.2] every right S-act is divisible. So we are done. □

Theorem 3.15. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are principally weakly injective.*
2. *All finitely generated right S-acts satisfying Condition (L_P), are principally weakly injective.*
3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are principally weakly injective.*
4. *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). If (3) holds, then all principal right ideals of S are principally weakly injective. By [11, IV, 1.6] S is regular.

(4) $\Rightarrow$ (1). It follows from [11, IV, 1.6]. □

Theorem 3.16. *The following statements are equivalent:*

1. *All right S-acts satisfying Condition (L_P), are regular.*

2. *All finitely generated right S-acts satisfying Condition (L_P), are regular.*

3. *All right S-acts generated by at most two elements satisfying Condition (L_P), are regular.*

4. $S = \{0\}$ *or* $S = \{0, 1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). By assumption, all cyclic right S-acts are regular. By [3, Theorem 1.17] we get $S = \{0\}$ or $S = \{0, 1\}$.

(4) ⇒ (1). It follows from [11, IV, 14.4]. □

For fixed elements $u, v \in S$, define a binary relation $P_{u,v}$ on S with $(x, y) \in P_{u,v} \iff ux = vy \, (x, y \in S)$. Also, for $s, t \in S$, let $\mu_{s,t} = ker\lambda_s \vee ker\lambda_t$.

Recall that for any right ideal I of S, Rees congruence ρ_I on S defined by $(x, y) \in \rho_I$ if $x = y$ or $x, y \in I$.

Recall that an act is called cofree if it is isomorphic to the act $X^S = \{f | f$ is a mapping from S to $X\}$ for some nonempty set X, where fs is defined by $fs(t) = f(st), t \in S$, for every $f \in X^S, s \in S$.

Theorem 3.17. *The following statements are equivalent:*

1. *All fg-weakly injective right S-acts satisfy Condition (L_P).*

2. *All weakly injective right S-acts satisfy Condition (L_P).*

3. *All injective right S-acts satisfy Condition (L_P).*

4. *All cofree right S-acts satisfy Condition (L_P).*

5. *For any $s, t \in S$, there exist $u, v \in S$ such that:*

   (i) $P_{u,v} \subseteq P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$.

   (ii) $ker\lambda_u \subseteq \rho_{sS}$.

   (iii) $ker\lambda_v \subseteq \rho_{tS}$.

   (iv) $ker\lambda_{us} \cup ker\lambda_{vt} \subseteq \mu_{s,t}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are clear.

(4) ⇒ (5). Suppose that $s, t \in S$ and let S_1 and S_2 be two sets such that $|S_1| = |S| = |S_2|$. Assume that $\alpha : S \to S_1$ and $\beta : S \to S_2$ are bijections. Let $X = S/\mu_{s,t} \sqcup S_1 \sqcup S_2$. Define the mappings $f, g : S \longrightarrow X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}}, & \text{if } x \in sS (x = sy) \\ \alpha(x), & \text{if } x \notin sS \end{cases}$$

and

$$g(x) = \begin{cases} [y]_{\mu_{s,t}}, & \text{if } x \in tS(x = ty) \\ \beta(x), & \text{if } x \notin tS \end{cases}$$

If $y_1, y_2 \in S$ are such that $sy_1 = sy_2$, then $(y_1, y_2) \in ker\lambda_s \subseteq \mu_{s,t}$ which implies that $f(sy_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = f(sy_2)$.

Similarly if $ty_1 = ty_2$ then $g(ty_1) = g(ty_2)$. So f and g are well-defined. Also $fs = gt$ by definition. Since X^S satisfies Condition (L_P), there exist $u, v \in S$ and the map $h : S \to X$ such that $g = hv$ and $f = hu$. Now, we show that the conditions (i) to (iv) are true.

(i) let $(l_1, l_2) \in P_{u,v}$, $l_1, l_2 \in S$. Then $ul_1 = vl_2$, and so

$$f(l_1) = hu(l_1) = h(ul_1) = h(vl_2) = hv(l_2) = g(l_2).$$

The equality $f(l_1) = g(l_2)$ implies that there exist $y_1, y_2 \in S$ such that $l_1 = sy_1, l_2 = ty_2, (y_1, y_2) \in \mu_{s,t}$. Hence $(l_1, l_2) \in P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$.

(ii) Assume that the condition (ii) does not hold. Then there exist $p_1, p_2 \in S$ such that $(p_1, p_2) \in ker\lambda_u \setminus \rho_{sS}$. It follows that

$$(\exists p_1, p_2 \in S)(up_1 = up_2 \wedge p_1 \neq p_2 \wedge (p_1 \notin sS \vee p_2 \notin sS)).$$

Hence $f(p_1) = hu(p_1) = h(up_1) = h(up_2) = hu(p_2) = f(p_2)$. Since $p_1 \neq p_2$ and α is injective, $p_1, p_2 \in sS$ which is a contradiction.

(iii) The argument is similar to (ii).

(iv) Suppose that $(l_1, l_2) \in ker\lambda_{us}$, $l_1, l_2 \in S$. So $usl_1 = usl_2$ and we have $f(sl_1) = hu(sl_1) = h(usl_1) = h(usl_2) = hu(sl_2) = f(sl_2)$. According to the definition of f, we have $sl_1 = sl_2$ or $[l_1]_{\mu_{s,t}} = [l_2]_{\mu_{s,t}}$. If $sl_1 = sl_2$ then $(l_1, l_2) \in ker\lambda_s \subseteq \mu_{s,t}$. If $[l_1]_{\mu_{s,t}} = [l_2]_{\mu_{s,t}}$ then $(l_1, l_2) \in \mu_{s,t}$. So $ker\lambda_{us} \subseteq \mu_{s,t}$. Similar argument shows that $ker\lambda_{vt} \subseteq \mu_{s,t}$. Consequently $ker\lambda_{us} \cup ker\lambda_{vt} \subseteq \mu_{s,t}$.

(5) $\Rightarrow$ (1). Suppose that A_S is fg-weakly injective and for $a, a' \in A_S, s, t \in S, as = a't$. By assumption there exist $u, v \in S$ such that the conditions (i) to (iv) hold. Define $\varphi : uS \cup vS \to A_S$ by

$$\varphi(x) = \begin{cases} ap, & \text{if } x \in uS(x = up), \\ a'r, & \text{if } x \in vS(x = vr), \end{cases}$$

for every $x \in uS \cup vS$. To show that φ is well defined, we consider three cases as follows:

Case 1. If there exist $p, r \in S$ such that $up = vr$, then by condition (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{1,s}, (y_1, y_2) \in \mu_{s,t}, (y_2, r) \in P_{t,1}$. Thus $p = sy_1, r = ty_2$ and $(y_1, y_2) \in (ker\lambda_s \vee ker\lambda_t)$. So, there exist $z_1, z_2, ..., z_n \in S$ such that

$$\begin{array}{cccc} sy_1 = sz_1 & sz_2 = sz_3 & \dots & sz_{n-1} = sz_n \\ tz_1 = tz_2 & & & tz_n = ty_2. \end{array}$$

Now, we have $ap = asy_1 = asz_1 = a'tz_1 = a'tz_2 = ... = a'tz_n = a'ty_2 = a'r$.

Case 2. Assume that there exist $p_1, p_2 \in S$ such that $up_1 = up_2$. If $p_1 = p_2$, then $ap_1 = ap_2$. If $p_1 \neq p_2$, then by condition (ii), there exist $y'_1, y'_2 \in S$ such that $p_1 = sy'_1$ and $p_2 = sy'_2$. From $up_1 = up_2$ we have $usy'_1 = usy'_2$ and so $(y'_1, y'_2) \in ker\lambda_{us} \subseteq \mu_{s,t}$. This means that there exist $z_1, z_2, ..., z_n \in S$ such that

$$\begin{array}{ccccc} sy'_1 = sz_1 & sz_2 = sz_3 & \dots & sz_{n-1} = sz_n & \\ & tz_1 = tz_2 & & & tz_n = ty'_2. \end{array}$$

Hence $ap_1 = asy'_1 = asz_1 = a'tz_1 = ... = a'tz_n = a'ty'_2 = asy'_2 = ap_2$.

Case 3. If there exist $r_1, r_2 \in S$ such that $vr_1 = vr_2$, then by a similar argument we get $a'r_1 = a'r_2$. Hence φ is well defined. Clearly φ is a homomorphism. Since A_S is fg-weakly injective, there exists a homomorphism $\psi : S_S \to A_S$ which extends φ. Put $a'' = \psi(1)$. Then $a = \varphi(u) = \psi(u) = \psi(1)u = a''u$ and $a' = \varphi(v) = \psi(v) = \psi(1)v = a''v$. Thus A_S satisfies Condition (L_P). □

Corollary 3.1. *If S is a commutative monoid, then all cofree S-acts satisfy Condition (L_P) if and only if S is a group.*

Proof. Let $s \in S$. By Theorem 3.17, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$ $(\mu_{s,s} = ker\lambda_s \vee ker\lambda_s = ker\lambda_s)$. If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which implies that $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Commutativity of S implies that $uv = vu$ which yields $(v, u) \in P_{u,v}$ and so $u = v$. Hence $ker\lambda_u = P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$. Since $u.1 = u.1$, there exist $y_1, y_2 \in S$ such that $(1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, 1) \in P_{s,1}$. Hence $1 = sy_1 = sy_2$ and S is a group.

Conversely, suppose that S is a group. By Theorem 3.1, all right S-acts satisfy Condition (L_P). So all cofree right S-acts satisfy Condition (L_P). □

Corollary 3.2. *If S is a finite monoid, then all cofree S-acts satisfy Condition (L_P) if and only if S is a group.*

Proof. By Theorem 3.17, for any $s \in S$ there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{sS}$ and $ker\lambda_{us} \cup ker\lambda_{vs} \subseteq ker\lambda_s.(\mu_{s,s} = ker\lambda_s \vee ker\lambda_s = ker\lambda_s.)$ If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which means $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Suppose that $l_1, l_2 \in S$ are such that $ul_1 = ul_2$ and $l_1 \neq l_2$. Then there exist $y_1, y_2 \in S, l_1 = sy_1, l_2 = sy_2$ which implies that $usy_1 = usy_2$. The last equality implies that $(y_1, y_2) \in ker\lambda_{us} \subseteq ker\lambda_s$. So $sy_1 = sy_2$ that is, $l_1 = l_2$ which is a contradiction. Hence u is left cancellable. Let $S = \{1, x_1, x_2, ..., x_n\}$ (note that the elements of S are distinct). It is clear that $uS = \{u, ux_1, ux_2, ..., ux_n\} = S$. So $v \in uS$. If there exists $i \leq n$ such that $ux_i = v$, then $(x_i, 1) \in P_{u,v} \subseteq \Delta_S$ which implies that $x_i = 1$, a contradiction. Hence $v = u$ and by a similar argument to Corollary 3.1, we get that s has an inverse. Thus, S is a group.

The converse has been proved in Theorem 3.1. □

Corollary 3.3. *If S is an idempotent monoid, then all cofree S-acts satisfy Condition (L_P) if and only if $S = \{1\}$.*

Proof. Suppose that all cofree S-acts satisfy Condition (L_P). We claim that $S = \{1\}$. Assume that $S \neq \{1\}$. So there exists $e \in S \setminus \{1\}$. By Theorem 3.17, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,e} \circ ker\lambda_e \circ P_{e,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{eS}$. Obviously, $(u, 1) \in ker\lambda_u \subseteq \rho_{eS}$, and so $u = 1$ or there exist $y_1, y_2 \in S$ such that $u = ey_1$ and $1 = ey_2$. Since $e \neq 1$, we get that $u = 1$ and similarly $v = 1$. By a similar argument to the Corollary 3.1, e has a right inverse. Hence $e = 1$, which is a contradiction. Thus $S = \{1\}$ and we are done.

The converse is a part of the Theorem 3.1. □

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