Some remarks on fully stable gamma modules

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Abstract. In this work we study full stability in the gamma module theory. A gamma module M is fully stable, if for each gamma submodule N of M, $\theta(N) \subseteq N$ for each gamma homomorphism θ of N into M. Several properties and characterizations of this classes of gamma modules have been studied. The advantages of these characterizations have been considered. Finding some sources of full stability and discuss the direct sum in fully stable gamma modules, by the way we show that in fully stable gamma modules, each gamma submodule has a unique complement. Finally characterize full stability by some of their generalizations and relate with the (SIP) and (SSP) properties.
Keywords: gamma modules, fully stable gamma modules, uniserial gamma module, $I-$multiplication gamma modules, fully ds$-$(essential) stable gamma modules, the (SIP) and (SSP) properties.

1. Introduction

In 1964, N. Nobusawa gave, as a generalization of the concept of ring , the notion of gamma rings [6]. in 1966 W.E. Barnes generalized this concept and obtained some basic properties gamma rings [3].

Let R and Γ be two additive Abeline groups. R is called a $\Gamma-$ring if there is a mapping: $R \times \Gamma \times R \to R$, $(r, \alpha, \overline{r}) \to r\alpha\overline{r}$ such that the following hold for all $r_1, r_2, r_3 \in R$, $\alpha, \beta \in \Gamma$.

i. $(r_1 + r_2)\alpha r_3 = r_1\alpha r_3 + r_2\alpha r_3$

ii. $r_1(\alpha + \beta)r_2 = r_1\alpha r_2 + r_1\beta r_2$

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iii. $r_1\alpha(r_2+r_3)=r_1\alpha r_2+r_1\alpha r_3$

iv. $(r_1\alpha r_2)\beta r_3=r_1\alpha(r_2\beta r_3)$.

A subset W of a $\Gamma-$ring R is called right (left) $\Gamma-$ideal of R if W is a subgroup of R and $W\Gamma R\subseteq W(RW\subseteq W)$, Where $R\Gamma W=\{W\alpha r|r\in R,\alpha\in\Gamma,w\in W\}$, W is called $\Gamma-$ideal if it is both right and left $\Gamma-$ideal of R,in particular $W\alpha R$ is called aright (left) $\alpha-$ideal of R Where $\alpha\in\Gamma$ if W is asubgroup of R and $W\alpha R(R\alpha W)\subseteq W$.

In 2010, R. Ameri, R. Sadeqhi extend the concept of modules to gamma modules [2]. Let R be a $\Gamma-$ring. A (left) $R_\Gamma-$module is an additive Abelian group M, if there exist a mapping: $R\times\Gamma\times M\to M$ $r\gamma m$ denote the image of (r,γ,m) such that the following hold for all $m,m_1,m_2\in M$, $\gamma,\gamma_1,\gamma_2\in R$ and $r,r_1,r_2\in R$:

i. $r\gamma(m_1+m_2)=r\gamma m_1+r\gamma m_2$

ii. $(r_1+r_2)\gamma m=r1\gamma m+r_2\gamma m$

iii. $r(\gamma_1+\gamma_2)m=r\gamma_1 m+r\gamma_2 m_1$

iv. $r_1\gamma_1(r_2\gamma_2 m)=(r_1\gamma_1 r_2)\gamma_2 m$.

A right $R_\Gamma-$module is defined a similar way. A left $R_\Gamma-$module M is unitary if there are element 1 in R and $\alpha_0\in\Gamma$ such that $1\alpha_0 m=m$ for all m in M. For more detail of gamma module see [2].

In 1991, M. S. Abbas introduced and studied the notion of fully stable modules [1] Let M be an $R-$module. Asubmodule N of M is called stable if $f(N)\subseteq N$ for each homomorphism f of N into M. If each submodule of M is stable, then M is called fully stable.

In this work, we consider the notion of full stability in the category of gamma modules. A left $R_\Gamma-$module M is called fully stable if $f(N)\subseteq N$. for each $R_\Gamma-$submodule N of M and $R_\Gamma-$homomorphism $f:N\to M$ Let M be an $R_\Gamma-$module. For an arbitrary fixed α in Γ, a subset A of R and a subset W of M we define $\ell_R^\alpha(W)=\{r\in R|r\alpha W=0\}$ and $\Upsilon_M^\alpha(A)=\{m\in M\mid A\alpha m=0\}$.

We gave several properties and characterizations of this class of gamma modules a left $R_\Gamma-$module M is fully stable if and only if $\gamma_M^{\alpha_0}\left(\ell_R^{\alpha_0}(R\alpha_0 a)\right)=R\alpha_0 a$ for all a in M, that is fully stable $R_\Gamma-$modules are precisely those $R_\Gamma-$modules in which each α_0-cyclic $R_\Gamma-$submodule is a right annihilatar of some right α_0-ideal of R. with certain conditions this characterization is extended for α_0-finitely generated $R_\Gamma-$submodules. $I-$multiplication $R_\Gamma-$module and uniserial $R_\Gamma-$module with descending chain condition for α_0-cyclic $R_\Gamma-$submodules are fully stable.

By examples the full stability of the direct sum of fully stable $R_\Gamma-$modules being not true, and investigate conditions under which the full stability is closed under direct-sum. We see that in fully stable $R_\Gamma-$modules each direct summand has a unique complement direct summand, this result motivate us to consider some generalizations of fully stable gamma module in fact we restrict the stability only on certain class of gamma submodule. By means of these generalizations we see that every fully stable gamma modules satisfies the (SIP) and (SSP) properties and every regular multiplication gamma modules is fully stable.

2. Fully stable gamma modules

Definition 2.1. *An $R_\Gamma-$submodule N of an $R_\Gamma-$module of M is called stable if $\theta(N) \subseteq N$ for each $R_\Gamma-$homomorphism θ of N into M. An $R_\Gamma-$module is called fully stable if all of its $R_\Gamma-$submodules are stable. A $\Gamma-$ring R is right (left) fully stable if R is a right (left) fully stable $R_\Gamma-$module.*

Let M be an $R_\Gamma-$module and an arbitrary fixed element. An $R_\Gamma-$submodule N is called $\alpha-$cyclic, if there exists an element $m \in M$ such that $N = R\alpha m$. Its an easy matter to see that an $R_\Gamma-$module M is fully stable if and only if each α_0-cyclic $R_\Gamma-$submodule of M is stable. Every $R_\Gamma-$submodule of fully stable $R_\Gamma-$module is fully stable.

Let M be an $R_\Gamma-$module, and N a non-empty subset of M. For arbitrary fixed $\alpha \in \Gamma$. N is called $R^\alpha_\Gamma-$submodule of M if N is a subgroup of M and $R\alpha N \subseteq N$. For a subset A of R and a subset W of M, we define $\ell^\alpha_R(W) = \{r \in R | r\alpha W = 0\}$ and $\Upsilon^\alpha_M(A) = \{m \in M | A\alpha m = 0\}$.

Then its an easy matter to see the following

Lemma 2.2. *Let M be an $R_\Gamma-$module with subset A and W of R and M respectively and α an arbitrary fixed element in Γ. Then:*

1. *$\ell^\alpha_R(W)$ is a left α or $\Gamma-$ideal of R.*
2. *If W is an $R^\alpha_\Gamma-$submodule of M, then $\ell^\alpha_R(W)$ is right $\alpha-$ideal of R.*
3. *If A is a right α or $\Gamma-$ideal of R, then $\Upsilon^\alpha_M(A)$ is an $R^\alpha_\Gamma-$submodule of M.*

If A is a right α or $\Gamma-$ideal of R, then clearly $\Upsilon^\alpha_M(A)$ is a stable $R^\alpha_\Gamma-$submodule of M. In the case that each α_0-cyclic $R_\Gamma-$submodule of M is of the form $\Upsilon^{\alpha_0}_M(I)$ for some right α_0-ideal of R, then M is fully stable. In the following we show that fully stable $R_\Gamma-$modules are precisely those $R_\Gamma-$modules in which each α_0-cyclic $R_\Gamma-$submoule is a right annihilator of some right α_0-ideal of R.

Theorem 2.3. *The following statements are equivalent for a an $R_\Gamma-$module M:*

1. *M is fully stable.*
2. *$X \subseteq Y$ for every $R_\Gamma-$submodules X and Y of M in which X is an $R_\Gamma-$homomorphic image of Y.*

3. *For each* a, b *in* M*,* $b \neq R\alpha_0 a$ *implies that* $\ell_R^{\alpha_0}(R\alpha_0 a) \nsubseteq \ell_R^{\alpha_0}(R\alpha_0 b)$*.*

4. $\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(R\alpha_0 a)\right) = R\alpha_0 a$*, for all* $a \in M$*.*

Proof. $(1 \to 2)$ let X and Y be an $R_\Gamma-$submodules of M and $\theta : Y \to X$ an $R_\Gamma-$epimorphism. Then $X = \theta(Y) \subseteq Y$.

$(2 \to 3)$Assume that there are a, b in M with $b \notin R\alpha_0 a$ and $\ell_R^{\alpha_0}(R\alpha_0 a) \subseteq \ell_R^{\alpha_0}(R\alpha_0 b)$, then $R\alpha_0 b$ is an $R_\Gamma-$homomorphic image of $R\alpha_0 a$ By (2) $R\alpha_0 b \subseteq R\alpha_0 a$, and hence $b \in R\alpha_0 a$ which is a contradiction.

$(3 \to 4)$ Assume that there exists $m \in \ell_{\mathrm{M}}^{\alpha_0}\left(\Upsilon_R^{\alpha_0}(R\alpha_0 a)\right)$ and $m \notin R\alpha_0 a$. By (3) $\ell_R^{\alpha_0}(R\alpha_0 a) \nsubseteq \ell_R^{\alpha_0}(R\alpha_0 m)$ and hence there is $s \in \ell_R^{\alpha_0}(R\alpha_0 a)$, and $s \notin \ell_R^{\alpha_0}(R\alpha_0 m)$ so $s\alpha_0 m = 0$ which is a contradiction.

Thus $\ell_{\mathrm{M}}^{\alpha_0}\left(\Upsilon_R^{\alpha_0}(R\alpha_0 a)\right) \subseteq R\alpha_0 a$. Thus the other inclusion is always true.

$(4 \to 1)$ its obvious. □

Corollary 2.4. *Let* M *be a fully stable* $R_\Gamma-$*module. Then distinct* R_Γ*-submodules of* M *are not isomorphic.*

Proof. Assume that there are distinct $R_\Gamma-$submodules X and Y of M and X is isomorphic to Y, then each of them is an $R_\Gamma-$homomorphic image of the other and hence $X \subseteq Y$ and $Y \subseteq X$ which is a contradiction. □

Example 2.5.

1. It is clear that every simple $R_\Gamma-$module is fully stable, but semisimple $R_\Gamma-$modules (in which every $R_\Gamma-$submodule is direct summand) may not be fully stable generally, for example, let R be a simple $\Gamma-$ring and $M = R \oplus R$ as $R_\Gamma-$module. Consider the $R_\Gamma-$submodule $N = R \oplus 0$ of M and define $\theta : N \to M$ by $\theta(r, 0) = (0, r)$ for all $r \in R$. Then $\theta(N) \nsubseteq N$.

2. Let R be a $\Gamma-$ring. Then every Abelian group G can be made into an $R_\Gamma-$module by defining $r\alpha g = 0$ for all $r \in R$, $\alpha \in \Gamma$ and $g \in G$. Then M is fully stable.

3. Let $R = \mathbb{Z}$ and $\Gamma = S$ be arbitrary subring of $\mathbb{Z}$. Then every arbitrary Abelian group M is $\mathbb{Z}_S-$module by define the mapping $(x, \overline{\alpha}, m) \to x\overline{\alpha}m$ for all $x \in \mathbb{Z}$, $\alpha \in S$ and $m \in M$. Consider the following cases:

   (a) If $M = \mathbb{Z}_{p^\infty}$ and $\theta : \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^\infty}$ an $R_\Gamma-$homomorphism, then the elements of $\mathbb{Z}_{p^k}$ are those elements of order less than or equal to p^k, hence $\theta(\mathbb{Z}_{p^k}) \subseteq \mathbb{Z}_{p^k}$. It follows that $\mathbb{Z}_{p^\infty}$ is a fully stable $\mathbb{Z}_S-$module.

   (b) Let $M = \mathbb{Z}$. For each non-zero elements $n, \overline{n} \in M$ with $\overline{n} \notin R\alpha_0 n$. It is always $\ell_{\mathbb{Z}}^{\alpha_0}(R\alpha_0 n) = \ell_{\mathbb{Z}}^{\alpha_0}(R\alpha_0 \overline{n}) = 0$, thus $\mathbb{Z}$ it is not fully stable $\mathbb{Z}_S-$module. By a similar way, we can see $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both not fully stable $\mathbb{Z}_S-$module.

4. Every fully stable $R-$module is fully stable $R_\Gamma-$module.

We will eventually attempt to take advantage of condition (4) in theorem (2.3). to begin our attempt we need the following proposition:

Proposition 2.6. *Let M be a fully stable $R_\Gamma-$module such that for each $x \in M$ and left $\Gamma-$ideal I of R, each $R_\Gamma-$homomorphism $\theta : I\alpha_0 x \to M$ can be extended to an $R_\Gamma-$homomorphism $\theta : R\alpha_0 x \to M$. If an $R_\Gamma-$submodule N of M satisfies the $\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(N)\right) = N$, then so does $N + R\alpha_0 x$.*

Proof. Set $A = \ell_R^{\alpha_0}(N)$ and $B = \ell_R^{\alpha_0}(R\alpha_0 x)$. Then by the hypothesis $\Upsilon_M^{\alpha_0}(A) = N$ and $\Upsilon_M^{\alpha_0}(B) = R\alpha_0 x$. Since $\ell_R^{\alpha_0}(N + R\alpha_0 x) = A \cap B$, it is enough to see that $\Upsilon_M^{\alpha_0}(A \cap B) \subseteq N + R\alpha_0 x$. Let $y \in \Upsilon_M^{\alpha_0}(A \cap B)$. Define the well defined $R_\Gamma-$homomorphism $\theta : A\alpha_0 x \to M$ by $\theta(a\alpha_0 x) = a\alpha_0 y$ for a$\in A$. Then there is an $R_\Gamma-$homomorphism $\overline{\theta} : R\alpha_0 x \to M$ which extends θ. Full stability of M implies that $\overline{\theta}(x) \in R\alpha_0 x$. For each $a \in A$, $a\alpha_0\overline{\theta}(x) = \overline{\theta}(a\alpha_0 x) = a\alpha_0 y$. Then $\overline{\theta}(x) - y \in \Upsilon_M^{\alpha_0}(A) = N$, so $y \in N + R\alpha_0 x$. □

Let M be an $R_\Gamma-$module and α be an arbitrary fixed element of Γ. An $R_\Gamma-$submodule N is called $\alpha-$finitely generated if there are $x_1, x_2, ..., x_n$ in M such that $N = R\alpha x_1 + R\alpha x_2 + \ldots + R\alpha x_n$. An $R_\Gamma-$module is called $\alpha-$Noetherian if each $R_\Gamma-$submodule is $\alpha-$finitely generated.

If M is an $R-$module, then M is Noetherian if and only if each submodule of M is $1-$finitely generated.

By using the principle of induction on the generators of an $\alpha-$finitely generated $R_\Gamma-$submodule with the help of Proposition 2.1 and Theorem 2.1 we have the following:

Theorem 2.7. *Let M be an $R_\Gamma-$module such that for x in M and left $\Gamma-$ideal I of R, each $R_\Gamma-$homomorphism of $I\alpha_0 x$ into M can be extended to an $R_\Gamma-$homomorphism of $R\alpha_0 x$ to M. Then M is fully stable if and only if each α_0-finitely generated $R_\Gamma-$submodule of M satisfies the double annihilator condition.*

Recall that an $R_\Gamma-$module M is quasi-injectiv, if for each $R_\Gamma-$submodule A of M, each $R_\Gamma-$homomorphism from A into M can be extended to an $R_\Gamma-$endomorphism of M [7].

The concept of quasi-injective $R_\Gamma-$module satisfies the condition of theorem (2.2), so we have the following, corollary

Corollary 2.8. *The following conditions are equivalent for quasi-injective α_0-Noetherian $R_\Gamma-$module M:*

1. *M is fully stable.*
2. *$\Upsilon_M^{\alpha_0}\left(\ell_R^{\alpha_0}(N)\right) = N$ for each $R_\Gamma-$submodule N of M.*

we have mentioned that an $R_\Gamma-$module M is fully stable if and only if for each m in M $R_\Gamma-$homomorphism θ : R$\alpha_0 m \to$ M, there exists $r \in R$ such that $\theta(m) = r\alpha_0 m$. We denote this condition by (B).

Proposition 2.9. *Let M be an $R_\Gamma-$module with $\ell_R^{\alpha_0}(N \cap K) = \ell_R^{\alpha_0}(N) + \ell_R^{\alpha_0}(K)$ for all α_0-finitely generated $R_\Gamma-$submodule N and K of M. Then M is fully stable if and only if condition (B) satisfied for α_0-finitely generated $R_\Gamma-$submodules.*

Proof. Assume that condition (B) holds for α_0-finitely generated R_Γ-submodule, in particular for α_0-cyclic hence by theorem (2.1) M is fully stable. Conversely. assume that M is fully stable. Let N be α_0-finitely generated $R_\Gamma-$submodule of M and $f : N \to M$ an $R_\Gamma-$homomorphism. Now $N = R\alpha_0 x_1 + R\alpha_0 x_2 + \ldots + R\alpha_0 x_n$ for some $x_1, x_2, \ldots, x_n \in N$. We eventually use induction of the number of generators of N. For $n = 1$, this is just theorem (2.1). Suppose that M satisfies condition (B) for all $R_\Gamma-$submodules α_0-generated by m elements for $m \leq n-1$. Then there are $r, s \in R$ such that $f(x) = r\alpha_0 x$ for all $x \in R\alpha_0 x_1 + R\alpha_0 x_2 + \ldots + R\alpha_0 x_{n-1}$ and $f(x') = s\alpha_0 x'$ for all $x' \in R\alpha_0 x_n$. For each $y \in (R\alpha_0 x_1 + R\alpha_0 x_2 + \ldots + R\alpha_0 x_{n-1}) \cap R\alpha_0 x_n$, $f(y) = r\alpha_0 y = s\alpha_0 y$, then $r - s \in \ell_R^{\alpha_0}((R\alpha_0 x_1 + R\alpha_0 x_2 + \ldots + R\alpha_0 x_{n-1}) \cap R\alpha_0 x_n)$. Then there are $u \in \ell_R^{\alpha_0}(R\alpha_0 x_1 + R\alpha_0 x_2 + \ldots + R\alpha_0 x_{n-1})$ and $v \in \ell_R^{\alpha_0}(R\alpha_0 x_n)$ such that $r - s = u + v$ put $t = r - u = s + v$. For each $z \in N, z = \sum_{i=1}^{n} r_i\alpha_0 x_i$ for some $r_i \in R$.

$$
\begin{aligned}
f(z) = f\left(\sum_{i=1}^{n} r_i\alpha_0 x_i\right) &= f\left(\sum_{i=1}^{n-1} r_i\alpha_0 x_i\right) + f(r_n\alpha_0 x_n) \\
&= r\alpha_0\left(\sum_{i=1}^{n-1} r_i\alpha_0 x_i\right) + s\alpha_0(r_n\alpha_0 x_n) \\
&= r\alpha_0\left(\sum_{i=1}^{n-1} r_i\alpha_0 x_i\right) - u\alpha_0\left(\sum_{i=1}^{n-1} r_i\alpha_0 x_i\right) + s\alpha_0(r_n\alpha_0 x_n) + v\alpha_0(r_n\alpha_0 x_n) \\
&= (r-u)\alpha_0\left(\sum_{i=1}^{n-1} r_i\alpha_0 x_i\right) + (s+v)\alpha_o(r_n\alpha_0 x_n) = t\alpha_0 z.
\end{aligned}
$$

□

Corollary 2.10. *The following are equivalent for α_0-Noetherian $R_\Gamma-$module M.*

1. *M is fully stable.*
2. *Condition (B) holds for all $R_\Gamma-$submodules of M.*

In the following We prepare same sources of fully stable gamma modules. A left $\Gamma-$ideal A of a $\Gamma-$ring R is called idempotent if $A = A\Gamma A$ [5]. An $R_\Gamma-$module M is called $I-$multiplication if for each $R_\Gamma-$submodule N of M, there is an idempotent left $\Gamma-$ideal I of R such that $N = I\Gamma M$.

Proposition 2.11. *Every $I-$multiplication $R_\Gamma-$module is fully stable.*

Proof. Let M be $I-$multiplication $R_\Gamma-$module. For each $R_\Gamma-$submodule N of M and $R_\Gamma-$homomorphism $\theta : N \to M, N = I\Gamma M$ for some idempotent $\Gamma-$ideal I of R. Hence $\theta(N) = \theta(I\Gamma M) = \theta(I\Gamma I\Gamma M) = I\Gamma\theta(N) \subseteq I\Gamma M = N$. □

An $R_\Gamma-$module M is called uniserial if $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ for all two $R_\Gamma-$submodules N_1 and N_2 of M.

Proposition 2.12. *Every uniserial $R_\Gamma-$module M with descending chain condition for α_0-cyclic $R_\Gamma-$submodules is fully stable. In particular, if R is a finite $\Gamma-$ring in which each left $-$ideal is α_0-principle, then R is left fully stable.*

Proof. Assume that $a, b \in M$ with $\ell_R^{\alpha_0}(\mathrm{R}\alpha_O a) \subseteq \ell_R^{\alpha_0}(b)$ and $b \notin R\alpha_0 a$. Then $R\alpha_0 a \subseteq R\alpha_0 b$ such that $a = a\alpha_0 b$. Consider the descending chain condition $R\alpha_0(r\alpha_0 b) \supseteq R\alpha_0(r\alpha_0 r\alpha_0 b) \supseteq \dots$ Then $R\alpha_0((r\alpha_0 r)^n\alpha_0 b) = R\alpha_0((r\alpha_0 r)^{n+1}\alpha_0 b)$ for same positive integer n and so $(r\alpha_0 r)^n\alpha_0 b = t\alpha_o(r\alpha_0 r)^{n+1}\alpha_0 b$ for some $t \in R$. Thus $(r\alpha_0 r)^{n-1}r\alpha_0 a = t\alpha_0(r\alpha_0 r)^n r\alpha_0 a$. Since $\ell_R^{\alpha_0}(R\alpha_0 a) \subseteq \ell_R^{\alpha_0}(R\alpha_0 b)$, then $(r\alpha_0 r)^{n-1}r\alpha_0 b = t\alpha_o(r\alpha_0 r)^n\alpha b$. By using a similar argument to obtain $r\alpha_0 b = t\alpha_0 r\alpha_0 r\alpha_0 b$, so $a = t\alpha_0 r\alpha_0 a$ a. Then $b = t\alpha_0 r\alpha_0 r\alpha_0 b = t\alpha_0 r\alpha_0 a \in R\alpha_0 a$ which is a contradiction. Thus M is fully stable. □

The above proposition shows in another way that $\mathbb{Z}^\infty$ is fully stable $\mathbb{Z}_S$-module for every subring S of $\mathbb{Z}$. The finite condition in the particular case is essential for example Z as $\mathbb{Z}_S-$module.

Corollary 2.13. *Let R be a $\Gamma-$ring with descending chain condition on α_0-principle left $\Gamma-$ideals. If M a uniserial $R_\Gamma-$module, then M is fully stable.*

Proof. It is enough to show that M satisfies the descending chain condition on α_0-cyclic. If $m_1, m_2 \in M$ with $R\alpha_0 m_2 \subseteq R\alpha_0 m_1$, then there is $r \in R$ such that $R\alpha_0 m_2 = R\alpha_0 r\alpha_0 m_1$. Thus every descending chain of α_0-cyclic $R_\Gamma-$submodule has the form $R\alpha_0 m \supseteq R\alpha_0 r\alpha_0 m \supseteq R\alpha_0 r_2\alpha_0 r_1\alpha_0 m \supseteq \dots$

Consider the descending chain $R \supseteq R\alpha_0 r_1 \supseteq R\alpha_0 r_1\alpha_0 r_2 \supseteq \dots$

By hypothesis there is a positive integer n such that $R\alpha_0 r_1\alpha_0 r_2 \dots \alpha_0 r_n = R\alpha_0 r_1\alpha_0 r_2 \dots \alpha_0 r_{n+1}$.

Hence M satisfies the descending chain condition for α_0-cyclic R_Γ-submodule . □

In the following we discuss the direct sum of fully stable gamma modules. First, let R be a fully stable left $\Gamma-$ring. Then $M = R \oplus R$ is a free $R_\Gamma-$module of rank two which is not fully stable.

Proposition 2.14. *Let $M = M_1 \oplus M_2$ where M_1, M_2 are fully stable $R_\Gamma-$modules. If $\ell_R^{\alpha_0}(M_1) + \ell_R^{\alpha_0}(M_2) = R$, then M is fully stable.*

Proof. Let K be an $R_\Gamma-$submodule of M. We show that $K = N_1 \oplus N_2$ for some $R_\Gamma-$submodules N_1 and N_2 of M_1 and M_2 respectively. If $k \in K$, then $k = x + y$ where $x\imath M_1$. and $y \in M_2$. Moreover there are $a \in \ell_R^{\alpha_0}(M_1)$ and

$b \in \ell_R^{\alpha_0}(M_2)$ with $a+b=1$. Let $N_1 = \ell_R^{\alpha_0}(M_2)\,\alpha_0$x and $N_1 = \ell_R^{\alpha_0}(M_1)\,\alpha_0$y. Now $x = 1\alpha_0 x = (a+b)\,\alpha_0 x = b\alpha_0 x \in M_1$ and $y = a\alpha_0 y \in M_2$. Then $k = b\alpha_0 x + a\alpha_0 y \in N_1 \oplus N_2$. Thus $K \subseteq N_1 + N_2$. For other inclusion,let $W \in N1 + N2$, then $W = c\alpha_0 x + d\alpha_0 y$ for som $c \in \ell_R^{\alpha_0}(M_2)$ and $d \in \ell_R^{\alpha_0}(M_1)$. $W = c\alpha_O x + d\alpha_o x + c\alpha_o y + d\alpha_0 y = (c+d)\,\alpha_0\,(x+y) \in K$. Therefore $K = N_1 \oplus N_2$. Let $\theta : K \to M$ be an $\theta : K \to M$homomorphism, $J_1 : N_1 \to N_1 \oplus N_2$, $J_2 : N_2 \to N_1 \oplus N_2$ be the injection, and that $\pi_1 : M \to M_1$, $\pi_2 : M \to M_2$ be the projection. Put $\theta_1 = \pi_1 \circ \theta \circ J_1$ and $\theta_2 = \pi_2 \circ \theta \circ J_2$. Then $\theta = \theta_1 + \theta_2$ and $\theta\,(K) = \theta\,(N_1 \oplus N_2) = \theta_1\,(N_1) \oplus \theta_2\,(N_2) \subseteq N_1 \oplus N_2$ and or $(K) \subset K$. This shows that M is fully stable. □

Corollary 2.15. *Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where M_i are $R_\Gamma-$modules, $i = 1, 2, ..., n$ and $\ell_R^{\alpha_0}(M_i) + \cap_{i \neq i}\left(\ell_R^{\alpha_0}(M_j)\right) = R$. Then M is a fully stable $R_\Gamma-$module if and only if each M_i is fully stable.*

Lemma 2.16. *Let $M = A \oplus B$ where A and B are two $R_\Gamma-$submodules of M. If N is a stable $R_\Gamma-$submodule of M, then $N = (A \cap N) \oplus (B \cap N)$.*

Proof. Let $\pi_A : M \to A$ and $\pi_B : M \to B$ be the projections of M onto A and B respectively. Stability of N implies that $\pi_A\,(N) \subseteq A \cap N$ and $\pi_B\,(N) \subseteq B \cap N$. $N = \pi_A\,(N) \oplus \pi_B\,(N) \subseteq (A \cap N) \oplus (B \cap N)$.

If M is a left $R_\Gamma-$module, then $T = End_{R_\Gamma}(M) = \{\alpha |\propto$ is an R_Γ endomorphism of $M\}$ is an Abelian group with usual addition of functions. And T is $\Gamma-$ring with mapping $(f, \alpha, g) \to f\alpha g$, where $f\alpha g\,(m) = g\,(1\alpha f\,(m))$ for all $f, g \in$ T$, \alpha \in \Gamma$ and $m \in M$. Furthermore M is a right $T_\Gamma-$module by $(m, \alpha, f) \to m\alpha f$ where $m\alpha f = f\,(1\alpha m)$ for all $f \in$ T$, \alpha \in \Gamma$, and $m \in M$. □

Let R be a $\Gamma-$ring. An element r of R is called α_0-idempotent if $r = r\alpha_0 r$ Two $R_\Gamma-$endomorphisms f and g are called sum -1 orthogonal α_0-idempotent, if $f + g = 1_M$, $f\alpha_0 g = g\alpha_0 f = 0$ and f, g are α_0-idempotents. We note in the above lemma that π_A and π_B are sum -1 orthogonal α_0-idempotents. Generally we have the following:

Proposition 2.17. *Let M be an $R_\Gamma-$module. Then M is a direct sum of two $R_\Gamma-$submodules if and only if there exist sum -1 orthogonal α_0- idempotents in T.*

Proof. It is enough to assume that there are sum -1 α_0-idempotents fand g of M. Set $N = f(M)$ and $K = g(M)$, then $M = N + K$. Let $x \in N \cap K$. Then $x = f(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. $f(x) = g(x) = 0$ and hence $x = 0$. □

Theorem 2.18. *A direct summand $R_\Gamma-$submodule N of an $R_\Gamma-$module M has the property $M = N \oplus K = N \oplus K'$ implies $K = K'$ if and only if K is a stable in M.*

Proof. Assume that K is a stable $R_\Gamma-$submodule of M, then by lemma (2.16), $K = (K \cap N) \oplus (K \cap K') = K \cap K'$. Hence $K \subseteq K'$. Similarly $K' \subseteq K$ and hence $K = K'$.

Conversely, let $M = N \oplus K$, π_N and π_K the projections of M onto N and K respectively. If K is not stable then there is an $R_\Gamma-$homomorphism $\theta : K \to M$ with $\theta(K) \nsubseteq K$. We may extend θ to M by putting $\theta(n) = 0$ for each $n \in N$. Then $\theta\alpha_0\pi_K = \theta$ and $\theta\alpha_0\pi_N = 0$. The two $R_\Gamma-$endomorphisms $(\pi_N + \pi_N\alpha_0\theta)$ and $(\pi_K - \pi_N\alpha_0\theta)$ are sum -1 orthogonal α_0-idempotent of M By proposition (2.6), M is a direct sum of the $R_\Gamma-$submodules $(\pi_N + \pi_N\alpha_0\theta)(M)$ and $(\pi_K + \pi_N\alpha_0\theta)(M)$, but $(\pi_N + \pi_N\alpha_0\theta)(M) = N$ and $(\pi_K - \pi_N\alpha_0\theta)(M) \nsubseteq K$. Thus they define a direct sum decomposition $M = N \oplus K'$ where $K' \neq K$ which is a contradiction. □

One can restate theorem (2.3) as follows, a complement direct summand of a direct summand of an $R_\Gamma-$module M is unique if and only if it is stable in M. Hence we have the following corollary:

Corollary 2.19. *In a fully stable $R_\Gamma-$modules each direct summand has a unique complement direct summand.*

Now we turn to a unique complement of an arbitrary gamma submodules. First we say that an $R_\Gamma-$module M satisfies condition $(*)$ if for every pair of $R_\Gamma-$submodules N_1 and N_2 of M with $N_1 \cap N_2 = 0$, we have $Hom(N_1, N_2) = Hom(N_2, N_1) = 0$.

Proposition 2.20. *If an $R_\Gamma-$module M satisfies condition $(*)$, then every R_Γ-submodule of M has a unique complement.*

Proof. Let N be an $R_\Gamma-$submodule of M. Define $N' = \{m \in M | \forall n(\neq 0) \in N, (r, \alpha) \in R \times \Gamma, \exists (s, \beta) \in R \times \Gamma / S\beta n \neq 0, s\beta r\alpha m = 0\}$, we show that N' is the unique complement of N. First we show that N' is an $R_\Gamma-$submodule of M. Let $m_1, m_2 \in N', 0 \neq n \in N$ and $(r, \alpha) \in R \times \Gamma$. Then there exists $(s, \beta) \in R \times \Gamma$ such that $s\beta n \neq 0$ and $s\beta r\alpha m_1 = 0$. Since $m_2 \in N'$, for $0 \neq s\beta n \in N$ and $(s\beta r, \alpha) \in R \times \Gamma$, there exists $(t, \gamma) \in R \times \Gamma$ such that $t\gamma\,(s\beta n) \neq 0$ and $t\gamma\,(s\beta r)\,\alpha m_2 = 0$. Thus there exist $(t\gamma s, \beta) \in R \times \Gamma$ such that $(t\gamma s)\,\beta n \neq 0$ and $(t\gamma s)\,\beta r\alpha\,(m_1 + m_2) = 0$ and this shows that $m_1 + m_2 \in N'$. Its an easy matter to see that $R\Gamma N' \subseteq N'$. Its clear that $N \cap N' = 0$. Finally assume $N \cap W = 0$ for some $R_\Gamma-$submodule W of M. If $W \nsubseteq N'$, then there is $w \in W$ and $w \notin N'$, so there exist $0 \neq n \in N$ and $(a, \alpha) \in R \times \Gamma$ such that for any $(r, \beta) \in R \times \Gamma$, the condition $r\beta a\alpha w = 0$ implies that $r\beta n = 0$. This follows that $Hom\,(R\Gamma a\alpha w, R\Gamma n) \neq 0$, but $R\Gamma a\alpha w \cap N = 0$ which contradicts the condition $(*)$. Thus $W \subseteq N'$ and so N' is the unique complement of N. □

It is clear that fully stable $R_\Gamma-$module are satisfying the condition $(*)$, Then we have the following:

Corollary 2.21. *Every $R_\Gamma-$submodule of a fully stable $R_\Gamma-$module has a unique complement.*

3. Some generalizations

Let M be an $R_\Gamma-$module. An $R_\Gamma-$submodule N of M is called essential if N has nontrivial intersection with every nonzero $R_\Gamma-$submodule of M (see [7])

First we consider the following:

Definition 3.1. *An $R_\Gamma-$submodule N of M is called fully ess−stable if each essential $R_\Gamma-$submodule of M is stable.*

It's clear that in a semisimple $R_\Gamma-$module M, the only essential R_Γ-submodule is M itself. So every semisimple $R_\Gamma-$module is fully essential stable, but there are many semisimple $R_\Gamma-$modules which are not fully stable.

Lemma 3.2. *Each $R_\Gamma-$submodule of an $R_\Gamma-$module is direct summand of an essential $R_\Gamma-$submodule.*

Proof. Let M be an $R_\Gamma-$module and N an $R_\Gamma-$submodule of M. We shall see that $N \oplus N^c$ is essential in M. Let K be an $R_\Gamma-$submodule of M with $K \cap (N \oplus N^c) = 0$. Let $a = x + y \in N \cap (N^c \oplus K)$ where $x \in N^c$ and $y \in K$, so $y = a - x \in K \cap (N \oplus N^c) = 0$, so $a = x \in (N \oplus N^c) = 0$. By maximality of N^c, we have $N^c = N^c \oplus K$ and hence $K = 0$. □

Theorem 3.3. *An $R_\Gamma-$module M is fully stable if and only if M is fully ess−stable and satisfies the condition $(*)$.*

Proof. Let N be an $R_\Gamma-$submodule of M and $\theta : N \to M$ an R_Γ-homomorphism. θ can be extended to $\overline{\theta} : N \oplus N^c \to M$ in the susal way. By the hypothesis $\theta(N) = \overline{\theta}(N \oplus N^c) \subseteq N \oplus N^c$. $\pi_{N^c} \circ \theta : N \to N^c$ where $\pi_(N^c)$ is the projection of M onto N^c. By the condition $(*)$ we have $Hom(N, N^c) = 0$ and hence $\theta(N) \subseteq N$. □

Corollary 3.4. *Every semisimple $R_\Gamma-$module with condition $(*)$ is fully stable.*

We have proved in that fully stable $R_\Gamma-$module, each direct summand has a unique complement direct summand. In the following we consider some generalization of fully stable $R_\Gamma-$modules in which precisely each direct summand has a unique complement direct summand.

Definition 3.5. *An $R_\Gamma-$module M is called fully direct−summand stable (simply fully ds−stable) if every direct summand of M is stable. A $\Gamma-$ring R is called right (left) fully ds−stable if it is fully ds−stable right (left) $R_\Gamma-$module.*

It is clear that every fully stable $R_\Gamma-$module is fully ds−stable, but in general the converse is not true. An $R_\Gamma-$module M is uniform if each non−zero $R_\Gamma-$submodule has non−zero intersection with every non−zero $R_\Gamma-$submodule of M. Thus every uniform $R_\Gamma-$module is fully ds−stable, but in particular the $\mathbb{Z}_\mathbb{Z}-$module $\mathbb{Z}$ is fully ds−stable which is not fully stable.

Remark. 1. It is an easy matter to see that an R_Γ-module M is fully ds$-$stable if and only if each direct summand N is fully invariant ($f(N) \subseteq N$, for each $R_\Gamma-$endomorphism of M). Thus every duo $R_\Gamma-$module (in which every $R_\Gamma-$submodule is fully invariant) is fully ds$-$stable and hence we have the following implications:

Fully stable $R_\Gamma-$module $\Rightarrow$ duo $R_\Gamma-$module $\Rightarrow$ fully ds$-$stable $R_\Gamma-$module. Note that the $\mathbb{Z}_\mathbb{Z}-$module $\mathbb{Q}$ is fully ds$-$stable which is not duo.

2. By the above every commutative $\Gamma-$ring is fully ds$-$stable. Let $R = \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$ be the polynomial $R-$ring in two indeterminates x and y over $\mathbb{Z}_2$ modulo the ideal $\langle x^2, y^2\rangle$ and $\overline{R} = R/\langle x,y\rangle$. Since $\ell'_{\overline{R}}(\gamma_{\overline{R}}(\langle x\rangle)) = \ell'_{\overline{R}}(\langle \overline{\overline{x}}, \overline{\overline{y}}\rangle) = \langle \overline{\overline{x}}, \overline{\overline{y}}\rangle \neq \langle \overline{\overline{x}}\rangle$ where $\overline{\overline{x}}$ is an element of $\overline{R}$. Then $\overline{R}$ is not fully stable $\overline{R}-$ring while commutativity of $\overline{R}$ gives that $\overline{R}$ is fully ds$-$stable $\overline{R}-$ring.

The proof of the following proposition is as theorem (2.3), but replace that $R_\Gamma-$submodule by direct summand.

Proposition 3.6. *An $R_\Gamma-$module M is fully ds$-$stable if and only if for each direct summand K of M, there exists unique complement H of K in M such that $M = K \oplus H$.*

Proposition 3.7. *Let $M = N \oplus L$ be an $R_\Gamma-$module. Then N is fully invariant in M if and only if $Hom(N, L) = 0$.*

Proof. Let $\pi_L : M \to L$ be the projection of M onto L and θ an R_Γ-endomorphism of M. If $Hom(N, L) = 0$, then $\pi_L\theta$ must be trivial and hence $\theta(N) \subseteq N$. Conversely, let $w \in Hom(N, L)$. Then w can be extending to $\overline{w} : M \to M$ by $\overline{w}(n + \ell) = w(n)$. If N is fully invariant in M then for all $x \in M, w(x) = \overline{w}(x) \in N$. But $w(x) \in L$ and $N \cap L = 0$. Thus $Hom(N, L) = 0$. □

In the following we give a characterization of fully ds$-$stable gamma modules based on the above proposition and the fact that a direct summand is stable if and only if it is fully invariant.

Proposition 3.8. *An $R_\Gamma-$mdoule M is fully ds$-$stable if and only if for every decomposition $M = N \oplus L$, $Hom(N, L) = 0$.*

In the following we give some useful property of fully ds$-$stable gamma modules. We introduce the following:

Definition 3.9. *An $R_\Gamma-$module M is said to have the summand intersection property (SIP), if the intersection of any two direct summands of M is a direct summand.*

Theorem 3.10. *An $R_\Gamma-$module M has (SIP) if and only if for each decomposition $M = N \oplus K$ and $R_\Gamma-$homomorphism $\theta : N \to K$, $ker(\theta)$ is direct summand of M.*

Proof. Assume that M has (SIP), $M = N \oplus K$ and $\theta : N \to K$ an R_Γ-homomorphism. Let $L = \{x + \theta(x) | x \in N\}$. Since $r\alpha (x + \theta (x)) = r\alpha x + r\alpha\theta (x) = r\alpha x + \theta (r\alpha x)$ for all $r \in R$ and $\alpha \in \Gamma$, then L is an $R_\Gamma-$submodule of M. We claim that $M = L \oplus K$, let $x \in M$, there are $n \in N$ and $k \in K$ such that $x = n + k$. $x = n + f(n) - f(n) + k$, so $x \in L + K$ and hence $M = L + K$. Now, let $x \in L \cap K$. Then $x = n + f(n)$ for some $n \in N$ and hence $n = x - \theta (n) \in N \cap K = 0$. Thus $x = 0$ and hence $M = L \oplus K$. But M has (SIP), then $L \cap N$ is a direct summand of M. It is easily to show that $L \cap N = \ker (\theta)$, so $\ker (\theta)$ is a direct summand of M. Conversely, let N and K be two direct summands of M. Then $M = N \oplus N_1$ and $M = K \oplus K_1$ for some $R_\Gamma-$submodules N_1 and K_1 of M. Let $\pi_{N_1} : M \to N_1$ and $\pi_K : M \to K$ be the projections of M onto N_1 and K respectively. Define $h = (\pi_{N_1} \alpha \pi_K) |_N$. It is clear that h is defined from N onto N_1. By the hypothesis $\ker (h)$ is a direct summand of M. Notice that $\ker (h) = (N \cap K) \oplus (N \cap K_1)$. Now since $N \cap K$ is a direct summand of $\ker (h)$, then $N \cap K$ is a direct summand of M. □

Proposition 3.11. *Every fully ds−stable $R_\Gamma-$module has the (SIP) property.*

Proof. Assume that M is fully ds−stable $R_\Gamma-$module and let $M = A \oplus B$ be a decomposition of M. Proposition (3.3) implies that $Hom(A, B) = 0$, that is the kernel of every $R_\Gamma-$homomorphism from A to B is A which is a direct summand of M. So theorem (3.2) shows that M has the property (SIP).

The converse of proposition (3.4) may not be true, for example, let F be a field and $M = F \oplus F$ as F_F-module. It is clear that M is semisimple and hence has the property (SIP). But M is not fully ds−stable since let $S = \{(x, 0) | x \in F\}$ and $S' = \{(0, y) | y \in F\}$. Then $M = S \oplus S$", but S' is not stable in M.

We noticed that every fully stable gamma module is fully ds−stable and the converse is not true. in the following, we obtain a condition under which the converse is true. First recall that an $R_\Gamma-$module M is regular if for each $m \in M$, there are $\theta : M \to R$ and $\gamma \in \Gamma$ such that $m = \theta(m)\gamma m$ [7]. This is equivalent that every α_0-cyclic $R_\Gamma-$submodule of M is direct summand. □

Proposition 3.12. *Every regular fully ds−stable gamma module is fully stable.*

Proof. Let M be a regular fully ds−stable $R_\Gamma-$module and N be α_0-cyclic $R_\Gamma-$submodule of M. Regularity of M implies that N is a direct summand of M and hence stable. Thus M is fully stable. □

We have proved that every $I-$multiplication $R_\Gamma-$module is fully stable. An $R_\Gamma-$module M is called multiplication of for each $R_\Gamma-$submodule N of M, there is a $\Gamma-$ideal A of R such that $N = A\Gamma M$. [on multiplication $\Gamma-$modules]. Observe that the concepts of fully stable gamma modules and multiplication gamma modules are different, the $\mathbb{Z}_\mathbb{Z}-$module $\mathbb{Z}$ is multiplication which is not fully stable, while the $\mathbb{Z}_\mathbb{Z}-$module is fully stable which is not multiplication. In the following we have:

Proposition 3.13. *Every multiplication gamma module is fully ds−stable.*

Proof. Let M be a multiplication $R_\Gamma-$module, N a direct summand of M and $f : N \to M$ be a $R_\Gamma-$homomorphism. f can be extended to all M to an $R_\Gamma-$homomorphism $g : M \to M$ trivially. Since M is multiplication, then $N = A\Gamma M$ for some $\Gamma-$ideal A of R. Thus $f(N) = g(N) = g(A\Gamma M) = A\Gamma g(M) \subseteq A\Gamma M = N$. □

Corollary 3.14. *Every regular multiplication gamma module is fully stable.*

In the following part we consider the dual of the property (SIP) in gamma modules and its relation with full stability.

Definition 3.15. *An $R_\Gamma-$module M is said to have the sum summand property (SSP) if the sum of two direct summands of M is again a direct summand.*

Definition 3.16. *An $R_\Gamma-$module M is said to have:*

1. *(C_2) if any $R_\Gamma-$submodule of M which is isomorphic to a direct summand of M is itself direct summand.*

2. *(C_3) if for any two direct summands A and B with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.*

It is an easy matter of checking (as in the module theory) that if an $R_\Gamma-$module M has (C_2) then it has (C_3).

Theorem 3.17. *Every fully stable gamma module has the (SSP) property.*

Proof. Let M be a fully stable $R_\Gamma-$module and A, B be direct summands of M. By proposition (3.4), M has the (SIP) and hence $(A \cap B) \oplus L = M$ for some $R_\Gamma-$submodule L of M. By using gamma module law [7], we have $A = (A \cap B) \oplus (L \cap A)$ and $B = (A \cap B) \oplus (L \cap B)$ and hence $A + B = (A \cap B) + [(L \cap A) \oplus (L \cap B)]$

We claim that $(A \cap B) \cap [(L \cap A) \oplus (L \cap B)] = 0$

For if, $w \in (A \cap B) \cap [(L \cap A) \oplus (L \cap B)]$, then there exist $x \in L \cap A$ and $y \in L \cap A$ such that $w = x + y$. Now $y = w - x \in [(A \cap B) + (L \cap A)] \cap (L \cap B) \subseteq A \cap (L \cap B) = 0$. So $y = 0$ and $w = x \in (A \cap B) \cap (L \cap A) = A \cap B \cap L = 0$.

Thus $A + B = (A \cap B) \oplus [(L \cap A) \oplus (L \cap B)] = B \oplus (L \cap A)$.

Again since M has the SIP and L and A are direct summands of M, then $L \cap A$ is a direct summand of M. Corollary (2.4) implies that M has (C_2) and hence (C_3). It follows that $A + B = B \oplus (L \cap A)$ is a direct summand of M. Therefore, M has the (SSP). □

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