Fixed point results via tri-simulation function

Rqeeb Gubran*
Department of Mathematics
Aden University
Yemen
rqeeeb@gmail.com

W.M. Alfaqih
Department of Mathematics
Hajjah University
Yemen
waleedmohd2016@gmail.com

Mohammad Imdad
Department of Mathematics
Aligarh Muslim University
India
mhimdad@yahoo.co.in

Abstract. In this article, we introduce a new type of simulation functions involving three variables and utilize the same to unify several known contractions of the existing literature besides being general enough to yield new contractions. In doing so, we also introduce two new notions (namely: α-permissible and α-orbital permissible mappings) and investigate their relationship with some earlier relevant notions of the existing literature.

Keywords: simulation function, tri-simulation function, α-permissible mapping, α-orbital permissible mapping, α-admissible mapping.

1. Introduction

In 1922, Banach [1] proved that every contraction mapping S defined on a complete metric space (M, d) admits a unique fixed point. Since then, a multitude of contraction conditions enriching the natural contraction were introduced and utilized to generalize Banach celebrated principle (e.g. [2]-[4] and references therein). Of course, every new contraction gives rise yet another theorem which does require an independent proof as well. In 1997, Popa [5] initiated the idea of unifying several contraction conditions in one go. To accomplish this, he introduced the idea of the implicit function which is very extensively utilized in [6]-[11]. Another noted attempt to extend Banach contraction principle is essentially due to Wardowski [12] wherein the author initiated the idea of F-contractions which has been further studied in [13]-[15] besides several other

$*$. Corresponding author

ones. Last but not least, Khojasteh et al. [16] introduced the idea of simulation function which is also designed to unify several contractions.

In what follows, $\mathbb{R}_+$ stands for the set of non-negative real numbers whereas all other involved notions are used in their standard sense. For brevity, we write Su instead of writing $S(u)$. Further, for any u_0, $u_n := S^n u_0 := \underbrace{SS\ldots S}_{n\text{-times}} u_0$.

The following definition is due to Khojasteh et al. [16].

Definition 1.1 ([16]). *Let $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a mapping. Then ξ is called a simulation function if it satisfies the following conditions:*

$(\xi 1)$: $\xi(0,0)$*=0;*

$(\xi 2)$: $\xi(y,x) < x - y$, *for all* $x, y > 0$*;*

$(\xi 3)$: *if* $\{y_n\}$ *and* $\{x_n\}$ *are sequences in* $(0,\infty)$ *with* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$ *, then* $\limsup_{n\to\infty} \xi(y_n, x_n) < 0$.

Argoubi et al. [17] slightly sharpened Definition 1.1 by removing the condition $(\xi 1)$. Moreover, to avoid the symmetry of the simulation function, Roldán-López-de-Hierro et al. [18] modified the condition $(\xi 3)$ as follows:

$(\xi \acute{3})$: if $\{y_n\}$ and $\{x_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$ and $y_n < x_n$, for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \xi(y_n, x_n) < 0$.

For examples of the simulation functions, we refer the readers to [16]-[19].

Definition 1.2 ([16]). *A self-mapping S on a metric space (M,d) is said to be $\mathcal{Z}$-contraction with respect to a simulation function ξ if the following condition is satisfied:*

$$\xi(d(Su,Sv), d(u,v)) \geq 0, \text{ for all } u, v \in M. \tag{1.1}$$

It is worth noting that the idea of simulation function is general enough to unify several contractions of the existing literature but this idea, in its earlier form, is not applicable to contractions involving variables beyond $d(Su,Sv)$ and $d(u,v)$ such as in the following result:

Theorem 1.1 ([4]). *Let (M,d) be a complete metric space and $S : M \to M$ a continuous mapping satisfying $\alpha(u,v)d(Su,Sv) \leq \psi\big(d(u,v)\big)$, for all $u, v \in M$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$ and $\alpha : M \times M \to \mathbb{R}_+$. Assume that the following two conditions hold:*

(*i*) *there exists $u_0 \in M$ such that $\alpha(u_0, Su_0) \geq 1$;*

(*ii*) *S is α-admissible i.e.,*

$$\alpha(u,v) \geq 1 \Rightarrow \alpha(Su,Sv) \geq 1, \text{ for all } u, v \in M.$$

Then S has a fixed point.

As already pointed out, Theorem 1.1 can not be covered by a simulation function. Very recently, Karapinar [19] and later Gubran et al. [20] were able to do so by embedding admissible function into a simulation function. However, the only flaw which continues to persist is that it forces the expression $\alpha(u,v)d(Su,Sv)$ to behave as one element whenever it occurs in the contraction inequality.

The aim of this article is to enlarge the class of simulation functions and overcome the shortcoming mentioned earlier so that the involved terms can occur independently. Specifically speaking, we introduce a new type of simulation functions in three variables and utilize the same to unify several known contractions of the existing literature besides being general enough to yield new contractions.

2. Tri-simulation function

Motivated by forgoing observations, we introduce a new simulation function involving three variables and hence call the same as tri-simulation which runs as follows:

Definition 2.1. *Let $T : \mathbb{R}_+^3 \to \mathbb{R}$ be a mapping. Then T is called a tri-simulation function if it satisfies the following conditions:*

$(T1)$: *$T(z,y,x) < x - yz$, for all $x, y > 0, z \geq 0$;*

$(T2)$: *if $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ are sequences in $(0,\infty)$ such that $y_n < x_n$, for all $n \in \mathbb{N}$, $\lim_{n\to\infty} z_n \geq 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then*

$$\limsup_{n\to\infty} T(z_n, y_n, x_n) < 0.$$

Let us denote the set of all tri-simulation functions by $\mathcal{T}$. To substantiate the definition, let us furnish the following tri-simulation function examples:

Example 2.1. $T(z,y,x) = \lambda x - yz$, for all x, y and $z \in \mathbb{R}_+$, where $\lambda \in [0,1)$.

Example 2.2. $T(z,y,x) = \psi(x) - \phi(yz)$, for all x, y and $z \in \mathbb{R}_+$, where $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$.

Example 2.3. $T(z,y,x) = x - \psi(x) - yz$, for all x, y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi-continuous function such that $\psi(t) = 0$ if and only if $t = 0$.

Example 2.4. $T(z,y,x) = x\psi(x) - yz$, for all x, y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to [0,1)$ is such that $\lim_{t\to r^+} \psi(t) < 1$, for all $r > 0$.

Example 2.5. $T(z,y,x) = \frac{x}{x+1} - yz$, for all x, y and $z \in \mathbb{R}_+$.

Example 2.6. $T(z,y,x) = x - yz - \tau$, for all x, y and $z \in \mathbb{R}_+$, where $\tau > 0$.

Example 2.7. $T(z,y,x) = \psi(x) - z\phi(y)$, for all x, y and $z \in \mathbb{R}_+$, where $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$.

Example 2.8. $T(z,y,x) = x - \frac{f(y,x)}{g(y,x)} yz$, for all x, y and $z \in \mathbb{R}_+$, where $f, g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions w.r.t. each variable and such that $f(s,t) > g(s,t) > 0$, for all $s, t > 0$.

Example 2.9. $T(z,y,x) = \psi(x) - yz$, for all x, y and $z \in \mathbb{R}_+$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is Matkowski function i.e., non-decreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Observe that, $\psi(t) < t$, for all $t > 0$ (see [21], [22]).

Definition 2.2. *A self-mapping S on a metric space (M, d) is said to be $\alpha\mathcal{T}$-contraction with respect to $T \in \mathcal{T}$ if*

$$T(\alpha(u,v), d(Su,Sv), d(u,v)) \geq 0, \text{ for all } u, v \in M, \tag{2.1}$$

where $\alpha : M \times M \to \mathbb{R}_+$.

Remark 2.1. If S is an $\alpha\mathcal{T}$-contraction for some $T \in \mathcal{T}$, then (in view of the condition $(T1)$) we have

$$\alpha(u,v)d(Su,Sv) < d(u,v), \text{ for all distinct } u, v \in M. \tag{2.2}$$

3. α-admissibility

In this section, M stands for a non-empty set, $S : M \to M$ and $\alpha : M \times M \to \mathbb{R}_+$.

Popescu [3] relaxed the α-admissibility (see condition (ii) of Theorem 1.1) as follows:

Definition 3.1 ([3])**.** *S is called α-orbital admissible if for all $u, v \in M$,*

$$\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2v) \geq 1.$$

Karapinar et al. [2] introduced the following new and strong α-admissibility notion:

Definition 3.2 ([2])**.** *S is said to be triangular α-admissible if for all u, v and $w \in M$,*

(i) $\alpha(u,v) \geq 1 \Rightarrow \alpha(Su,Sv) \geq 1$;

(ii) $\{\alpha(u,w) \geq 1$ *and* $\alpha(w,v) \geq 1\} \Rightarrow \alpha(u,v) \geq 1$.

Later on, Popescu [3], enlarges the class of functions of Definition 3.2 as follows:

Definition 3.3 ([3])**.** *S is said to be triangular α-orbital admissible if for all $u, v \in M$,* (*i*) $\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2u) \geq 1$;

(*ii*) $\{\alpha(u, v) \geq 1$ *and* $\alpha(v, Sv) \geq 1\} \Rightarrow \alpha(u, Sv) \geq 1$.

In the same continuation, we introduce the following definition:

Definition 3.4. *S is said to be α-permissible if for all $m \geq n \geq 1$ and $u, v \in M$,*

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(S^n u, S^m v) \geq 1.$$

A relatively weaker version of Definition 3.4 is the following:

Definition 3.5. *S is said to be α-orbital permissible if for all $m \geq n \geq 1$ and $u \in M$,*

$$\alpha(u, Su) \geq 1 \Rightarrow \alpha(S^n u, S^m u) \geq 1.$$

Example 3.1. Let $M = \{0, 1, 2, 3\}$. Define S and α as follows:

$$S(u) = \begin{cases} 1, & \text{if } u = 1; \\ (u+1) \mod 4, & \text{otherwise.} \end{cases}$$

and

$$\alpha(u, v) = \begin{cases} 1, & \text{if } \frac{u+v}{2} \text{ is an integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, $\alpha(0, 2) = 1$ but $\alpha(S0, S^2 2) = 0$ so that S is not α-permissible mapping. However, it is α-orbital permissible as $\alpha(1, S1) = \alpha(1, 1) = 1$ and $S^n 1 = 1$ for all $n \in \mathbb{N}$. Observe that S is vacuously α-orbitally permissible for other cases.

The forthcoming two remarks shed light on certain salient features of Definition 3.5 which we intend to utilize in our later discussions.

Remark 3.1. α-admissibility and α-orbital permissibility are independent. Recall that, the α-orbital permissible mapping in Example 3.1 is not α-admissible as $\alpha(1, 3) = 1$ but $\alpha(S1, S3) = 0$. The following example illustrates the reverse part of the claim:

Example 3.2. Let us take $M = \{0, 1, 2, 3\}$. Define S and α as follows:

$$S(u) = (u+1) \mod 4$$

and

$$\alpha(u, v) = \begin{cases} 0, & \text{if } u + v \text{ is an even number;} \\ 1, & \text{otherwise.} \end{cases}$$

By a routine calculation one can verify that S is α-admissible. However, it is not α-orbital permissible as $\alpha(1, S1) = 1$ but $\alpha(S1, S^3 1) = 0$.

Remark 3.2. Every triangular α-orbital admissible mapping is α-orbital permissible and the reverse implication need not to be true in general. Indeed, if there exists no $u \in M$ such that $\alpha(u, Su) \geq 1$, the assertion holds vacuously. Otherwise, Lemma 7 due to Karapinar et al. [2] (or Lemma 8 due to Popescu [3]) supports the assertion. On the other hand, S in the following example is α-orbital permissible but not triangular α-orbital admissible:

Example 3.3. Let $M = A \cup B$ where $A = \{0, 1\}$ and $B = \{2, 3\}$. Define S and α as follows:

$$S(u) = \begin{cases} 1, & \text{if } u = 0, \\ 0, & \text{if } u = 1, \\ 3, & \text{if } u = 2, \\ 2, & \text{if } u = 3. \end{cases}$$

and

$$\alpha(u, v) = \begin{cases} 1, & \text{if (both } u \text{ and } v \text{ are either in } A \text{ or in } B\text{) or } (u, v) = (1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, $\alpha(1, S1) = 1$ and $\{\alpha(1, 2) = 1$ and $\alpha(2, S2) = 1\}$ but $\alpha(1, S2) = 0$.

The following diagram depicts the inter-relations amongst the six forgoing notions:

$$\begin{array}{ccc} \text{tringular } \alpha\text{-admissible} & \xrightarrow{\textit{implies}} & \text{tringular } \alpha\text{-orbital admissible} \\ \Big\downarrow \textit{implies} & & \Big\downarrow \textit{implies} \\ \alpha\text{-admissibile} & \xrightarrow{\textit{implies}} & \alpha\text{-orbital admissibile} \\ \Big\uparrow \textit{implies} & & \Big\uparrow \textit{implies} \\ \alpha\text{-permissibile} & \xrightarrow{\textit{implies}} & \text{orbital } \alpha\text{-permissibile} \end{array}$$

4. Fixed point results

Our main result in this article runs as follows:

Theorem 4.1. *Let (M, d) be a complete metric space and $S : M \to M$ an $\alpha\mathcal{T}$-contraction under some tri-simulation function T. Suppose that*

(a) S is triangular α-permissible;

(b) there exists $u_0 \in M$ such that $\alpha(u_0, Su_0) \geq 1$;

(c) S is continuous.

Then S has a fixed point w (say) in M. Moreover, for every such u_0 in M, the Picard sequence $\{u_n := S^n u_0\}$ converges to w.

Proof. Choose $u_0 \in M$ such that $\alpha(u_0, u_1) \geq 1$. Then for all $m > n \geq 1$,

$$\alpha(u_n, u_m) \geq 1. \tag{4.1}$$

If $u_m = u_{m+1}$ for some $m \in \mathbb{N}$, then u_m is a fixed point of S so that we are done. Otherwise, let $d(u_n, u_{n+1}) > 0$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} 0 &\leq T(\alpha(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})) \\ &< d(u_n, u_{n+1}) - \alpha(u_n, u_{n+1}) d(u_{n+1}, u_{n+2}), \end{aligned} \tag{4.2}$$

which shows that $\{d(u_n, u_{n+1})\}$ is a strictly decreasing sequence of positive real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then on letting $n \to \infty$ on both sides of (4.2), we get $\lim_{n\to\infty} \alpha(u_n, u_{n+1}) = 1$. Moreover, by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})) < 0,$$

a contradiction. Therefore, for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{4.3}$$

Furthermore, we prove that $\{u_n\}$ is a bounded sequence. To establish the assertion, let on a contrary that $\{u_n\}$ is unbounded. Then, there exists a subsequence $\{u_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(u_{n_k}, u_{n_{k+1}}) > 1 \tag{4.4}$$

and

$$d(u_{n_k}, u_m) \leq 1, \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

Now, using triangular inequality, we have

$$\begin{aligned} 1 &< d(u_{n_k}, u_{n_{k+1}}) \\ &\leq d(u_{n_k}, u_{n_{k+1}-1}) + d(u_{n_{k+1}-1}, u_{n_{k+1}}) \\ &\leq 1 + d(u_{n_{k+1}-1}, u_{n_{k+1}}), \end{aligned}$$

which, on letting $k \to \infty$ in 4.3, gives rise

$$\lim_{k\to\infty} d(u_{n_k}, u_{n_{k+1}}) = 1.$$

By (2.2), we have

$$\alpha(u_{n_k-1}, u_{n_{k+1}-1}) d(u_{n_k}, u_{n_{k+1}}) < d(u_{n_k-1}, u_{n_{k+1}-1}).$$

Now, on using (4.1) and (4.4), we have

$$\begin{aligned} 1 &< \alpha(u_{n_k-1}, u_{n_{k+1}-1}) d(u_{n_k}, u_{n_{k+1}}) \\ &< d(u_{n_k-1}, u_{n_{k+1}-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + d(u_{n_k}, u_{n_{k+1}-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + 1, \end{aligned}$$

which, on letting $k \to \infty$ and using (4.3), gives

$$\lim_{k\to\infty} d(u_{n_k-1}, u_{n_{k+1}-1}) = \lim_{k\to\infty} \alpha(u_{n_k-1}, u_{n_{k+1}-1}) = 1.$$

Hence, on using $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_{n_k-1}, u_{n_{k+1}-1}), d(u_{n_k}, u_{n_{k+1}}), d(u_{n_k-1}, u_{n_{k+1}-1})) < 0,$$

which is a contradiction. Therefore, $\{u_n\}$ is a bounded sequence.

Now, let $c_n =: \sup\{d(u_i, u_j) : i, j \geq n\}$. Observe that, $\{c_n\}$ is decreasing sequence of non-negative real numbers which is bounded due to the boundedness of $\{u_n\}$. Therefore, there exists some $c \geq 0$ such that $\lim_{n\to\infty} c_n = c$. If $c \neq 0$, then by the definition of $\{c_n\}$, for every $k \in \mathbb{N}$ there exist m_k, n_k with $m_k > n_k \geq k$ such that

$$c_k - \frac{1}{k} \leq d(u_{m_k}, u_{n_k}) \leq c_k,$$

which gives rise

$$\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = c. \tag{4.5}$$

Owing to (4.1) and (2.2), we have

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &\leq \alpha(u_{n_k-1}, u_{m_k-1}) d(u_{n_k}, u_{m_k}) \\ &< d(u_{n_k-1}, u_{m_k-1}) \\ &\leq d(u_{n_k-1}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_k-1}), \end{aligned}$$

which is (in view of (4.3) and (4.5)) on letting $k \to \infty$, gives rise

$$\lim_{k\to\infty} d(u_{n_k-1}, u_{m_k-1}) = c \text{ and } \lim_{k\to\infty} \alpha(u_{n_k-1}, u_{m_k-1}) = 1.$$

As S is $\alpha\mathcal{T}$-contraction w.r.t. T, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_{n_k-1}, u_{m_k-1}), d(u_{n_k}, u_{m_k}), d(u_{n_k-1}, u_{m_k-1})) < 0,$$

a contradiction which shows that $c = 0$. Consequently, $\{u_n\}$ is a Cauchy sequence. The completeness of M ensures the existence of some $w \in M$ such that

$$\lim_{n\to\infty} u_n = w.$$

Now, the continuity of S implies that

$$\lim_{n\to\infty} u_n = S \lim_{n\to\infty} u_{n-1} = Sw.$$

Now, the uniqueness of the limit concludes the proof. □

Theorem 4.2. *Theorem 4.1 remains true if one replaces condition* (c) *by the following: If* $\{u_n\}$ *is a sequence in* M *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $u_n \to w \in M$ *as* $n \to \infty$*, then there exists a subsequence* $\{u_{n(k)}\}$ *of* $\{u_n\}$ *such that* $\alpha(u_{n(k)}, w) \geq 1$ *for all* k.

Proof. On the lines of the proof of Theorem 4.1, we know that the Picard sequence $\{u_n\}$ based on u_0 converges to some $w \in M$. From (4.1) and the new condition (c), there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, w) \geq 1$ for all k. In view of (1.1), for all k, we get that

$$\begin{aligned} 0 &\leq T(\alpha(u_{n(k)}, w), d(Su_{n(k)}, Sw), d(u_{n(k)}, w)) \\ &= T(\alpha(u_{n(k)}, w), d(u_{n(k)+1}, Sw), d(u_{n(k)}, w)) \\ &< d(u_{n(k)}, w) - \alpha(u_{n(k)}, w) d(u_{n(k)+1}, Sw) \end{aligned}$$

so that

$$d(u_{n(k)+1}, Sw) = d(Su_{n(k)}, Sw) \leq \alpha(u_{n(k)}, w) d(Su_{n(k)}, Sw) \leq d(u_{n(k)}, w).$$

Letting $k \to \infty$, we have $d(w, Sw) = 0$. □

Observe that, every contraction condition (of the form (1.1)) obtained via a simulation function can be deduced using a suitable tri-simulation function. Indeed, for if the contraction condition obtained via a simulation function ξ, then $T(z, y, x) := \xi(x, yz)$ satisfies conditions $(T1)$ and $(T2)$. Further, inequality (2.1) remains true for $\alpha \equiv 1$.

Now, to exhibit the advantage of Theorem 4.1 (the same can be done for Theorem 4.2), we derive the following two consequences. We begin with the following result which remains a modified version of Theorem 1.1:

Corollary 4.1. *Let* (M, d) *be a complete metric space and* S *a self-mapping on* M *satisfying*

$$\alpha(u, v) d(Su, Sv) \leq \psi\big(d(u, v)\big), \text{ for all } u, v \in M,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing function such that* $\lim_{n\to\infty} \psi^n(t) = 0$*, for all* $t > 0$ *and* $\alpha : M \times M \to \mathbb{R}_+$*. If* S *satisfies conditions* (a) $-$ (c) *of Theorem 4.1, then* S *has a fixed point.*

Proof. The proof follows directly in view of Example 2.9 and Theorem 4.1. □

Remark 4.1. The assumption $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$ in Corollary 4.1 is relatively weaker than the assumption $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$ assumed in Theorem 1.1.

Remark 4.2. In view of Remark 3.2, Corollary 4.1 remains weaker than the one deduced via Theorem 1.4 of Karapinar [19].

The following consequence can be proved in view of Theorem 4.1 and Example 2.7:

Corollary 4.2. *Let (M,d) be a complete metric space and S a self-mapping on M satisfying*

$$\alpha(u,v)\phi\big(d(Su,Sv)\big) \leq \psi\big(d(u,v)\big), \forall u,v \in M, \tag{4.6}$$

where $\phi,\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$, for all $t > 0$ and $\alpha : M \times M \to \mathbb{R}_+$. If S satisfies conditions (a)–(c) *of Theorem 4.1, then S has a fixed point.*

Notice that, Corollary 4.2 can not be deduced using Theorem 1.4 of Karapinar [19] which also substantiate the utility of our extension carried out in this note.

Similarly, one can also derive an existence result corresponding to all other tri-simulation functions defined in this article or elsewhere.

Theorem 4.3. *The fixed point of S ensured by Theorem 4.1 (or Theorem 4.2) remains unique provided one of the following conditions hold:*

(*i*) $\alpha(u,v) \geq 1$ *for all* $u,v \in Fix(S) := \{x \in M : Sx = x\}$.

(*ii*) S *is* α*-permissible and for all* $u,v \in M$ *there exists* $z \in M$ *such that* $\alpha(u,z) \geq 1$ *and* $\alpha(v,z) \geq 1$.

Proof. Let u and v be two distinct fixed points of S. If the condition (i) is satisfied, then

$$\begin{aligned}0 &\leq T(\alpha(u,v), d(Su,Sv), d(u,v))\\ &= T(\alpha(u,v), d(u,v), d(u,v))\\ &< d(u,v) - \alpha(u,v)d(u,v),\end{aligned}$$

a contradiction so that $u = v$ and we are done with the condition (i).

Alternately, if condition (ii) holds, there exists $w \in M$ such that $\alpha(u,w) \geq 1$ and $\alpha(v,w) \geq 1$. If w equals one of these two fixed points (say u), then, processing as in the case of condition (i), we can show that $w = v$ which is a contradiction. Thus, we assume that u, v and w are distinct points. Owing to the α-permissibility of S, we have $\alpha(u,w_n) \geq 1$ and $\alpha(v,w_n) \geq 1$, for all $n \geq 1$. Now, we assert that

$$\lim_{n\to\infty} w_n = u.$$

If $w_m = u$, for some $m \in \mathbb{N}$, then the assertion immediately follows. Otherwise, assume that $d(u,w_n) > 0$, for all $n \in \mathbb{N}$. Now,

$$\begin{aligned}0 &\leq T\big(\alpha(u_n,w_n), d(u_{n+1},w_{n+1}), d(u_n,w_n)\big)\\ &< d(u,w_n) - \alpha(u,w_n)d(u,w_{n+1}).\end{aligned}$$

That is $\{d(u, w_n)\}$ is a strictly decreasing sequence of non-negative real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T\big(\alpha(u, w_n), d(u, w_{n+1}), d(u, w_n)\big) < 0,$$

a contradiction which establishes the assertion. Likewise, we can show that

$$\lim_{n\to\infty} w_n = v.$$

Now, the uniqueness of the limit implies $u = v$ which concludes the proof. □

References

[1] Stefan Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math., 3 (1922), 133-181.

[2] Erdal Karapınar, Poom Kumam, and Peyman Salimi, *On α-ψ-meir-keeler contractive mappings,* Fixed Point Theory Appl., Art ID: 94, 2013.

[3] Ovidiu Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces,* Fixed Point Theory Appl., Art. ID 190, 2014.

[4] Bessem Samet, Calogero Vetro, and Pasquale Vetro, *Fixed point theorems for α-ψ-contractive type mappings,* Nonlinear Anal. Theory Methods Appl., 75 (2012), 2154-2165.

[5] Valeriu Popa, *Fixed point theorems for implicit contractive mappings,* Stud. Cerc. St. Ser. Mat. Univ. Bacau, 7 (1997), 127-133.

[6] Javid Ali and M Imdad, *An implicit function implies several contraction conditions,* Sarajevo J. Math., 4 (2008), 269-285.

[7] Vasile Berinde, *Approximating fixed points of implicit almost contractions,* Hacet. J. Math. Stat., 41 (2012), 93-102.

[8] Vasile Berinde and Francesca Vetro, *Common fixed points of mappings satisfying implicit contractive conditions,* Fixed Point Theory Appl., Art. ID 105, 2012.

[9] Mohammad Imdad, Rqeeb Gubran, Md Ahmadullah, *Using an implicit function to prove common fixed point theorems,* arXiv preprint arXiv:1605.05743, 2016.

[10] Mohammad Imdad, Santosh Kumar, and MS Khan, *Remarks on some fixed point theorems satisfying implicit relations,* Rad. Mat, 11 (2002), 135-143.

[11] Valeriu Popa, Mohammad Imdad, and Javid Ali, *Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces,* Bull. Malays. Math. Sci. Soc., 33 (2010), 105-120.

[12] Dariusz Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory Appl., Art. ID 94, 2012.

[13] Mohammad Imdad, Rqeeb Gubran, Mohammad Arif, and Dhananjay Gopal, *An observation on α-type F-contractions and some ordered-theoretic fixed point results,* Mathematical Sciences, 11 (2017), 247-255.

[14] Rqeeb Gubran, Mohammad Imdad, I. A. Khan, and W. M. Alfaqieh, *Order-theoretic common fixed point results for f-contractions,* Bulletin of Mathematical Analysis and Applications, 10 (2017), 80-88.

[15] Hossein Piri and Poom Kumam, *Wardowski type fixed point theorems in complete metric spaces,* Fixed Point Theory Appl., Art. ID 45, 2016.

[16] Farshid Khojasteh, Satish Shukla, and Stojan Radenović, *A new approach to the study of fixed point theory for simulation functions,* Filomat, 29 (2015), 1189-1194.

[17] Hajer Argoubi, Bessem Samet, and Calogero Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order,* J. Nonlinear Sci. Appl., 8 (2015), 1082-1094.

[18] Antonio Francisco Roldán-López-de Hierro, Erdal Karapıinar, Concepción Roldán-López-de Hierro, and Juan Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions,* J. Comput. Appl. Math, 275 (2015), 345-355.

[19] Erdal Karapınar, *Fixed points results via simulation functions,* Filomat, 30 (2016), 2343-2350.

[20] Rqeeb Gubran, W. M. Alfaqieh, and Mohammad Imdad, *Common fixed point results for α-admissible mappings via simulation function,* The Journal of Analysis, 25 (2017), 281-290.

[21] Janusz Matkowski, *Integrable solutions of functional equations,* Dissertation, 127 (1975), 1-68.

[22] Mohammad Imdad and Rqeeb Gubran, *Ordered-theortic fixed point results for monotone generalized boyd-wong and Matkowski type contractions,* Journal of Advanced Mathematical Studies, 10 (2017), 49-61.

Accepted: 04.05.2019