Oscillation of second-order nonlinear neutral dynamic equations with "Maxima" on time scales with nonpositive neutral term

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Abstract. In this paper, we establish some new oscillation criteria for second order nonlinear neutral delay dynamic equation of the form

$$(r(t)((m(t)y(t) - p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0$$

on a time scale $\mathbb{T}$. The present results not only generalize and extend some existing results but also can be applied to some of the oscillation problems that are not covered before. Finally, we give some examples to illustrate our main results.

Keywords: oscillation, maxima, neutral equation, delay equation, dynamic equation, time scale.

1. Introduction

The theory of time scales was introduced by Hilger [13] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary

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nonempty closed subset of the reals. The cases when time scale equals to the reals or to the integers, the obtained results represent the classical theories of differential and difference equations. Many other interesting time scales exist (e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ which has important applications in quantum theory, $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = H_n$ the set of harmonic numbers). For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [10, 11]. In the last few years, the qualitative theory of differential equations with "Maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real systems (see, for example [4,14]). The oscillatory behavior of solutions of differential equations with "Maxima" is discussed in [2, 5-9, 15, 17,18,20], and the references cited therein. In [2], the authors discussed the oscillatory behavior of the second order quasilinear neutral differential equation with "Maxima"

$$(r(t)((y(t) + p(t)y(\tau(t)))')^{\gamma})' + q(t) \max_{s\in[t-\delta,t]} y^{\beta}(s) = 0, \ t \geq t_0 \geq 0, \tag{1.1}$$

where $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} dt < \infty$. The authors in [17] introduced sufficient conditions for the oscillation of the equation

$$(r(t)((y(t) + p(t)y(\tau(t)))')^{\gamma})' + q(t) \max_{s\in[t-\delta,t]} y^{\gamma}(s) = 0, \ t \geq t_0 \geq 0, \tag{1.2}$$

where, $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} dt < \infty$. The oscillatory and asymptotic behavior of the second order neutral delay difference equation

$$\Delta(a_n \Delta(x_n + p_n x_{n-\tau})) + q_n \max_{[n-\delta,n]} x_n^{\gamma} = 0, \quad n \in \mathbb{N}, \tag{1.3}$$

where $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$ studied in [1]. Also, in [16] Qi Li et al. considered the differential equation

$$(r(t)((y(t) - p(t)y(\tau(t)))')^{\gamma})' + q(t)f(y(\delta(t))) = 0, \ t \geq t_0 > 0, \tag{1.4}$$

where, $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} dt = \infty$. The authors in [3, 12, 19] improved the results obtained in [16]. They introduced sufficient conditions for oscillation of all solutions of Eq. (1.4).

In this paper, we study the oscillation of the second order nonlinear neutral dynamic equation with "Maxima" of the form

$$(r(t)((m(t)y(t) - p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0, \ t \in \mathbb{T}, \tag{1.5}$$

where $\mathbb{T}$ is a time scale. Throughout this paper, we consider the following hypotheses:

(A_1) γ and β_i, $i = 1, 2, ..., n$ are quotient of odd positive integers.

(A_2) $\tau(t), \delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$, $\delta_i(t) \leq t$, $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}(\delta_i(t)) = \infty, i = 1, 2, ..., n$.

(A_3) $r(t)$, $m(t)$, $p(t)$ and $q_i(t)$, $i = 1, 2, ..., n$ are real valued rd-continuous positive functions defined on $\mathbb{T}$, $\lim_{t\to\infty} p(t) = p_0$ and $\lim_{t\to\infty} m(t) = m_0 > 0$ and $m_0 > p_0$. Also, $m(t) > p(t)$.

Also, the following condition is taken into consideration:

$$\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty. \tag{1.6}$$

Now, defining

$$x(t) := m(t)y(t) - p(t)y(\tau(t)). \tag{1.7}$$

Eq. (1.5) reduces to

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) = 0. \tag{1.8}$$

2. Preliminaries and Lemmas

Lemma 2.1. *Assume that conditions $(A_1) - (A_3)$ and (1.6) hold. If $y(t)$ is an eventually positive solution of Eq. (1.5), then $x(t)$ satisfies one of the following two cases:*

(c_1) $x(t) > 0$, $x^{\Delta}(t) > 0$ *and* $(r(t)((x^{\Delta}(t))^{\gamma}))^{\Delta} < 0$.

(c_2) $x(t) < 0$, $x^{\Delta}(t) > 0$ *and* $(r(t)((x^{\Delta}(t))^{\gamma}))^{\Delta} < 0$.

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Proof. Since, $y(t)$ is an eventually positive solution of Eq. (1.5), then, by (A_2) there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta_i(t)) > 0$, $i = 1, 2, ..., n$ for $t \geq t_1$. From Eq. (1.8),

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} = -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s) < 0.$$

Hence, $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing and of one sign, therefore there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for $t \geq t_2$. If $x^{\Delta}(t) > 0$, then either (c_1) or (c_2) occurs. Now, we prove that $x^{\Delta}(t) < 0$ cannot occur. If $x^{\Delta}(t) < 0$, then for $t \geq t_2$,

$$r(t)(x^{\Delta}(t))^{\gamma} \leq -C < 0, \tag{2.1}$$

where $C := -r(t_2)(x^\Delta(t_2))^\gamma > 0$, integrating inequality (2.1) from t_2 to t and using condition (1.6), we get

$$x(t) \leq x(t_2) - C^{\frac{1}{\gamma}} \int_{t_2}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s \to -\infty \; as \; (t \to \infty).$$

Hence,

$$\lim_{t\to\infty} x(t) = -\infty.$$

Now, we consider the following two cases.
Case(1): If $y(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} y(t_k) = \infty$, where $y(t_k) = \max\{y(s) : t_0 \leq s \leq t_k\}$. For sufficiently large k, we have $\tau(t_k) > t_0$. Using condition A_2, we get

$$y(\tau(t_k)) = \max\{y(s) : t_0 \leq s \leq \tau(t_k)\} \leq \max\{y(s) : t_0 \leq s \leq t_k\} = y(t_k).$$

Hence, $x(t_k) = m(t_k)y(t_k) - p(t_k)y(\tau(t_k)) \geq m(t_k)y(t_k) - p(t_k)y(t_k) \geq m(t_k)(1 - \frac{p(t_k)}{m(t_k)})y(t_k) > 0$, which contradicts $\lim_{t\to\infty} x(t) = -\infty$.
Case(2): If $y(t)$ is bounded, then using condition (A_3), $x(t)$ is bounded. This contradicts $\lim_{t\to\infty} x(t) = -\infty$. Hence, $x^\Delta(t) < 0$ cannot occur.

Remark 2.1. From Eq. (1.8), we have

$$(r(t)(x^\Delta(t))^\gamma)^\Delta = -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s). \tag{2.2}$$

If (c_1) holds. From the definition of $x(t)$, we have

$$\min_{s\in[\delta_i(t),t]} (\frac{x^{\beta_i}(s)}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} (\frac{x^{\beta_i}(s)}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s), \; i = 1, 2, ..., n.$$

Since $x^\Delta > 0$, then

$$x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}) \leq \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s), \; i = 1, 2, ..., n.$$

Hence,

$$-\sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}) \geq -\sum_{i=1}^{n} q_i(t) \max_{s\in[\delta_i(t),t]} y^{\beta_i}(s). \tag{2.3}$$

From Eq. (2.2) and inequality (2.3), we get

$$(r(t)(x^\Delta(t))^\gamma)^\Delta \leq -\sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}). \tag{2.4}$$

Lemma 2.2. *The function $y(t)$ is a negative solution of Eq. (1.5) if and only if $-y(t)$ is a positive solution of the equation*

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+\sum_{i=1}^{n}q_i(t)\min_{s\in[\delta_i(t),t]}y^{\beta_i}(s)=0. \tag{2.5}$$

Proof. The assertion of Lemma 2.2 can be verified easily.

Define,

$$R(t)=r^{\frac{1}{\gamma}}(t)\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s,$$
$$A(l,a)=\int_{a}^{l}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s$$

and $\eta(t,a)=\frac{A(\delta(t),a)}{A(\sigma(t),a)}$.

Lemma 2.3. *Assume that the conditions* $(A_1)-(A_3)$ *and* (1.6) *hold. If* $y(t)$ *is an eventually positive solution of Eq.* (1.5) *and* (c_1) *in Lemma 2.1 holds, then*

(1) $x(t)\geq R(t)x^{\Delta}(t)$ *for* $t\geq t_1$.

(2) $x(\delta(t))\geq\eta(t,t_2)x(\sigma(t))$ *for* $t\geq\delta(t)\geq t_2\geq t_1$.

Proof. Since (c_1) holds,
(1) for $t\geq t_1$, we have

$$x(t)=x(t_1)+\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta s\geq\ r^{\frac{1}{\gamma}}(t)x^{\Delta}(t)\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s=R(t)x^{\Delta}(t),$$

(2) for $t>t_2\geq t_1$, we have

$$x(\sigma(t))-x(\delta(t))=\int_{\delta(t)}^{\sigma(t)}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta s\leq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))\int_{\delta(t)}^{\sigma(t)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s.$$

Therefore,

$$\frac{x(\sigma(t))}{x(\delta(t))}\leq 1+\frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))}A(\sigma(t),\delta(t)). \tag{2.6}$$

Also,

$$x(\delta(t))\geq x(\delta(t))-x(t_2)=\int_{t_2}^{\delta(t)}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s$$
$$\geq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))\int_{t_2}^{\delta(t)}\frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s.$$

Hence,

$$\frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))} \leq \frac{1}{A(\delta(t),t_2)}. \tag{2.7}$$

From inequalities (2.6) and (2.7), we get

$$\frac{x(\sigma(t))}{x(\delta(t))} \leq 1 + \frac{A(\sigma(t),\delta(t))}{A(\delta(t),t_2)} = \frac{A(\sigma(t),\delta(t)) + A(\delta(t),t_2)}{A(\delta(t),t_2)} = \frac{A(\sigma(t),t_2)}{A(\delta(t),t_2)}.$$

Hence,

$$x(\delta(t)) \geq \eta(t,t_2)x(\sigma(t)). \tag{2.8}$$

This completes the proof.

Remark 2.2. Without loss of generality we can deal only with the positive solution of Eq. (1.5), since the proof for the negative solution is similar.

3. Main results

Theorem 3.1. *Assume that conditions* $(A_1) - (A_3)$ *and* (1.6) *hold,* $\gamma \geq \beta_i$, $i = 1,2,...,n$ *and there exists a positive rd-continuous* Δ*-differentiable function* $z(t)$ *such that for a constant* $b > 0$*,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} [Q(u) - \frac{z_+^{\Delta}(u)}{A^{\gamma}(u,t_1)}]\Delta u = \infty, \tag{3.1}$$

where

$$Q(u) = z(\sigma(u)) \sum_{i=1}^{n} \frac{q_i(u)\eta_i^{\beta_i}(u,t_2)}{b^{\gamma-\beta_i}(A(\sigma(u),t_1))^{\gamma-\beta_i}} \min_{s\in[\delta_i(u),u]} (\frac{1}{m^{\beta_i}(s)}),$$

$$A(\sigma(u),t_1) = \int_{t_1}^{\sigma(u)} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s, z_+^{\Delta}(u) = \max\{z^{\Delta}(u), 0\}$$

and $\eta_i(t,a) = \frac{A(\delta_i(t),a)}{A(\sigma(t),a)}$, $i = 1,2,...,n$. *Then, every solution of Eq.* (1.5) *is either oscillatory or tends to zero.*

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5), then by $(A_1) - (A_3)$ there exists $t_1 \in \mathbb{T}$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta_i(t)) > 0$, $i = 1,2,...,n$ for $t \geq t_1$ and Lemma 2.1 holds. If (c_1) holds, define the function $w(t)$ by the Riccati substitution

$$w(t) := z(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \quad for \quad t \geq t_1, \tag{3.2}$$

then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= z^{\Delta}(t)(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}) + z(\sigma(t))(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)})^{\Delta} \\ &= z^{\Delta}(t)(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}) \\ &+ z(\sigma(t))(\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}x^{\gamma}(t) - (x^{\gamma}(t))^{\Delta}r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}) \\ &= z^{\Delta}(t)(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}) \\ &+ z(\sigma(t))\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{x^{\gamma}(\sigma(t))} - z(\sigma(t))\frac{(x^{\gamma}(t))^{\Delta}r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \end{aligned} \tag{3.3}$$

Using $\gamma > 0$ and Keller's chain rule, we get

$$\begin{aligned} (x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1 [x(t) + h\mu(t)x^{\Delta}(t)]^{\gamma-1}dh\, x^{\Delta}(t) \\ &= \gamma\int_0^1 [(1-h)x + hx^{\sigma}]^{\gamma-1}dh\, x^{\Delta}(t). \end{aligned}$$

Hence,

$$(x^{\gamma}(t))^{\Delta} \geq \begin{cases} \gamma x^{\gamma-1}(t)x^{\Delta}(t) & \text{if } \gamma \geq 1, \\ \gamma(x^{\sigma})^{\gamma-1}(t)x^{\Delta}(t) & \text{if } \gamma < 1. \end{cases} \tag{3.4}$$

From inequalities (2.4), (3.3) and (3.4), we get

$$w^{\Delta}(t) \leq \begin{cases} z_{+}^{\Delta}(t)(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}) \\ -z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)x^{\beta_i}(\delta_i(t))}{x^{\gamma}(\sigma(t))}\min_{s\in[\delta_i(t),t]}(\frac{1}{m^{\beta_i}(s)}) \\ -z(\sigma(t))\frac{\gamma x^{\gamma-1}x^{\Delta}(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}, \text{ if } \gamma \geq 1, \\ z_{+}^{\Delta}(t)(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}) \\ -z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)x^{\beta_i}(\delta_i(t))}{x^{\gamma}(\sigma(t))}\min_{s\in[\delta_i(t),t]}(\frac{1}{m^{\beta_i}(s)}) \\ -z(\sigma(t))\frac{\gamma(x^{\sigma})^{\gamma-1}x^{\Delta}(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}, \text{ if } \gamma < 1. \end{cases} \tag{3.5}$$

Using Lemma 2.3 in inequality (3.5), we get

$$w^{\Delta}(t) \leq \frac{z_{+}^{\Delta}(t)}{A^{\gamma}(t,t_1)} - z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)\eta_i^{\beta_i}(t,t_2)}{x^{\gamma-\beta_i}(\sigma(t))}\min_{s\in[\delta_i(t),t]}(\frac{1}{m^{\beta_i}(s)}). \tag{3.6}$$

Since,

$$x(t)-x(t_1)=\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta(s)\leq r^{\frac{1}{\gamma}}(t_1)x^{\Delta}(t_1)A(t,t_1),$$

then, there exists $T\in[t_1,\infty)_{\mathbb{T}}$ and a suitable constant $b>0$ such that

$$x(t)\leq bA(t,t_1) \text{ for } t\in[T,\infty)_{\mathbb{T}}. \tag{3.7}$$

From inequalities (3.6) and (3.7), we have

$$w^{\Delta}(t)\leq\frac{z_{+}^{\Delta}(t)}{A^{\gamma}(t,t_1)}-z(\sigma(t))\sum_{i=1}^{n}\frac{q_i(t)\eta_i^{\beta_i}(t,t_2)}{b^{\gamma-\beta_i}A^{\gamma-\beta_i}(\sigma(t),t_1)}\min_{s\in[\delta_i(t),t]}(\frac{1}{m^{\beta_i}(s)}). \tag{3.8}$$

Integrating from $t_3=\max\{t_1,t_2,T\}$ to t and taking the limit supremum of both sides as $t\to\infty$, we get a contradiction with condition (3.1).

If (c_2) holds, then $\lim_{t\to\infty}y(t)=0$ (the proof is similar to that of Lemma 2.2 [16]). This completes the proof.

Theorem 3.2. *Assume that conditions* $(A_1)-(A_3)$ *and* (1.6) *hold,* $\gamma\geq\beta_i$, $\gamma\geq 1$, $i=1,2,...,n$ *and there exists a positive rd-continuous* Δ*-differentiable function* $z(t)$ *such that for a constant* $b>0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q_1(u)-\frac{r(u)(z_{+}^{\Delta}(u))^2}{4\gamma z(u)R^{\gamma-1}(u)}]\Delta u=\infty, \tag{3.9}$$

where

$$Q_1(u)=z(u)\sum_{i=1}^{n}\frac{q_i(u)\eta_i^{\beta_i}(u,t_2)}{b^{\gamma-\beta_i}(A(\sigma(u),t_1))^{\gamma-\beta_i}}\min_{s\in[\delta_i(u),u]}(\frac{1}{m^{\beta_i}(s)}),$$

$$A(\sigma(u),t_1)=\int_{t_1}^{\sigma(u)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s,\ z_{+}^{\Delta}(u)=\max\{z^{\Delta}(u),0\}$$

and $\eta_i(t,a)=\frac{A(\delta_i(t),a)}{A(\sigma(t),a)}$, $i=1,2,...,n$. *Then, every solution of Eq.* (1.5) *is either oscillatory or tends to zero.*

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5), then by $(A_1)-(A_3)$ there exists $t_1\in\mathbb{T}$ sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta_i(t))>0$, $i=1,2,...,n$ for $t\geq t_1$ and Lemma 2.1 holds. If (c_1) holds, define the function $w(t)$ by the Riccati substitution

$$w(t):=\frac{z(t)}{x^{\gamma}(t)}r(t)(x^{\Delta}(t))^{\gamma}\ \ for\ \ t\geq t_1, \tag{3.10}$$

then $w(t)>0$ and

$$w^{\Delta}(t)=\frac{z(t)}{x^{\gamma}(t)}(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}$$

$$+r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma}(\frac{z^{\Delta}(t)x^{\gamma}(t)-z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}). \tag{3.11}$$

Using inequalities (2.4) and (3.4),

$$
\begin{aligned}
w^{\Delta}(t) \leq & \frac{-z(t)}{x^{\gamma}(t)} \sum_{i=1}^{n} q_i(t) x^{\beta_i}(\delta_i(t)) \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)}) \\
& + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{z(t)}{z(\sigma(t))} w^{\sigma} \gamma \frac{x^{\Delta}}{x}.
\end{aligned} \tag{3.12}
$$

Using Lemma 2.3 and inequality (3.7), we get

$$
w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{\gamma z(t)}{z(\sigma(t))} w^{\sigma} \frac{r(t)(x^{\Delta})^{\gamma} R^{\gamma-1}(t)}{r(t) x^{\gamma}}, \tag{3.13}
$$

where,

$$
Q_1(t) := \sum_{i=1}^{n} \frac{z(t) q_i(t) \eta_i^{\beta_i}(t,t_2)}{b^{\gamma-\beta_i} (A(\sigma(t),t_1))^{\gamma-\beta_i}} \min_{s\in[\delta_i(t),t]} (\frac{1}{m^{\beta_i}(s)})
$$

Since $(r(t)(x^{\Delta}(t)^{\gamma})^{\Delta} < 0$ and $x^{\Delta}(t) > 0$,

$$
w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))} w^{\sigma} - \frac{\gamma z(t)}{r(t) z^2(\sigma(t))} R^{\gamma-1}(t) (w^{\sigma})^2. \tag{3.14}
$$

Hence,

$$
w^{\Delta}(t) \leq -Q_1(t) + \frac{r(t)(z_+^{\Delta}(t))^2}{4\gamma z(t) R^{\gamma-1}(t)} \tag{3.15}
$$

Integrating from $t_1$to t and taking the limit supremum of both sides as $t \to \infty$, we get a contradiction with condition (3.9).

If (c_2) holds, the proof is similar to that of Theorem 3.1. This completes the proof.

Define $D = \{(t,s) \in T^2 : t \geqslant s \geqslant 0\}$ and $\mathbb{H} = \{H(t,s) \in C_{rd}(\mathbb{D}, \mathbb{R}) : H(t,t) = 0,\ H(t,s) > 0$ and H^{Δ_s} exists for $t > s \geq 0\}$.

Theorem 3.3. *Assume that conditions $(A_1)-(A_3)$ and (1.6) hold, $\gamma \geq \beta_i, \gamma \geq 1$, $i = 1,2,...,n$ and there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a function $H \in \mathbb{H}$ and constant $b > 0$,*

$$
\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,u) Q_1(u) - \frac{r(u) z^2(\sigma(u)) C^2(t,u)}{4\gamma z(u) R^{\gamma-1}(u) H(t,u)}] \Delta u = \infty, \tag{3.16}
$$

where $Q_1(u)$ is the same as in Theorem 3.2 and

$$
C(t,u) = [H^{\Delta_u}(t,u) + \frac{z_+^{\Delta}(u)}{z(\sigma(u))} H(t,u)].
$$

Then, every solution of Eq. (1.5) is either oscillatory or tends to zero.

Proof. Suppose that $y(t)$ is an eventually positive solution of Eq. (1.5). Proceeding as in the proof of Theorem 3.2, we get

$$w^{\Delta}(t) \leq -Q_1(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))}w^{\sigma} - \frac{\gamma z(t)}{r(t)z^2(\sigma(t))}R^{\gamma-1}(t)(w^{\sigma})^2.$$

For a function $H \in \mathbb{H}$, we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)Q_1(s)\Delta s \leq & -\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_2}^{t} \frac{z_+^{\Delta}(s)}{z(\sigma(s))}H(t,s)w^{\sigma}(s)\Delta s \\ & -\int_{t_2}^{t} \frac{\gamma z(s)R^{\gamma-1}(s)}{r(s)z^2(\sigma(s))}(w^{\sigma}(s))^2 H(t,s)\Delta s. \qquad (3.17)\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}-\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s &= -H(t,s)w(s)\mid_{t_2}^{t} + \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &= H(t,t_2)w(t_2) + \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s. \qquad (3.18)\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{t_2}^{t} H(t,s)Q_1(s)\Delta s \leq & H(t,t_2)w(t_2) + \int_{t_2}^{t}[H^{\Delta_s}(t,s) + \frac{z_+^{\Delta}(s)}{z(\sigma(s))}H(t,s)]w^{\sigma}(s)\Delta s \\ & -\int_{t_2}^{t} \frac{\gamma z(s)R^{\gamma-1}(s)}{r(s)z^2(\sigma(s))}H(t,s)(w^{\sigma}(s))^2\Delta s \qquad (3.19) \\ \leq & H(t,t_2)w(t_2) + \int_{t_2}^{t} \frac{r(s)c^2(t,s)z^2(\sigma(s))}{4\gamma z(s)R^{\gamma-1}(s)H(t,s)}\Delta s.\end{aligned}$$

Taking the limit supremum of both sides as $t \to \infty$, we get a contradiction to condition (3.16). If (c_2) holds, the proof is similar to that of Theorem 3.1. This completes the proof.

Theorem 3.4. *Assume that conditions $(A_1) - (A_3)$ and (1.6) hold, $\beta_i \geq \gamma$, $i = 1, 2, ..., n$ and there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a constant $b > 0$,*

$$\limsup_{t\to\infty} \int_{t_0}^{t}[Q_2(u) - \frac{z_+^{\Delta}(u)}{A^{\gamma}(u,t_1)}]\Delta u = \infty, \qquad (3.20)$$

where

$$Q_2(u) = \sum_{i=1}^{n} z(\sigma(u))q_i(u)\eta_i^{\beta_i}(u,t_2)b^{\beta_i-\gamma} \min_{s\in[\delta_i(u),u]}(\frac{1}{m^{\beta_i}(s)}),$$

$A(u,t_1) = \int_{t_1}^{u}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s$, *$z_+^{\Delta}(u) = \max\{z^{\Delta}(u), 0\}$ and $\eta_i(t,a) = \frac{A(\delta_i(t),a)}{A(\sigma(t),a)}$, $i = 1, 2, ..., n$. Then, every solution of Eq. (1.5) is either oscillatory or tends to zero.*

Proof. The proof is similar to that of Theorem 3.1.

4. Examples

To illustrate our results, we give some examples of second order neutral delay differential equations with "Maxima" which cannot be studied by the previous known criteria of oscillation.

Example 4.1. Consider the equation

$$(4.1)\qquad ((((2+\frac{1}{t})y(t)-(1+\frac{1}{t^2})y(\frac{t}{2}))')^{\frac{5}{3}})'+\lambda\sqrt{t}\max_{s\in[\frac{t}{3},t]} y^{\frac{3}{5}}(s)=0,\ t\in[1,\infty).$$

Here, $\gamma=\frac{5}{3}$, $\beta=\frac{3}{5}$, $r(t)=1$, $m(t)=2+\frac{1}{t}$, $p(t)=1+\frac{1}{t^2}$, $q(t)=\lambda\sqrt{t}$, $\tau(t)=\frac{t}{2}$ and $\delta(t)=\frac{t}{3}$. Hence, $\gamma>\beta$, $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt=\infty$ and $m(t)>p(t)$.

Also,

$$A(\sigma(u),t_1)=A(u,t_1)=\int_{t_1}^{u}(\frac{1}{r(t)})^{\frac{1}{\gamma}}dt=u-t_1,$$

and

$$\eta(u,t_2)=\frac{u-3t_2}{3(u-t_2)}.$$

Choosing $z(t)=1$, then $z'=0$ and

$$Q(u)=\frac{\lambda u^{\frac{1}{2}}(u-3t_2)^{\frac{3}{5}}u^{\frac{3}{5}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}(u-t_2)^{\frac{3}{5}}(u-t_1)^{\frac{16}{15}}(2u+3)^{\frac{3}{5}}}>\frac{\lambda(u-3t_2)^{\frac{17}{10}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}u^{\frac{34}{15}}}.$$

Hence,

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q(u)-\frac{z'_+(u)}{A^{\gamma}(u,t_1)}]\Delta u>\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda(u-3t_2)^{\frac{17}{10}}}{3^{\frac{3}{5}}b^{\frac{16}{15}}u^{\frac{34}{15}}}du=\infty,$$

if $\lambda>0$. Then, by Theorem 3.1 every solution of Eq. (4.1) is either oscillatory or tends to zero if $\lambda>0$.

Example 4.2. Consider the equation

$$(4.2)\qquad (((y(t)-\frac{1}{2}y(t-1))')^{\frac{7}{5}})'+\max_{s\in[t-5,t]}y(s)+\max_{s\in[\frac{t}{2},t]}y^{\frac{1}{7}}(s)=0,\ t\in[1,\infty),$$

choosing $z(t)=1$, one can easily verify that condition (3.1) is satisfied. Then, by Theorem 3.1 every solution of Eq. (4.2) is either oscillatory or tends to zero.

Example 4.3. Consider the equation

$$(4.3)\qquad (((3y(t)-(1+\frac{1}{t})y(t-2))')^{\frac{1}{7}})'+\lambda t\max_{s\in[t-3,t]}y(s)=0,\ t\in[1,\infty),$$

choosing $z(t)=1$, one can easily verify that "condition (3.25) is satised if $\lambda>0$. Then, by Theorem 3.4 every solution of Eq. (4.3) is either oscillatory or tends to zero if $\lambda>0$.

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